Remainder estimates for the approximation numbers of weighted Hardy operators acting on L^2.

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Abstract

We consider the weighted Hardy integral operator $T : L^2(a,b) \to L^2(a,b)$, $-\infty \leq a < b \leq \infty$, defined by

$$(Tf)(x) = v(x) \int_a^x u(t) f(t) dt.$$

In [EEH1] and [EEH2], under certain conditions on u and v, upper and lower estimates and asymptotic results were obtained for the approximation numbers $a_n(T)$ of T. In this paper we show that under suitable conditions on u and v,

$$\limsup_{n\to\infty} n^{1/2} \left| \frac{1}{\pi} \int_a^b |u(t)v(t)| dt - n a_n(T) \right|$$
$$\leq c(\|u'\|_{2/3} + \|v'\|_{2/3})(\|u\|_2 + \|v\|_2) + \frac{3}{\pi}\|uv\|_1,$$

where $\|w\|_p = (\int_a^b |w(t)|^p dt)^{1/p}$.

1 Introduction.

Let u and v be real-valued, measurable functions on $(a,b), -\infty \leq a < b \leq \infty$. The weighted Hardy integral operator T is defined at any real-valued, measurable function f on (a,b) for which $\int_a^x |u(t)f(t)|dt < \infty$, $x \in (a,b)$, by

$$(Tf)(x) = v(x) \int_a^x u(t) f(t) dt, \qquad x \in (a,b). \tag{1. 1}$$

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This paper is devoted to the study of the asymptotic behavior of the approximation numbers of T as an operator from $L^2(a,b)$ to itself. We recall that, given any positive integer n, the n^{th} approximation number of T, denoted $a_n(T)$, is defined by

$$a_n(T) = \inf \|T - F\|,$$

where the infimum is taken over all bounded linear maps $F : L^2(a,b) \to L^2(a,b)$ having rank less than n. In fact, the approximation numbers may be defined in exactly the same way for any bounded linear map acting between Banach spaces.

In this Hilbert space setting, approximation numbers are the same as the singular values of T. For general information on approximation numbers see [EE]; in particular one finds there a proof of the classical theorem that a linear operator T on a Hilbert space H to itself is compact if and only if $\lim_{n\to\infty} a_n(T) = 0$. This is the case for T defined by (1. 1) if $u, v \in L^2(a,b)$; indeed, as detailed in section 2, one then has

$$\lim_{n\to\infty} n a_n(T) = \frac{1}{\pi}\int_a^b |u(t)v(t)|dt, \tag{1. 2}$$

a fact established under weaker conditions on u and v in [EEH2]. (For related results see also [NS].)

For $1 < p < \infty$, $p \neq 2$ and under appropriate conditions on u and v, it was proved in [EEH2] that for some constant α_p, depending only on p,

$$\frac{1}{4}\alpha_p \int_0^\infty |u(t)v(t)|dt \le \liminf_{n\to\infty} n a_n(T)$$

$$\le \limsup_{n\to\infty} n a_n(T) \le \alpha_p \int_0^\infty |u(t)v(t)|dt.$$

These results were extended in [EHL1] to include the limiting cases $p = 1$ and $p = \infty$.

In the recent paper [EHL2] it was proved that a result like (1. 3) is valid also for $1 < p < \infty$ on metric trees under certain restrictions on u and v, i.e.

$$\lim_{n\to\infty} n a_n(T) = C(p) \int_\Gamma |u(t)v(t)|dt,$$

where Γ is a metric tree (recall that an interval is a special case of a metric tree, so that all previous results follow from this paper).

Further extensions were given in [LL] to deal with the cases in which T is viewed as a map from L^p to L^q, for any $p, q \in [1, \infty]$.

In the present paper we focus exclusively on the case $p = 2$. A refinement of (1. 2) is given in our

Main Result. *Let* $-\infty \le a < b \le \infty$. *Suppose that* $u, v \in L^2(a,b)$ *and* $u', v' \in L^{2/3}(a,b) \cap C(a,b)$. *Then*

$$\limsup_{n\to\infty} n^{1/2} \left| \frac{1}{\pi}\int_a^b |u(t)v(t)|dt - na_n(T) \right| \tag{1. 3}$$
$$\le \frac{3}{\pi}\int_a^b |u(t)v(t)|dt + 3\sqrt{2}(\|u'\|_{2/3} + \|v'\|_{2/3})(\|u\|_2 + \|v\|_2),$$

