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NON-SEMISIMPLE TOPOLOGICAL QUANTUM FIELD THEORIES FOR 3-MANIFOLDS WITH CORNERS

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Abstract. — In this book we describe extended topological quantum field theories (TQFT's) as double functors between two naturally defined double categories: one of topological nature, made of 3-manifolds with corners, the other of algebraic nature, made of linear categories, functors, vector spaces and maps. The conventional notion of TQFT's of Atiyah's, as well as the notion of a modular functor from axiomatic conformal field theory are unified in this concept. We construct a large class of such extended TQFT's, assigning a double functor to every abelian modular category, which does not have to be semisimple. Most of the known quantum invariants turn out to be special cases of our construction.

Résumé (*Théories des champs quantiques topologiques non-semisimples pour les variétés de dimension 3 à coins*)

Dans ce livre nous décrivons théories des champs quantiques topologiques comme des foncteurs doubles d'une catégorie double de la nature topologique (formée par les variétés de dimension 3 à coins) dans une autre catégorie double de la nature algébrique (formée par les catégories linéaires, des foncteurs, des espaces vectoriels et applications). La définition traditionnelle d'Atiyah, ainsi que la notion de foncteur modulaire de théorie axiomatique des champs conformes sont unifiées dans ce concept. Nous construisons une large classe de telles théories des champs quantiques topologiques. Chaque foncteur double correspond à une catégorie abélienne modulaire, qui ne doit pas être semisimple. La plupart des invariants quantiques connus se réduit aux cas spéciales de notre construction.

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CHAPTER 0

INTRODUCTION AND SUMMARY OF RESULTS

In the last decade quantum field theory and string theory have strongly impacted many areas of mathematics, especially the geometry and topology of low dimensional manifolds. In particular, a wealth of intriguing mathematical structures were discovered to be inherent to so called *topological quantum field theories* (TQFT's) and *conformal field theories* (CFT's). Originally, these notions refer to a class of concrete physical quantum field theories, among which three dimensional Chern-Simon's theory and two dimensional rational conformal field theory are some of the most prominent ones. It was realized by many people that the abstract setting of category theory makes it possible to efficiently organize the zoo of data and structures of these field theories. Eventually, TQFT's evolved into purely mathematical notions, defined axiomatically in the language of categories and functors. Axiomatic TQFT's and similar theories are, therefore, in nature rather similar to other functors in algebraic topology, such as homology. Atiyah was the first mathematician to cast the notion of TQFT's into an axiomatic framework in his seminal work [**Ati88**]. Let us start by recalling the essential ingredients of his theory, which applies to quantum field theories on manifolds with smooth boundaries. Subsequently, we generalize the axiomatics, in order to include manifolds with corners and unify TQFT's and CFT's into one single theory.

0.1. Atiyah's TQFT Axioms via Categories:

According to the axioms of Atiyah [**Ati88**], a TQFT $\mathcal{V}$ in dimension $(d+1)$ assigns to a d-dimensional oriented manifold $\Sigma^{(d)}$ a vector space $\mathcal{V}(\Sigma^{(d)})$, and to an oriented $(d+1)$-dimensional manifold, whose boundary is a disjoint union of d-dimensional manifolds $-\Sigma_0^{(d)}$ and $\Sigma_1^{(d)}$, a linear map $\mathcal{V}(M^{(d+1)}) : \mathcal{V}(\Sigma_0^{(d)}) \to \mathcal{V}(\Sigma_1^{(d)})$. The manifold $-\Sigma_0^{(d)}$ is $\Sigma_0^{(d)}$ with the opposite orientation. The *gluing axiom* in [**Ati88**] requires that if we glue two such $(d+1)$-manifolds together along a common boundary component, the linear map for the composite has to be the composition of the linear maps of the individual $(d+1)$-manifolds.

Using the language of categories and functors, as in [**Mac88**], we can state Atiyah's axioms very concisely as follows:

0.1.1 Definition ([Ati88]). A topological quantum field theory in dimension d is a functor between *symmetric monoidal categories* [**Mac88**] as follows:

$$\mathcal{V} \ : \ \mathbf{Cob}^0_{(d+1)} \longrightarrow \Bbbk\text{-vect}\,.$$

Here $\Bbbk$-vect denotes the category, whose objects are finite dimensional vector spaces over a perfect field $\Bbbk$, for instance, a field of characteristic 0. The set of morphisms between two vector spaces is simply the set of linear maps with the usual composition. The category $\mathbf{Cob}^0_{(d+1)}$ has as objects closed oriented d-dimensional manifolds. A morphism between two such d-manifolds $\Sigma_0^{(d)}$ and $\Sigma_1^{(d)}$ is a $(d+1)$-cobordism, meaning an oriented $(d+1)$-dimensional manifold, $M^{(d+1)}$, whose boundary $\partial M^{(d+1)} = -\Sigma_0^{(d)} \sqcup \Sigma_1^{(d)}$ is the disjoint union of the two surfaces. Given another cobordism $N^{(d+1)}$, between $\Sigma_1^{(d)}$ and $\Sigma_2^{(d)}$ in the above sense, we define the composite by $M^{(d+1)} \circ N^{(d+1)} = M^{(d+1)} \cup_{\Sigma_1^{(d)}} N^{(d+1)}$. The union $\cup_{\Sigma_1^{(d)}}$ stands for the quotient space of disjoint union, in which we have glued the two $(d+1)$-manifolds along the common boundary component $\Sigma_1^{(d)}$. Atiyah's gluing axiom is thus implied by functoriality: $\mathcal{V}(M \circ N) = \mathcal{V}(M) \cdot \mathcal{V}(N)$.