so that

$$a_n(T) = \frac{1}{\pi n}\int_a^b |u(t)v(t)|dt + O(n^{-3/2}).$$

The following is an outline of our proof of this result. First of all, we focus attention on a particular class of operators of finite rank; namely, those of the form

$$F = \sum_{k=1}^n P_k T_k. \tag{1. 4}$$

Here, P_k is the projection operator onto functions of the form $\alpha v \chi_{I_k}$, $\alpha \in \mathbf{R}$, where the $I_k = (a_k, b_k)$ are pairwise-disjoint subintervals of (a,b), with $(a,b) \subset \cup_{k=1}^n \overline{I_k}$, and

$$(T_k f)(x) = \chi_{I_k}(x)v(x)\int_{a_k}^x u(t)f(t)dt, \qquad x \in I_k,$$

whence,

$$(P_k T_k f)(x) = w_{I_k}(f)v(x) \equiv \frac{\int_{I_k}(\int_{a_k}^y u(t)f(t)dt)v^2(y)dy}{\int_{I_k} v^2(y)dy} v(x), \qquad x \in I_k,\ k = 1, 2, ...$$

Of special interest are those F having the operator norms of all (but possibly one) of $T_k - P_k T_k$ equal. Denoting the common value of the norms by ε and the corresponding n in (1. 4) by $N(\varepsilon)$, we are able to get, arguing as in [EEH1] and [EEH2], that ε is essentially equal to $a_{N(\varepsilon)}$, that

$$\lim_{\varepsilon\to\infty} \varepsilon N(\varepsilon) = \frac{1}{\pi}\int_a^b |u(t)v(t)|dt \tag{1. 5}$$

and hence, from a number of technical lemmas, that (1. 2) holds.

Now, it seems that the convergence in (1. 5) is most rapid when the weights u and v are constant on the I_k; indeed, if u and v take the constant values u_k and v_k on I_k, then, as we show in Lemma 3.1,

$$\|T_k - P_k T_k\| = \frac{1}{\pi}|u_k||v_k||I| = \frac{1}{\pi}\int_{I_k} |u(t)v(t)|dt,$$

$k = 1, 2, ..., n$. To refine (1. 2) we then measure how far u and v are from step functions by counting the minimum number, $m(\alpha)$, of steps required to so approximate both u and v to within $\alpha > 0$ in $L^2(a,b)$. The quantity αm_α is naturally estimated in terms of the derivates u' and v'; in fact, as we show in Theorem 3.8,

$$\limsup_{\alpha\to 0_+} \alpha m_\alpha \le \sqrt{2}(\|u'\|_{2/3} + \|v'\|_{2/3}).$$

Finally, connecting $N(\varepsilon)$ and m_α by setting $m_\alpha = \sqrt{N(\varepsilon)}$, the two strands of the argument are brought together to yield (1. 3).

2 Preliminaries.

Throughout the paper we shall assume that $-\infty \le a < b \le \infty$ and that

$$u, v \in L^2(a,b), \qquad \text{and } u^2, v^2 > 0 \text{ on } (a,b). \tag{2. 1}$$

It then follows from [EGP, Theorem 1] that the operator T in (1. 1) is compact from $L^2(a,b)$ to itself; see also [OK].

We next introduce a function A which will play a key role in the paper. Given $I = (c,d) \subset (a,b)$, set

$$A(I) := \sup_{\|f\|_{2,I}\le 1} \|Tf - w_I(f)v\|_{2,I}$$

where $\|g\|_{2,I} = \|g\chi_I\|_2$, $\|g\|_2 = (\int_a^b |g|^2)^{1/2}$,

$$w_I(f) = \frac{1}{v(I)} \int_I \left(\int_c^x f(t)u(t)dt \right) v^2(x)dx,$$

and $v(I) = \|v\chi_I\|_2^2$. The function $A(I)$ is equal to the function $K(I)$ (when $p = 2$) from [EEH1] and [EEH2]. We observe that $w_I(f)v$ is the projection of Tf on the subspace $\{\alpha v; \alpha \in \mathbf{R}\}$ of $L^2(I)$, so that

$$A(I) = \sup_{\|f\|_{2,I}\le 1} \inf_{\alpha\in R} \|Tf - \alpha v\|_{2,I}.$$

Lemma 2.1 *Let $I = (c,d) \subset (a,b)$ and $v^2(I) < \infty$. Then*

$$A(I) = A^*(I) := \sup_{\|f\|_{2,I}\le 1, f\ge 0} \|T(f \operatorname{sgn} u) - w_I(f \operatorname{sgn} u)v\|_{2,I}.$$

Proof. Since

$$\min_{x\in I} \int_c^x u(t)g(t)dt \le w_I(g) \le \max_{x\in I} \int_c^x u(t)g(t)dt, \qquad \text{for } \operatorname{supp} g \subset I,$$

we have

$$A(I) = \sup_{\|f\|_{2,I}\le 1} \inf_{y\in I} \|(Tf)(x) - v(x) \int_a^y u(t)f(t)dt\|_{2,I}$$

and

$$A^*(I) = \sup_{\|f\|_{2,I}\le 1, f\ge 0} \inf_{y\in I} \|T(f \operatorname{sgn} u)(x) - v(x)\int_c^y u(t)f(t)\operatorname{sgn} u(t)dt\|_{2,I}.$$

Clearly, $A^*(I) \le A(I)$. Again, let us take any f with $\|f\|_{2,I} \le 1$. Then, for $x, y \in I$,

$$(Tf)(x) - v(x)\int_c^y u(t)f(t)dt = v(x)\int_y^x u(t)f(t)dt,$$

so

$$\begin{aligned} |(Tf)(x) - v(x)\int_c^y u(t)f(t)dt| &= |v(x)\int_y^x u(t)f(t)dt| \\ &\le |v(x)|\int_{\min[x,y]}^{\max[x,y]} u(t)|f(t)|\operatorname{sgn} u(t)dt \\ &\le |T(|f|\operatorname{sgn} u)(x) - v(x)\int_c^y u(t)|f(t)|\operatorname{sgn} u(t)dt|. \end{aligned}$$

We conclude $A(I) = A^*(I)$, which completes the proof. □

Lemma 2.2 *Given that u and v satisfy (2. 1) and $a \le c < d \le b$, we have:*

1. *The function $A(., d)$ is non-increasing and continuous on (a, d).*
2. *The function $A(c, .)$ is non-decreasing and continuous on (c, b).*
3. $\lim_{y\to a_+} A(a, y) = \lim_{y\to b_-} A(y, b) = 0$.

Proof. The proof of 1 illustrates the techniques necessary to prove 2 and 3 also. That $A(., d)$ is non-increasing is easy to see. To get continuity from the left, fix $y \in (a, d)$. Then, there exists $h_0 > 0$ such that for $0 < h < h_0$

$$\begin{aligned} & A^2(y, d) \le A^2(y-h, d) \\ = & \sup_{\|f\|_{2,(y-h,d)}\le 1} \inf_{|\alpha|\le \|u\|_{2,(y-h_0,d)}} \|v\left[\int_{y-h}^{\cdot} u(t)f(t)dt - \alpha\right]\|^2_{2,(y-h,d)} \\ = & \sup_{\|f\|_{2,(y-h,d)}\le 1} \inf_{|\alpha|\le \|u\|_{2,(y-h_0,d)}} \left[\|v\left[\int_{y-h}^{y} u(t)f(t)dt - \alpha\right]\|^2_{2,(y-h,y)}\right. \\ & \left. + \|v\left[\int_{y}^{\cdot} u(t)f(t)dt - \alpha + \int_{y-h}^{y} u(t)f(t)dt\right]\|^2_{2,(y,d)}\right] \\ \le & \sup_{\|f\|_{2,(y-h,d)}\le 1} \inf_{|\alpha|\le \|u\|_{2,(y-h_0,d)}} \left[2\|v\int_{y-h}^{\cdot} u(t)f(t)dt\|^2_{2,(y-h,y)} + 2\alpha^2\|v\|^2_{2,(y-h,y)}\right. \\ & \left. + \|v\left[\int_{y}^{\cdot} u(t)f(t)dt - \alpha + \int_{y-h}^{y} u(t)f(t)dt\right]\|^2_{2,(y,d)}\right] \\ \le & 2\|u\|^2_{2,(y-h,y)}\|v\|^2_{2,(y-h,y)} + 2\|u\|^2_{2,(y-h_0,d)}\|v\|^2_{2,(y-h,y)} + 2\|u\|^2_{2,(y-h,y)}A^2(y,d) + A^2(y,d). \end{aligned}$$

It follows that

$$\lim_{h\to 0_+} A(y-h,d) = A(y,d).$$

In the same way we see that

$$\lim_{h\to 0_+} A(y+h,d) = A(y,d),$$

and now the proof of the Lemma is complete. □

Lemma 2.3 *Let $I=(c,d)$ and $J=(c',d')$ be subintervals of (a,b), with $J\subset I$, $|J|>0$ and $|I-J|>0$. Then*

$$A(I) > A(J) > 0. \tag{2. 2}$$

Proof. Let $0\le f\in L^2(J)$, $\|f\|_{2,I}\le 1$. Then

$$|\{x\in J : |\int_{c'}^{x} u(t)f(t)dt - w_J(f)|^2 v^2(x) > 0\}| > 0,$$

which means $\|Tf - w_J(f)v\|_{2,J} > 0$ and hence $A(J)>0$.