The term *monoidal* in Definition 0.1.1 means that $\mathcal{V}$ respects the natural tensor structures on the two categories. The tensor product on $\mathbf{Cob}^0_{(d+1)}$ is given by disjoint union, and the product on $\Bbbk$-vect – by the usual tensor product $\otimes_{\Bbbk}$. These conditions allow us to infer the remaining set of axioms from [**Ati88**].

Note, that in their original form Atiyah's axioms associate to any $(d+1)$-manifold M with boundary a *vector* $\mathcal{V}(M)$ in $\mathcal{V}(\partial M)$. The assignment of linear maps and the composition rule follow from the additional axioms for tensor products and duals via the identifications $\mathcal{V}(\partial M) = \mathcal{V}(-\Sigma_0 \sqcup \Sigma_1) \simeq \mathcal{V}(\Sigma_0)^* \otimes \mathcal{V}(\Sigma_1) \simeq \mathrm{Hom}(\mathcal{V}(\Sigma_0), \mathcal{V}(\Sigma_1))$.

The axiomatic formulation in [**Ati88**] was mainly inspired by Witten's investigation [**Wit89**] of the Chern-Simons field theory, giving rise to a TQFT with 3-dimensional cobordisms ($d = 2$). Although the functional integral formulation used by Witten is a priori not rigorous, it lends itself nicely to illustrate the implied properties of a TQFT as outlined in Appendix A. Shortly after Witten's work Reshetikhin and Turaev, in their ground breaking paper [**RT91**], managed to give a rigorous definition of the same 3-manifold invariants using quantum groups. The generalization of their approach to cobordisms and TQFT's is realized by Turaev in his book [**Tur94**].

At around the same time Moore, Seiberg, Segal, and others found similar categorical structures for rational conformal field theories on 2-dimensional surfaces, see [**MS89**] and references therein. Witten [**Wit89**] also realized that the restriction of a Chern-Simons theory on a 3-manifold M to its boundary ∂M yields precisely such a CFT. The important new ingredient in CFT is that one considers surfaces with boundaries so that surfaces can be "sewn together" along their boundary components. In the physical interpretation the holes or punctures in the 2-dimensional surface give the locations where charges are inserted into the theory. The corresponding observables in Chern-Simons theory are currents in 3-dimensions that run along

Wilson lines, thus creating a charge where they end in the bounding surface. See Appendix A for a more detailed exposition. This leads us to consider 3-cobordisms, from which we excise tubular neighborhoods of embedded lines. The excision at an end point of a line on the surface thus results in the removal of a disc from the surface at this location.

In order to generalize Atiyah's axioms to include also the sewing operations of the related CFT's besides the gluing axioms of Chern-Simons theory, we need to extend them to the 3-manifolds with corners obtained from tubular excisions along lines. As a consequence, the notion of a TQFT can no longer be formulated by ordinary categories and functors, but we have to pass to higher category theory. The definition of an extended TQFT as a double functor of double categories is the subject of the next paragraph.

0.2. Extended TQFT's via Double Categories:

The fact that a unification of Atiyah's TQFT axioms and the axioms of CFT requires the use of some sort of higher category theory was realized by many people independently. The simplest generalization is that of a *2-category*, which allows us to talk about morphisms between morphisms. Let us recall the definition (see e.g. [**Bén67**], [**KS74**] or [**KV94**]).

A 2-category $\mathfrak{C}$ firstly consists of an ordinary category $\mathfrak{C}_1$, with objects and 1-morphisms between them, as well as a composition operation between 1-morphisms if the target and source objects are matching. In addition, we associate to any two 1-morphisms $A_i : O_s \to O_t$ and $A_f : O_s \to O_t$ with the same source and target a set $\mathrm{Hom}_2(A_i, A_f)$ of 2-morphisms, denoted $B : A_i \Rightarrow A_f$. We have a (vertical) composition operation between two 2-morphisms if the target 1-morphism of the one coincides with the source 1-morphism of the other. The composition of 1-morphisms extends to a second type of (horizontal) composition of 2-morphisms over an intermediate object. Finally, the two compositions are required to be mutually distributive.

One standard example is given by the 2-category **Cat** of categories. The objects of **Cat** are essentially small categories, the 1-morphism class $\mathrm{Hom}_1(\mathcal{C}_s, \mathcal{C}_t)$ consists of functors from $\mathcal{C}_s$ to $\mathcal{C}_t$, and $\mathrm{Hom}_2(\mathcal{F}_i, \mathcal{F}_f)$, for functors between the same pair of categories, is the set of natural transformations from $\mathcal{F}_i$ to $\mathcal{F}_f$. The vertical and horizontal compositions of natural transformations are given in standard ways, see [**Mac88**]. We shall be interested in its 2-subcategory $\mathbf{AbCat}^l$ of essentially small, $\Bbbk$-linear, *abelian* categories with length, *left exact* functors, and natural transformations, where $\Bbbk$ is a field. A *category with length* has all objects of finite length, and homomorphism $\Bbbk$-spaces are finite dimensional.

Another 2-category of interest is that of *relative cobordisms*, $\mathfrak{C} = \mathbf{Cob}^{rel}_{(d+1)}$. The underlying category $\mathfrak{C}_1$ of objects and 1-morphisms is identical to the cobordism category $\mathbf{Cob}^0_{(d)}$ from above. For two d-dimensional cobordisms $M^{(d)}$ and $N^{(d)}$ between $(d-1)$-manifolds $X^{(d-1)}$ and $Y^{(d-1)}$, we can consider the *closed* d-manifold $S := M^{(d)} \cup Z^{(d-1)} \times [0,1] \cup N^{(d)}$, where $Z^{(d-1)}$ is the disjoint union $X^{(d-1)} \sqcup Y^{(d-1)}$. In the definition of S we make identifications $\partial M^{(d)} \cong Z^{(d-1)} \times 0$ and $\partial N^{(d)} \cong Z^{(d-1)} \times 1$.