Next, suppose $0\le f\in L^2(I)$, $\|f\|_{2,I}\le 1$, but with $\operatorname{supp} f\subset J$. Now,

$$\begin{aligned}\|Tf-w_I(f)v\|_{2,I}^2 &= \|Tf-w_I(f)v\|_{2,J}^2 + \int_{I-J} |(Tf)(x)-w_I(f)v(x)|^2 dx\\ &\ge \|Tf-w_J(f)v\|_{2,J}^2 + \int_{I-J} |(Tf)(x)-w_I(f)v(x)|^2 dx,\end{aligned}$$

as $w_J(f)$ is the point in $\{\alpha v : \alpha\in\mathbf{R}\}$ nearest to Tf in $L^2(J)$.

If $c<c'$, we get

$$\|Tf-w_I(f)v\|_{2,I}^2 \ge \|Tf-w_J(f)v\|_{2,J}^2 + w_I(f)^2\int_c^{c'} v^2(x)dx,$$

while if $c=c'$ so that $d'<d$, we have

$$\|Tf-w_I(f)v\|_{2,I}^2 \ge \|Tf-w_J(f)v\|_{2,J}^2 + \int_{d'}^{d} |\int_{c'}^{x} u(t)f(t)dt - w_I(f)|^2 v^2(x)dx.$$

Both $w_I(f)^2$ and $\int_{d'}^{d}|\int_{c'}^{x} u(t)f(t)dt - w_I(f)|^2 v^2(x)dx$ are positive continuous functions on the set $K_J=\{f\in L^2(J); f\ge 0, \|f\|_{2,J}\le 1\}$, which is compact and metrizable in the *weak** topology. We conclude there exists $k>0$, independent of $f\in K_J$, such that

$$\|Tf-w_I(f)v\|_{2,I}^2 \ge \|Tf-w_J(f)v\|_{2,J}^2 + k,$$

and hence $A(I)>A(J)$. □

Remark 2.4 *It follows from the continuity of A that for sufficiently small $\varepsilon > 0$ there is an a_1, $a < a_1 < b$, for which $A(a_1, b) = \varepsilon$. Indeed, since T is compact, there exists a positive integer $N(\varepsilon)$ and points $b = a_0 > a_1 > \ldots > a_{N(\varepsilon)} = a$ with $A(a_i, a_{i-1}) = \varepsilon, i = 1, 2, \ldots, N(\varepsilon) - 1$ and $A(a, a_{N(\varepsilon)-1}) \leq \varepsilon$.*

Lemma 2.5 *The number $N(\varepsilon)$ is a non-increasing function of ε which takes on every sufficiently large integer value.*

Proof. Fix $c, a < c < b$. Then, (2. 2) ensures $A(c, b) = \varepsilon_0 > 0$. Moreover, as observed in Remark 2.4 there is a positive integer $N(\varepsilon_0)$ and a partition $b = a_0 > a_1 > \ldots > a_{N(\varepsilon_0)} = a$ such that $A(a_i, a_{i-1}) = \varepsilon_0,\ i = 1, 2, \ldots, N(\varepsilon_0) - 1$ and $A(a, a_{N(\varepsilon_0)-1}) \leq \varepsilon_0$. Let $d \in (a, a_{N(\varepsilon_0)-1})$. According to Lemma 2.3, $A(d, a_{N(\varepsilon_0)-1}) = \varepsilon_0' < \varepsilon_0$ and the procedure outlined above applied with ε_0' gives $\infty > N(\varepsilon_0') \geq N(\varepsilon_0) + 1$. If $N(\varepsilon_0') = N(\varepsilon_0) + 1$, stop.