The 2-morphisms $M \to N$ are given by $(d+1)$-dimensional manifolds W with the boundary $\partial W \cong S$.

The (vertical) composition of 2-morphisms over 1-morphisms is given by gluing the $(d+1)$-manifolds together over the bounding d-manifolds. For the (horizontal) composition over objects we glue the $(d+1)$-manifolds together along the cylinders over the respective source and target $(d-1)$-manifolds.

In this book we define TQFT's using certain generalizations of 2-categories and 2-functors, namely *double categories* and *double functors*. Double categories were introduced by Ehresmann in [**Ehr63b**]. Let us give a version of his definition.

0.2.1 Definition. A double category $\mathfrak{D}$ consists firstly of a class $\mathfrak{D}_0$ of objects.

For any pair of objects, X and Y, there are sets $\mathrm{Hom}_1^v(X,Y)$ and $\mathrm{Hom}_1^h(X,Y)$ of vertical and horizontal 1-morphisms or 1-arrows. The objects and the vertical 1-morphisms by themselves form an ordinary category $\mathfrak{D}_1^v$, and an analogous category $\mathfrak{D}_1^h$ for the horizontal 1-morphisms.

We call a square $\mathbf{S}$ a set of four objects X_{ij}, with $i,j = 0,1$, two vertical 1-morphisms $g_j \in \mathrm{Hom}^v(X_{0j}, X_{1j})$ and two horizontal 1-morphisms $f_j \in \mathrm{Hom}^h(X_{j0}, X_{j1})$ so that they can be arranged in a square diagram as follows:

$$\mathbf{S} \;=\; \begin{tikzcd} X_{00} \arrow[r, "f_0"] \arrow[d, "g_0"'] & X_{01} \arrow[d, "g_1"'] \arrow[dl, Rightarrow, "\alpha"'] \\ X_{10} \arrow[r, "f_1"] & X_{11} \end{tikzcd} \quad , \qquad \alpha \in \mathrm{Hom}_2(\mathbf{S}). \tag{0.2.1}$$

For any square, $\mathbf{S}$, one has a set $\mathrm{Hom}_2(\mathbf{S})$ of 2-morphisms. We often include an element $\alpha \in \mathrm{Hom}_2(\mathbf{S})$ in the diagrammatic notation as above.

If for two squares, $\mathbf{S}$ and $\mathbf{S}'$, we have $f_1 = f'_0$ for the vertical 1-morphisms, then we define the horizontal composite square $\mathbf{S}' \circ_h \mathbf{S}$ to be the one with vertical 1-morphisms f_0 and f'_1, and horizontal 1-morphisms $g'_0 \circ_h g_0$ and $g'_0 \circ_h g_0$. A double category is equipped with a horizontal composition

$$\circ_h : \mathrm{Hom}_2(\mathbf{S}') \times \mathrm{Hom}_2(\mathbf{S}) \to \mathrm{Hom}_2(\mathbf{S}' \circ_h \mathbf{S}) \quad : \quad (\alpha, \beta) \;\mapsto\; \alpha \circ_h \beta$$

Analogously, there is a vertical composition $\gamma \circ_v \alpha$ declared if the target horizontal 1-morphism of α coincides with the source horizontal 1-morphism of γ.

We require both compositions to give rise to categories $(\mathfrak{D}_2^h, \circ_h)$ and $(\mathfrak{D}_2^v, \circ_v)$, whose objects are the vertical and horizontal 1-morphisms respectively. In particular, $\circ_h$ and $\circ_v$ are associative.

Finally, the interchange law for double categories states that two composition are mutually distributive. More precisely, suppose four 2-morphisms α, β, γ, and δ have

1-morphisms that coincide as indicated in the square of squares below:

$$\begin{tikzcd} {} \arrow[r] \arrow[d] \arrow[dr, phantom, "\alpha"] & {} \arrow[r] \arrow[d] \arrow[dr, phantom, "\beta"] & {} \arrow[d] \\ {} \arrow[r] \arrow[d] \arrow[dr, phantom, "\gamma"] & {} \arrow[r] \arrow[d] \arrow[dr, phantom, "\delta"] & {} \arrow[d] \\ {} \arrow[r] & {} \arrow[r] & {} \end{tikzcd} \quad .$$

We require that the operations of performing the horizontal and vertical compositions can be interchanged:

$$(\delta \circ_h \gamma) \circ_v (\beta \circ_h \alpha) \quad = \quad (\delta \circ_v \beta) \circ_h (\gamma \circ_v \alpha) \quad . \qquad \square \tag{0.2.2}$$

In Appendix B.2 we recall the original definition of Ehresmann [**Ehr63b**], which does not require the distinction between 0-, 1-, and 2-morphisms.

From a double category $\mathfrak{D}$ we can readily extract a 2-category if we consider only squares, for which we have $X_{0j} = X_{1j}$ and g_j is the identity for $j = 0, 1$. Conversely, if we have a 2-category $\mathfrak{C}$ we can construct a double category $\mathfrak{D} = \mathcal{Q}\mathfrak{C}$, called the double category of *quintets* of $\mathfrak{C}$, as follows.

We choose both the horizontal and vertical categories to be identical to the category underlying $\mathfrak{C}$, that is, $(\mathcal{Q}\mathfrak{C})_1^v = \mathfrak{C}_1$ and $(\mathcal{Q}\mathfrak{C})_1^h = \mathfrak{C}_1$. For a square $\mathbf{S}$ of 1-morphisms as in (0.2.1) the associated 2-morphism sets are

$$\mathrm{Hom}_2^{\mathfrak{D}}(\mathbf{S}) \quad = \quad \mathrm{Hom}_2^{\mathfrak{C}}(g_1 \circ f_0, f_1 \circ g_0) \ .$$

The horizontal composition of 2-morphisms in $\mathfrak{D} = \mathcal{Q}\mathfrak{C}$ is the obvious composite $g_1' \circ f_0' \circ f_0 \Rightarrow f_1' \circ g_0' \circ f_0 = f_1' \circ g_1 \circ f_0 \Rightarrow f_1' \circ f_1 \circ g_0$.

This construction yields the first example relevant to our definition of a TQFT, namely the double category $\mathcal{Q}\mathbf{AbCat}^l$, of $\Bbbk$-linear abelian categories, left exact functors and natural transformations. The precise definition of the topological double category $\widetilde{\boldsymbol{Cob}}_{\mathbf{3}}^{\sqcap}$ used in our definition of a TQFT is more involved. In outline it is as follows:

The set of objects is given as $\{S^{\sqcup a} : a \in \mathbb{Z}_{\geqslant 0}\}$, where $S^{\sqcup a}$ is a chosen union of a circles. The set of horizontal 1-morphisms $\mathrm{Hom}_1^h(a, b)$ between two such 1-manifolds consists of connected surfaces, whose boundary is parametrized by (and is homeomorphic to) $-S^{\sqcup a} \sqcup S^{\sqcup b}$. All such surfaces of the same genus g are homeomorphic, so we may leave only one representative in each homeomorphism class, parametrized by $g \in \mathbb{Z}_{\geqslant 0}$. We find technically more convenient to keep several isomorphic copies of horizontal morphisms – $\frac{1}{g+1}\binom{2g}{g}$ of them for genus g. The horizontal morphisms are parametrized by plane graphs G that look like nested arcs over a line. The standard surface Σ_G corresponding to the graph G is obtained from a thickening of G in $\mathbb{R}^3$.

The vertical 1-morphism set $\mathrm{Hom}_1^v(a, b)$ of $\widetilde{\boldsymbol{Cob}}_{\mathbf{3}}^{\sqcap}$ is empty if $a \neq b$. We identify the endomorphism set $\mathrm{Hom}_1^v(a, a) := S_a$ with the symmetric group of an a-element set. Hence, in a square $\mathbf{S}$ the two horizontal 1-morphisms Σ_G and Σ_H always lie in the same set $\mathrm{Hom}_1^h(a,b)$ as shown in the following diagram.

$$\mathbf{S} \;=\; \begin{tikzcd} a \arrow[r, "\Sigma_G"] \arrow[d, "\alpha"'] & b \arrow[d, "\beta"] \arrow[dl, Rightarrow, "\widetilde{M}"'] \\ a \arrow[r, "\Sigma_H"] & b \end{tikzcd} \quad .$$

The a source and b target bounding circles of each of the standard surfaces are numbered in a given way. We now sew the surfaces together by connecting the j-th source circle of Σ_G to the $\alpha(j)$-th source circle in the boundary of Σ_H by a cylinder $S^1 \times [0,1]$. Here $\alpha \in S_a$ is the left vertical 1-morphism of the diagram. Doing the same for the target circles, we obtain a closed surface $\Sigma_{\mathbf{S}}$. The cylinders are interpreted as Wilson lines in Chern-Simons theory.

A 2-morphism for a given square $\mathbf{S}$ is now a homeomorphism class of triples $\widetilde{M} = [(M,\phi,\alpha)]$, where each triple (M,ϕ,α) consists of a compact, oriented 3-manifold with corners, M, a homeomorphism, $\phi : \partial M \xrightarrow{\sim} \Sigma_{\mathbf{S}}$, and a 2-framing of its tangent bundle, $\alpha : TM \oplus TM \xrightarrow{\sim} \mathbb{R}^6 \times M$. The additional structure of a 2-framing is motivated by the Chern-Simons gauge theory. In this book we choose an equivalent description as an extensions by signatures of bounding 4-manifolds. In [**BHMV95**] yet another equivalent definition of this extension is given using so called p_1-structures.

If we disregard the signature extension of the 3-manifolds, we have natural and well defined vertical and horizontal compositions obtained by gluing two 3-manifolds together along the horizontal pieces Σ_G in their boundary or the cylindrical pieces respectively. One readily verifies that these compositions define a double category $\boldsymbol{Cob}_{\mathbf{3}}^{\cap}$. The compositions can be extended to 3-manifolds with 2-framings so that the axioms of a double category are fulfilled. In analogy to group theory we can thus view $\widetilde{\boldsymbol{Cob}}{}_{\mathbf{3}}^{\cap}$ as a central extension

$$1 \;\to\; \Omega_4 = \mathbb{Z} \;\hookrightarrow\; \widetilde{\boldsymbol{Cob}}{}_{\mathbf{3}}^{\cap} \;\twoheadrightarrow\; \boldsymbol{Cob}_{\mathbf{3}}^{\cap} \;\to\; 1\;, \tag{0.2.3}$$

where Ω_4 is the smooth 4-dimensional cobordism group generated by the said signature.

There is an obvious notion of a strict double functor between double categories. Its weak version – double pseudofunctor – is defined in Appendix B.2.

We are now in a position to give a definition of an extended topological quantum field theory:

0.2.2 Definition. An extended TQFT over a field $\Bbbk$ is a double pseudofunctor

$$\mathcal{V} \,:\, \widetilde{\boldsymbol{Cob}}{}_{\mathbf{3}}^{\cap} \longrightarrow \mathcal{Q}\mathbf{AbCat}^l\;,$$

between double categories as above, which is compatible with tensor structures on the level of objects.