Otherwise, define

$$\varepsilon_1 = \sup\{\varepsilon : 0 < \varepsilon < \varepsilon_0 \text{ and } N(\varepsilon) \geq N(\varepsilon_0) + 1\}.$$

We claim $N(\varepsilon_1) = N(\varepsilon_0) + 1$. Indeed, suppose $N(\varepsilon_1) \geq N(\varepsilon_0) + 2$ and the partition $b = a_0 > a_1 > \ldots > a_{N(\varepsilon_1)} = a$ satisfies $A(a_i, a_{i-1}) = \varepsilon_1, i = 1, 2, \ldots, N(\varepsilon_1) - 1$ and $A(a, a_{N(\varepsilon_1)-1}) \leq \varepsilon_1$. Decrease $a_{N(\varepsilon_1)-1}$ slightly to $a'_{N(\varepsilon_1)}$ so that both $A(a, a'_{N(\varepsilon_1)}) < \varepsilon_1$ and $A(a'_{N(\varepsilon_1)}, a_{N(\varepsilon_1)-1}) > \varepsilon_1$, continuing the process to get a partition of (a, b) having $N(\varepsilon_1)$ intervals such that $A(a, a'_{N(\varepsilon_1)}) < \varepsilon_1$ and $A(a'_i, a'_{i-1}) > \varepsilon_1, i = 1, 2, \ldots, N(\varepsilon_1) - 1, a'_0 = b$. Taking $\varepsilon_2 \leq \min_{2 \leq i \leq N(\varepsilon_1)-1} A(a'_i, a'_{i-1})$ we obtain $\varepsilon_2 > \varepsilon_1$ and $N(\varepsilon_2) \geq N(\varepsilon_0) + 2$, a contradiction. This establishes the claim.

An inductive argument completes the proof. □

The quantity $N(\varepsilon)$ is useful in the derivation of upper and lower estimates for the approximation numbers of T.

Lemma 2.6 *For all $\varepsilon \in (0, \|T\|)$,*

$$a_{N(\varepsilon)+2}(T) \leq \varepsilon \leq a_{N(\varepsilon)}(T).$$

Proof. This follows from [EEH1], Lemmas 6 and 7 (observe that $A(I) = K(I)$ for $p = 2$). □

A version of this result, with a slightly different $N(\varepsilon)$, was first proved in [EEH1] and was then extended in [EHL1]. For general u and v it is impossible to find a simple relation between ε and $N(\varepsilon)$, but by using the properties of A the behavior of $\varepsilon N(\varepsilon)$ when $\varepsilon \to 0_+$ can be determined.

Lemma 2.7 *Given $u, v \in L^2(a, b)$, we have*

$$\lim_{\varepsilon \to 0_+} \varepsilon N(\varepsilon) = \frac{1}{\pi} \int_a^b |u(t)v(t)| dt.$$

This result follows from an adaptation of the argument of [EEH2]; see, in particular, Theorem 16 of that paper. Together with Lemma 2.6 this shows, again using the techniques of [EEH2], that the following theorem holds.

Theorem 2.8 *Given $u, v \in L^2(a,b)$, the operator T defined in (1. 1) satisfies*

$$\lim_{n\to\infty} n a_n(T) = \frac{1}{\pi}\int_a^b |u(t)v(t)|dt.$$

A result of this type was established under weaker conditions on u and v in [EEH2].

3 Technical results.

Here we give some results of a technical nature which will prove very useful in the sequel. We begin with some information about the function A.

Lemma 3.1 *Let $I = (c,d) \subseteq (a,b)$ and suppose that u and v are constant functions over I. Then*

$$A(I,u,v) = |I||u||v|A((0,1),1,1)$$

and

$$A((0,1),1,1) = \frac{1}{\pi}.$$

Proof. By definition,

$$\begin{aligned} A(I,u,v) &= \sup_{f\in L^2(I)} \inf_{\alpha\in\mathbf{R}} \|Tf-\alpha v\|_{2,I}/\|f\|_{2,I} \\ &= \sup_{\|f\|_{2,I}\le 1} \inf_{\alpha\in\mathbf{R}} \|v\left(\int_c^{\cdot} uf dt - \alpha\right)\|_{2,I} \\ &= |v||u| \sup_{\|f\|_{2,I}\le 1} \inf_{\alpha\in\mathbf{R}} \|\int_c^{\cdot} f dt - \alpha\|_{2,I} \\ &= |v||u||I| \sup_{\|f\|_{2,(0,1)}\le 1} \inf_{\alpha\in\mathbf{R}} \|\int_0^{\cdot} f dt - \alpha\|_{2,(0,1)} \end{aligned}$$

Moreover, with the help of [EEH1], Lemma 6 and [EEH2], Lemma 2, and writing $F(x) = \int_0^x f(t)dt$, we see that:

$$A((0,1),1,1) = \sup_{\|f\|_{2,(0,1)}\le 1} \inf_{\alpha\in\mathbf{R}} \|F(.)-\alpha\|_{2,(0,1)} = \frac{1}{\pi}.$$

□

Next, we investigate the dependence of $A(I,u,v)$ on u and v.