The tensor structures referred to here are, on the level of objects, given for $\widetilde{\boldsymbol{Cob}}{}_{\mathbf{3}}^{\cap}$ by the disjoint unions of circles, and for $\mathcal{Q}\mathbf{AbCat}^l$ by Deligne's tensor product $\boxtimes$ of abelian categories. Hence,

$$\mathcal{V}(S^{\sqcup a}) \cong \mathcal{C} \boxtimes \ldots \boxtimes \mathcal{C}\;,$$

where $\mathcal{C}$ is the category associated to one circle. Since this category is of obvious interest as a generator, let us give a formal definition as follows:

0.2.3 Definition. Let $\mathcal{V}$ be an extended TQFT in the sense of Definition 0.2.2. The circle category of the TQFT double functor $\mathcal{V}$ is then defined as

$$\mathcal{C}_{\mathcal{V}} \quad := \quad \mathcal{V}(S^1) \quad .$$

0.2.4. Statement of Main Result on the Class of Extended TQFT's. — To state our main result let us note that circle categories always carry a monoidal structure. Moreover, they are braided and bounded abelian. A few additional assumptions lead to the following definition.

0.2.5 Definition. A modular category over a field $\Bbbk$ is a bounded abelian, rigid, monoidal, braided, balanced (ribbon) category $\mathcal{C}$ with a special Hopf pairing that is non-degenerate. The endomorphism ring of the unit object is supposed to be $\Bbbk$.

In particular, $\mathcal{C}$ is $\Bbbk$-linear. The precise definitions of these conditions, which are all rather natural, will be given in Chapter 4. The notion of an *abelian* category allows us to consider subobjects, quotients and decompositions of objects. The properties *monoidal* and *rigid* imply the existence of tensor products $X \otimes Y$ and duals $X^{\vee}$ of objects. These notions are part of classical category theory as described in [**Mac88**]. The word *braided* implies a natural isomorphism $c_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X$, which does not square to the identity. *Balanced* (or ribbon) refers to a natural isomorphisms $X \xrightarrow{\sim} X^{\vee\vee}$ compatible with the braiding. Braiding and balancing in categories were defined by many people, see, for example, [**JS91**] and [**RT90**]. We call *bounded* a category that is equivalent to a category of finite dimensional modules over a finite dimensional algebra. This turns out to be equivalent to the existence of the coend $\mathbb{F} := \int^{X \in \mathcal{C}} X \boxtimes X^{\vee}$ in $\mathcal{C} \boxtimes \mathcal{C}$. The details on coends can be found in Chapter 5. For example, a semisimple category is bounded precisely when the set of equivalence classes of simple objects is finite. The Hopf pairing $\omega : F \otimes F \to 1$, which we require to be non-degenerate, is defined universally on the coend $F := \otimes \mathbb{F} = \int^{X \in \mathcal{C}} X \otimes X^{\vee} \in \mathcal{C}$, which is a Hopf algebra.

0.2.6 Remark. Note that in our definition an abelian modular category does not have to be semisimple!

Now we state our main result.

0.2.7 Theorem. *For every modular category $\mathcal{C}$ there exists an extended TQFT $\mathcal{V}_{\mathcal{C}}$, which has $\mathcal{C}$ as circle category, meaning $\mathcal{C} = \mathcal{C}_{\mathcal{V}_{\mathcal{C}}}$.*

Replacing a modular abelian category $\mathcal{C}$ with an equivalent one, and replacing the structure of symmetric monoidal 2-category of $\mathbf{AbCat}^l$ with an equivalent one, we can achieve that $\mathcal{V}_{\mathcal{C}}$ is a strict double functor.

The only choice involved in the construction is that of a square root of some constant. In order to keep the diagrammatics simple, we first restrict to only one representative for each homeomorphism class of horizontal surfaces. The corresponding double category $\widetilde{\mathcal{C}ob}_3^{\,con}$ can be thought of as a double subcategory of $\widetilde{\mathcal{C}ob}_{\mathbf{3}}^{\cap}$, and for each modular $\mathcal{C}$ we construct a double pseudofunctor into $\mathcal{Q}\mathbf{AbCat}^l$.

0.2.8. Specializations and Generalizations. —

1. Conformal Field Theory. — In CFT one considers the double category **Surf** of surfaces, which has the same objects and 1-morphisms as $\widetilde{\mathcal{C}ob}_{\mathbf{3}}^{\cap}$. The 2-morphisms, however, are homeomorphisms between the surfaces. In Chapter 1 we identify the mapping class group of a surface with the invertible cobordism classes from this surface to itself. Hence, we can think of $\mathbf{Surf} \subset \widetilde{\mathcal{C}ob}_{\mathbf{3}}^{\cap}$ as a double subcategory. The restriction $\mathcal{V}_{\mathcal{C}}^{mod} : \mathbf{Surf} \to \mathcal{Q}\mathbf{AbCat}^l$ of a TQFT double functor, $\mathcal{V}_{\mathcal{C}}$, turns out to be a version of what Segal, Moore and Seiberg call a *modular 2-functor*. See again Appendix A for more context and details. The double functor $\mathcal{V}_{\mathcal{C}}^{mod}$ also implies projective representations of the mapping class groups that are compatible with respect to concatenations of surfaces.