Lemma 3.2 *Let $I = (c, d) \subset (a, b)$ and suppose that $v, u_1, u_2 \in L^2(I)$. Then*

$$|A(I, u_1, v) - A(I, u_2, v)| \leq \|u_1 - u_2\|_{2,I} \|v\|_{2,I}.$$

Proof. Without loss of generality we may suppose that $A(I, u_1, v) \geq A(I, u_2, v)$. Then

$$\begin{aligned}
A(I, u_1, v) &= \sup_{\|f\|_{2,I} \leq 1} \inf_{\alpha \in \mathbf{R}} \|v \left[\int_c^{\cdot} (u_1 - u_2 + u_2) f dt - \alpha \right] \|_{2,I} \\
&\leq \sup_{\|f\|_{2,I} \leq 1} \inf_{\alpha \in \mathbf{R}} \Big[\|v \int_c^{\cdot} (u_1 - u_2) f dt\|_{2,I} \\
&\qquad + \|v(\int_c^{\cdot} u_2 f dt - \alpha)\|_{2,I} \Big] \\
&\leq \sup_{\|f\|_{2,I} \leq 1} \inf_{\alpha \in \mathbf{R}} \Big[\|v\|_{2,I} \|u_1 - u_2\|_{2,I} \\
&\qquad + \|v(\int_c^{\cdot} u_2 f - \alpha)\|_{2,I} \Big] \\
&\leq \|v\|_{2,I} \|u_1 - u_2\|_{2,I} + A(I, u_2, v).
\end{aligned}$$

The result follows. □

Lemma 3.3 *Let $I = (c, d) \subset (a, b)$ and suppose that $u, v_1, v_2 \in L^2(I)$. Then*

$$|A(I, u, v_1) - A(I, u, v_2)| \leq \|u\|_{2,I} \|v_1 - v_2\|_{2,I}.$$

Proof. We may suppose that $A(I, u, v_1) \geq A(I, u, v_2)$. Then

$$\begin{aligned}
A(I, u, v_1) &= \sup_{\|f\|_{2,I} \leq 1} \inf_{\alpha \in \mathbf{R}} \|v_1 \left[\int_c^{\cdot} u f dt - \alpha \right] \|_{2,I} \\
&= \sup_{\|f\|_{2,I} \leq 1} \inf_{|\alpha| \leq \|u\|_{2,I} \|f\|_{2,I}} \|v_1 \left[\int_c^{\cdot} u f dt - \alpha \right] \|_{2,I} \\
&\leq \sup_{\|f\|_{2,I} \leq 1} \inf_{|\alpha| \leq \|u\|_{2,I}} \Big[\|(v_1 - v_2)[\int_c^{\cdot} u f dt - \alpha]\|_{2,I} \\
&\qquad + \|v_2[\int_c^{\cdot} u f dt - \alpha]\|_{2,I} \Big] \\
&\leq \|v_1 - v_2\|_{2,I} \|u\|_{2,I} + A(I, u, v_2).
\end{aligned}$$

The proof is complete. □

We now turn to the approximation of L^2 functions by step-functions.

Suppose $u, v \in L^2(a, b)$ and let $\alpha > 0$. We define a number $m_\alpha \in \mathbf{N}$ by the following requirements:

There exist two step-functions, u_α and v_α, each with m_α steps, say

$$u_\alpha(x) := \sum_{j=1}^{m_\alpha} \xi_j \chi_{w_\alpha(j)}(x), \qquad v_\alpha(x) := \sum_{j=1}^{m_\alpha} \psi_j \chi_{w_\alpha(j)}(x), \tag{3. 1}$$

where $\{w_\alpha(j)\}_{j=1}^{m_\alpha}$ is a family of non-overlapping intervals covering (a, b), such that

(i)

$$\max(\|u - u_\alpha\|_{2,(a,b)}^2, \|v - v_\alpha\|_{2,(a,b)}^2) \le \alpha^2; \tag{3. 2}$$

and

(ii) for any step-functions u'_α, v'_α with less than m_α steps, say n_α steps, $n_\alpha < m_\alpha$,

$$\max(\|u - u'_\alpha\|_{2,(a,b)}^2, \|v - v'_\alpha\|_{2,(a,b)}^2) > \alpha^2.$$