2. Atiyah's TQFT. — In the double category $\widetilde{\mathcal{C}ob}_{\mathbf{3}}^{\cap}$ we can consider the subcategory, in which all objects are empty 1-manifolds. This means we are dealing with closed surfaces and the only relevant composition is the gluing over these surfaces in vertical direction. We thus obtain a central extension $\widetilde{\mathbf{Cob}}_3^0$ of a version of $\mathbf{Cob}_3^0$. The functors associated to closed surfaces, seen as cobordisms between empty 1-manifolds, are naturally identified with vector spaces. As a result, we obtain an ordinary extended TQFT as a functor $\mathcal{V}_{\mathcal{C}}^0 : \widetilde{\mathbf{Cob}}_3^0 \to \mathbb{k}$-vect. To a *closed* manifold, seen as a cobordism between empty surfaces, the extended TQFT, as much as the ordinary TQFT, assigns a number – the associated invariant of the 3-manifold.

3. Reshetikhin-Turaev Theory. — Reshetikhin and Turaev proposed in [**RT91**] a construction that leads to an ordinary extended TQFT. The details are developed in Turaev's book [**Tur94**]. They use a *semisimple* modular category $\mathcal{C}$ for an input. Modularity is defined differently in [**Tur94**] and in our book, however, the both definitions are equivalent for semisimple categories as we show in Section 7.4.1. When $\mathcal{C}$ is a semisimple modular category, the restriction of the Reshetikhin-Turaev construction to connected surfaces is a TQFT, isomorphic to the above $\mathcal{V}_{\mathcal{C}}^0$. Besides, in semisimple case our construction extends to disconnected surfaces as well, giving a complete agreement of the theories. Semisimple categories can be produced, for instance, as semisimple trace quotients from non-semisimple representation categories of quantum groups [**RT91, And92, TW93**].

4. Disconnected Surfaces. — One assumption in our definition of $\widetilde{\mathcal{C}ob}_{\mathbf{3}}^{\cap}$ is that all surfaces representing 1-morphisms should be connected. The generalization of the necessary tangle presentations for closed surfaces can be found, for example, in [**Ker99**] or [**Tur94**]. It extends naturally to disconnected surfaces with boundaries. These allow us to also extend the construction of the double functors in the case, where $\mathcal{C}$ is a semisimple category. For non-semisimple categories but closed surfaces the relevant modification of the TQFT axioms and general constructions are described in [**Ker98b**].

5. Hennings Theory. — In [**Hen96**] Hennings defines an invariant directly from a possibly non-semisimple, quasi-triangular ribbon Hopf algebra $\mathcal{A}$, which naturally extends to a TQFT ([**Ker97**]). The invariants and TQFT's are again special cases of $\mathcal{V}^0_{\mathcal{C}}$, if we insert the representation category $\mathcal{C} = \mathcal{A}-$mod.

6. General Horizontal Surfaces. — An obvious question is whether it is possible and sensible to extend our construction to more general classes of surfaces representing the vertical 1-morphisms instead of simple cylinders connecting the boundary components of different surfaces. A small modification, that we can easily deal with in our formalism, is to allow a cylinder to connect a boundary component of surface to another one of the *same* surface. This means that the vertical category is identical with the category of 1-dimensional cobordisms $\mathbf{Cob}^0_{(1)}$ and we can have arbitrary objects in the corners of a square. Thus, a morphism in $\mathrm{Hom}^v_1(a,b)$ is a 1-manifold with a source endpoints and b target endpoints. The vertical surfaces are obtained by taking the Cartesian product with S^1. Clearly, the group of invertible 1-cobordisms $\mathrm{Aut}^v_1(a)$ is identical with S_a, which is the restriction we have used in the definition of $\widetilde{\mathcal{C}ob}^{\cap}_{\mathbf{3}}$. Further generalizations to other vertical surfaces are possible but quickly become impractical.

0.2.9. Strategy of Construction and Summary of Content. — The invariant of Witten, Reshetikhin and Turaev of a closed 3-manifold M is obtained by first presenting M via surgery along a framed link $\mathcal{L}$ in S^3. It turns out that one can find a combination of link invariants related to the Jones polynomial, which, evaluated on $\mathcal{L}$, depends only on the presented manifold M. In Witten's work [**Wit89**] the link invariants are obtained from Chern-Simons quantum field theory, and the coefficients for the combination of invariants are computed from conformal field theory. Unfortunately, these field theories are not rigorously defined. In their ground breaking paper [**RT91**] Reshetikhin and Turaev manage to give a rigorous construction of the same invariants for 3-manifolds without boundary using quantum groups. They are the first ones to design a strategy of constructing complex 3-manifold invariants via Kirby's calculus of links. Starting point is the construction of an invariant of links in S^3 using the methods developed in [**RT90**]. Then Reshetikhin and Turaev show that its value is the same for any two links equivalent with respect to Kirby moves [**RT91**]. Since the Kirby equivalence classes are in bijection with the homeomorphism classes of closed 3-manifolds, the link invariant is in fact a 3-manifold invariant. The generalization to TQFT's in the sense of Atiyah, using embedded ribbon graphs, was fully realized by Turaev in his book [**Tur94**] for the case of semisimple categories.

In our construction of the extended TQFT functor, proposed in Theorem 0.2.7, we follow an analogous strategy of first producing a combinatorial surgery presentation of the relative 3-cobordisms with corners and then assigning the algebraic data of an abelian modular category to it. The combinatorial data replacing a framed link also needs to encode the two composition structures. Hence it will be formulated as a double category $\mathcal{T}\boldsymbol{gl}^{\cap}$ itself, whose 2-morphisms are equivalence classes of tangles of certain type. The presentation should preserve the operational structure and, hence, it is given as a double functor $\mathfrak{Surg}$. Likewise, we formulate the assignment of the algebraic data to combinatorial tangles as a double functor $\mathcal{V}_{\mathcal{C}}^*$. In summary, we construct the TQFT functor $\mathcal{V}_{\mathcal{C}}$ as the composite of two double functors as follows:

$$\mathcal{V}_{\mathcal{C}} : \quad \widetilde{\boldsymbol{\mathcal{C}ob}}_{\mathbf{3}}^{\cap} \xrightarrow{\cong \mathfrak{Surg}^{-1}} \boldsymbol{\mathcal{T}gl}^{\cap} \xrightarrow{\mathcal{V}_{\mathcal{C}}^*} \mathcal{Q}\mathbf{AbCat}^{l} \tag{0.2.4}$$

The methods employed here are based on the techniques proposed in previous work of the authors. In this book the techniques are further developed and refined. The surgery presentation given by $\mathfrak{Surg}$ generalizes the one from [**Ker99**] for ordinary cobordisms between closed surfaces. In addition to this, we have to include the cylindrical boundary components in the definition and presentation of $\widetilde{\boldsymbol{\mathcal{C}ob}}_{\mathbf{3}}^{\cap}$ and to define the horizontal composition both in $\widetilde{\boldsymbol{\mathcal{C}ob}}_{\mathbf{3}}^{\cap}$ and $\boldsymbol{\mathcal{T}gl}^{\cap}$, so that we obtain double categories and double functors compatible with the 2-framing extension. The algebraic assignment $\mathcal{V}_{\mathcal{C}}^*$, for possibly non-semisimple $\mathcal{C}$, generalizes the methods used in the construction of 3-manifold invariants and representations of the mapping class groups in [**Lyu95a, Lyu96**]. In particular, we extend here the coend techniques to also construct functors and natural transformations, instead of just objects like $\mathbb{F}$ and F. The appearance of the symmetric group necessitates more careful investigations of its action on $\boxtimes$-products of abelian categories. We also expand and refine the theory of braided Hopf algebras in braided tensor categories, which allow very conceptual and concise invariance proofs of dictionary type.

We have organized this book by devoting one or two chapters to the construction and investigation of each of the five ingredients of (0.2.4).

The first three chapters of this book, therefore, concern themselves with the combinatorial representation $\mathfrak{Surg} : \boldsymbol{\mathcal{T}gl} \xrightarrow{\sim} \widetilde{\boldsymbol{\mathcal{C}ob}}_3^{con}$. The double categories $\boldsymbol{\mathcal{T}gl}$ and $\widetilde{\boldsymbol{\mathcal{C}ob}}_3^{con}$ differ from $\boldsymbol{\mathcal{T}gl}^{\cap}$ and $\widetilde{\boldsymbol{\mathcal{C}ob}}_{\mathbf{3}}^{\cap}$ appearing in (0.2.4) only in that they have one (instead of several) 1-morphism in each equivalence class. In Chapters 1 and 2 we discuss the definitions and characteristics of the double categories $\widetilde{\boldsymbol{\mathcal{C}ob}}_3^{con}$ and $\boldsymbol{\mathcal{T}gl}$ respectively. The double isomorphism functor $\mathfrak{Surg}$ is constructed in Chapter 3.

More specifically, we start in Chapter 1 with the discussion of the ordinary double category $\boldsymbol{\mathcal{C}ob}_3^{con}$ of relative 3-cobordisms. We discover that $\boldsymbol{\mathcal{C}ob}_3^{con}$ contains a canonical balanced braided tensor category. The mapping class group of a surface is identified with the group of invertible cobordisms of $\boldsymbol{\mathcal{C}ob}_3^{con}$ on that surface object. As a special subgroup we also discuss the image of the framed braid groups on a surface in the corresponding mapping class groups. In the last part of Chapter 1 we define the 2-framing extension $\widetilde{\boldsymbol{\mathcal{C}ob}}_3^{con}$ of $\boldsymbol{\mathcal{C}ob}_3^{con}$ using bounding 4-manifolds. Gluings are extended to 3-dimensional cylinders over the respective surfaces. We show that these operations factor into homeomorphism and cobordism classes, and verify that the composition structure on the classes fulfills the axioms of a double category.

In Chapter 2 the tangle double category $\boldsymbol{\mathcal{T}gl}$ is introduced. The 2-morphisms are given as equivalence classes of generic projections of *admissible tangles* with several types of strands. The equivalences are expressed in the form of a list of moves. A large part of this chapter is devoted to finding equivalent description of the category $\boldsymbol{\mathcal{T}gl}$. In particular, we show that $\boldsymbol{\mathcal{T}gl} \cong \boldsymbol{\mathcal{T}gl}_{S^2}^{\mathbf{BL}}$, where $\boldsymbol{\mathcal{T}gl}_{S^2}^{\mathbf{BL}}$ are classes of bridged link diagrams, in the sense of [**Ker98a**], in the thickened sphere $S^2 \times [0,1]$. The version $\boldsymbol{\mathcal{T}gl}_{S^2}^{\mathbf{BL}}$ will be closer to the surgery presentations of cobordisms, while $\boldsymbol{\mathcal{T}gl}$ is more adequate for the assignment of the algebraic data. Finally, we defined vertical and horizontal compositions for $\boldsymbol{\mathcal{T}gl}$ and prove that they do in fact give rise to a double category. The compositions are mild modifications of the usual stacking and juxtaposition operations.

In Chapter 3 the functor $\mathfrak{Surg} : \boldsymbol{\mathcal{T}gl}_{S^2}^{\mathbf{BL}} \overset{\sim}{\longrightarrow} \widetilde{\boldsymbol{\mathcal{C}ob}}_3^{con}$ is constructed by doing surgery along a link in a sum of handlebodies obtained from a respective tangle and thus generalizes the presentation [**Ker99**] for closed surfaces. We recall the standard tools such as surgery manipulations of handle attachments and Morse and Cerf theory, and review the resulting surgery calculi on non-simply connected manifolds. We prove that the surgery operation factors into an isomorphism $\mathfrak{Surg}$ on the equivalence classes of $\boldsymbol{\mathcal{T}gl}_{S^2}^{\mathbf{BL}}$ and $\widetilde{\boldsymbol{\mathcal{C}ob}}_3^{con}$. We also show that $\mathfrak{Surg}$ respects the vertical and horizontal compositions. Functoriality for the latter requires a more detailed analysis of the handle structure of the bounding 4-manifolds.