Thus, m_α is the minimum number of steps needed to approximate u and v is L^2 with the required accuracy. Note that, plainly,

$$\|u - u_\alpha\|_{2,(a,b)} \le \alpha, \qquad \|v - v_\alpha\|_{2,(a,b)} \le \alpha.$$

The best way to choose ξ_i and ψ_i for given $\{w_\alpha\}_{j=1}^{m_\alpha}$ is

$$\xi_i = \frac{1}{|w_\alpha(i)|} \int_{w_\alpha(i)} u(t)dt \qquad \psi_i = \frac{1}{|w_\alpha(i)|} \int_{w_\alpha(i)} v(t)dt,$$

when $w_\alpha(i)$ has finite length.

It turns out that the relationship between α and m_α is crucial for us; we next address this matter.

Lemma 3.4 *Suppose $u, v \in C(a, b) \cap L^2(a, b)$, at least one of them say u, being non-constant. Then, when α decreases to 0, m_α increases to ∞.*

Proof. We show that given $m \in N$ there exists $\alpha > 0$ having $m_\alpha > m$. The fact that u is continuous and nonconstant on (a, b) guarantees the existence of pairwise disjoint subintervals $I_1, I_2, \ldots, I_{2m}$ of (a, b) on each of which u is nonconstant.

Fix $\alpha > 0$ satisfying $\sum_{j=1}^m \|u - u_{I_{k_j}}\|_{2,I_{k_j}}^2 > \alpha^2$ for every set of m intervals from among $I_1, I_2, \ldots, I_{2m}$. Now, to any partition, $\{w_\alpha(j)\}_{j=1}^m$, of (a, b) into m non-overlapping subintervals there correspond $I_{k_1}, I_{k_2}, \ldots, I_{k_m}$ such that each I_{k_j} is subset of some $w_\alpha(i)$ and hence

$$\sum_{j=1}^m \|u - u_{w_\alpha(j)}\|_{2,w_\alpha(j)}^2 \ge \sum_{j=1}^m \|u - u_{w_\alpha(j)}\|_{2,I_{k_j}}^2 > \alpha^2.$$

Therefore $m_\alpha > m$. □

Lemma 3.5 *Suppose $u, v \in C(a,b) \cap L^2(a,b)$, at least one of them, say u, being nonconstant. Fix $\alpha > 0$ and set $\Lambda_\alpha = \{\beta; 0 < \beta \le \alpha$ and $m_\beta = m_\alpha\}$. Then, Λ_α is an interval with $\gamma = \inf \Lambda_\alpha$ and $\gamma \in \Lambda_\alpha$.*

Proof. Clearly, Λ_α is nonempty, since $\alpha \in \Lambda_\alpha$. Again, $m_{\lambda_1} \ge m_{\lambda_2}$ whenever $\lambda_1 < \lambda_2$, so Λ_α is convex and hence an interval, possibly equal to $\{\alpha\}$.

It follows from Lemma 3.4 that $\gamma > 0$. Now, if $\Lambda_\lambda = \{\alpha\}$, so that $\gamma = \alpha$, we are done. Otherwise, there exists a sequence $\{\alpha_n\}$ in Λ_α with $\alpha_n \searrow \gamma$. Let $u_{\alpha_n} = \sum_{j=1}^{m_\alpha} u_{w_{\alpha_n}(j)} \chi_{w_{\alpha_n}(j)}$ and $v_{\alpha_n} = \sum_{j=1}^{m_\alpha} v_{w_{\alpha_n}(j)} \chi_{w_{\alpha_n}(j)}$, as in (3. 1), so that

$$\max(\|u - u_{\alpha_n}\|^2_{2,(a,b)}, \|v - v_{\alpha_n}\|^2_{2,(a,b)}) \le \alpha_n^2.$$

Assume the notation has been chosen to ensure the end points of $w_{\alpha_n}(j) = (c_n^j, d_n^j)$ satisfy $a = c_n^1 < d_n^j \le c_n^{j+1} < d_n^{m_\alpha} = b$, $j = 1, 2, \ldots, m_\alpha - 1$.