Chapter 4 through 7 are concerned with the second composite $\mathcal{V}^* : \boldsymbol{\mathcal{T}gl} \to \mathcal{Q}\mathbf{AbCat}^l$ of the TQFT double functor as it is given in (0.2.4).

In Chapter 4 the algebraic building blocks for the construction of the functors $\mathcal{V}_{\mathcal{C}}$ are laid. In particular, we give a thorough discussion of the properties of ordinary braided tensor categories (BTC's) such as braided, reflexive balancings, braided Hopf algebras in BTC's and their integrals. Graphical calculi for both BTC's and Hopf algebras are introduced. We study Hopf pairings and find criteria of their non-degeneracy (side-invertibility) in terms of integrals. In the last section of this chapter we recall the basic definitions and properties of Deligne's tensor product $\boxtimes$ for abelian categories. We first consider only the 2-category of categories of modules over finite dimensional algebras inside a strict version of the category of vector spaces. For this strictified category we ensure that the 2-braiding induces a strict action of the symmetric group S_N on the multifold tensor products $\mathcal{C}_1 \boxtimes \mathcal{C}_2 \boxtimes \ldots \boxtimes \mathcal{C}_N$ of categories of modules. As a result, $\mathbf{AbCat}^l$ inherits the structure of a weak symmetric monoidal 2-category.

In Chapter 5 we begin with a discussion of a large class of coends in abelian tensor categories that are determined by an expression with operations such as $\otimes$, $\boxtimes$, and $\_^{\vee}$. In particular, the functors associated to horizontal 1-morphisms are obtained as coends of this form. We review the construction of the braided Hopf algebra structure for the special coend $F = \int^{X \in \mathcal{C}} X \otimes X^{\vee}$ in a *bounded*, abelian BTC $\mathcal{C}$. We construct a special Hopf pairing $\omega : F \otimes F \to 1$ for such a Hopf algebra F. Modularity of a bounded, ribbon category $\mathcal{C}$ means, by definition, non-degeneracy of the form ω. We prove that ω is non-degenerate if and only if integral-functionals factor through ω. In the modular case we prove that integrals of F are two-sided and that the natural transformation of the identity functor induced by the integral in $\mathrm{Hom}_{\mathcal{C}}(F, 1)$ factors through $1 \oplus \ldots \oplus 1$ for every object of $\mathcal{C}$.

In Chapter 6 we construct the double pseudofunctor $\mathcal{V}^* : \boldsymbol{\mathcal{T}gl} \to \mathcal{Q}\mathbf{AbCat}^l$ on tangles, which represent cobordisms. The proof of topological invariance, meaning the fact that $\mathcal{V}^*$ is well defined on equivalence classes of tangles, is obtained by a dictionary style translation of elementary moves to algebraic axioms. The proof, that the double functor is compatible with the vertical composition, is straightforward. The horizontal composition is, however, respected only up to isomorphism.

In the first part of Chapter 7 we lift the double pseudofunctor $\mathcal{V} : \widetilde{\boldsymbol{Cob}}_3^{con} \to \mathcal{Q}\mathbf{AbCat}^l$ to a double pseudofunctor $\mathcal{V} : \widetilde{\boldsymbol{Cob}}_{\mathbf{3}}^{\cap} \to \mathcal{Q}\mathbf{AbCat}^l$ using an analogous presentation via a tangle double category $\boldsymbol{\mathcal{T}gl}^{\cap}$. It can be made strict after replacing the structure of symmetric monoidal 2-category of $\mathbf{AbCat}^l$ with an equivalent one.

In the remainder of Chapter 7 we consider two special cases for the input category $\mathcal{C}$. The first is the example of a *semisimple* abelian category $\mathcal{C}$, for which our double functor extends the Reshetikhin-Turaev theory. In the second case we consider the Tannakian situation $\mathcal{C} = \mathcal{A}-$mod for a general quantum group $\mathcal{A}$, which yields an extension of the Hennings invariant. We discuss in detail the form of the braided Hopf algebras and their integrals for both types of categories.

In Appendix A we discuss the physical and historical background that leads us to defining an axiomatic field theory in terms of double functors. We start with an exposition of the topological aspects of Chern-Simons theory, that were investigated by Witten, and the functorial formulations of conformal field theories. Other various axiomatic frameworks, that attempt to unite and axiomatize these two theories, are presented and their relation to the double functor picture explained.

We recall the Ehresmann definition of double categories from [**Ehr63b**] in Appendix B.1. In Appendix B.2 we discuss weak versions of double functors – the pseudofunctors. The related notions – horizontal and vertical natural transformations – are also described. Our interest in those is explained by the fact that we first study a version of the TQFT functor $\mathcal{V}$, which is a double functor in the weak sense.

Finally, in Appendix C.1, we give a description of the category of multiple coends, which are associated to higher genus surfaces, together with natural isomorphisms between them. We do it in terms of the monoidal bicategory of thick tangles, which can be thought of as a free bicategory generated by a self-dual object. We obtain a combinatorial presentation of this bicategory in terms of generators and relations, which have graphical presentations. Coherence of the above mentioned functors and their isomorphisms is asserted in the form of a functor from the combinatorial bicategory to $\mathbf{AbCat}^l$.

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