There exists a sequence $n(k), k = 1, 2, \ldots$ of positive integers and numbers $c^1, c^2, \ldots, c^{m_\alpha}, d^1, d^2, \ldots, d^{m_\alpha}$ such that

$$\lim_k c^j_{n(k)} = c^j, \qquad \lim_k d^j_{n(k)} = d^j, \qquad j = 1, 2, \ldots, m_\alpha,$$

and

$$a = c^1 \le d^j \le c^{j+1} \le d^{m_\alpha} = b, \qquad j = 1, 2, \ldots, m_\alpha.$$

Observe that, setting

$$u_\gamma = \sum_{j=1}^{m_\alpha} u_{(c^j,d^j)} \chi_{(c^j,d^j)} \qquad \text{and} \qquad v_\gamma = \sum_{j=1}^{m_\alpha} v_{(c^j,d^j)} \chi_{(c^j,d^j)},$$

we have

$$\max(\|u - u_\gamma\|^2_{2,(a,b)}, \|v - v_\gamma\|^2_{2,(a,b)}) = \gamma^2,$$

which forces $m_\gamma = m_\alpha$. $\Box$

Lemma 3.6 *Suppose $u, v \in L^2(a,b) \cap C(a,b)$ are not equal to zero on (a,b), indeed, assume at least one of u and v nonconstant on (a,b). Then, there exists $\alpha_0 > 0$ such that given any α, $0 < \alpha < \alpha_0$, there exists a β, $0 < \beta < \alpha$, with $m_\beta = m_\alpha + 1$ or $m_\beta = m_\alpha + 2$.*

Proof. Say u is nonconstant on (a,b). We take α_0 to be the positive distance of u from the closed set $\{k\chi_I; k \in \mathbf{R}, 0 < |I| < \infty\}$ in $L^2(a,b)$. Observe that $m_\alpha \ge 2$ whenever $0 < \alpha < \alpha_0$.

Fix $\alpha, 0 < \alpha < \alpha_0$. By Lemma 3.5, $m_\gamma = m_\alpha$, where $\gamma = \inf \Lambda_\alpha$. Hence, there exists a partition $\{w_\gamma(j)\}_{j=1}^{m_\gamma}$ of (a,b) whose corresponding step functions, $u_\gamma = \sum_{j=1}^{m_\alpha} u_{w_\gamma(j)} \chi_{w_\gamma(j)}$ and $v_\gamma = \sum_{j=1}^{m_\alpha} v_{w_\gamma(j)} \chi_{w_\gamma(j)}$, satisfy

$$\max(\|u - u_\gamma\|^2_{2,(a,b)}, \|v - v_\gamma\|^2_{2,(a,b)}) = \gamma^2.$$

If $\|u-u_\gamma\|^2_{2,(a,b)} > \|v-v_\gamma\|^2_{2,(a,b)}$ then for some some j_0, $1 \le j_0 \le m_\alpha$,

$$\|u-u_{w_\gamma(j_0)}\|^2_{2,w_\gamma(j_0)} > 0.$$

It is possible to find a point c in the interval $w_\gamma(j_0) = (d,e)$ such that

$$\|u-u_{w_\gamma(j_0)}\|^2_{2,w_\gamma(j_0)} > \|u-u_{(d,c)}\|^2_{2,(d,c)} + \|u-u_{(c,e)}\|^2_{2,(c,e)}.$$

Let $w'_\gamma(j) = w_\gamma(j)$, $j = 1,2,\ldots,j_0-1,j_0+1,\ldots,m_\alpha$, $w'_\gamma(j_0) = (d,c)$ and $w'_\gamma(m_\alpha+1) = (c,e)$. Then, $\{w'_\gamma(j)\}_{j=1}^{m_\alpha+1}$ is a partition of (a,b) with associated step functions $u'_\gamma = \sum_{j=1}^{m_\alpha+1} u_{w'_\gamma(j)}\chi_{w'_\gamma(j)}$ and $v'_\gamma = \sum_{j=1}^{m_\alpha+1} v_{w'_\gamma(j)}\chi_{w'_\gamma(j)}$ such that

$$\max(\|u-u'_\gamma\|^2_{2,(a,b)}, \|v-v'_\gamma\|^2_{2,(a,b)}) = \beta^2 < \gamma^2,$$

and so $m_\beta = m_\alpha + 1$.

Similarly, when $\|v-v_\gamma\|_{2,(a,b)} > \|u-u_\gamma\|_{2,(a,b)}$, there is a $\beta \in (0,\alpha)$ with $m_\beta = m_\alpha+1$. Suppose, then, $\|v-v_\gamma\|_{2,(a,b)} = \|u-u_\gamma\|_{2,(a,b)} = \gamma^2 > 0$. As before, we can find an interval $w_\gamma(j_0) = (d_0,e_0)$ and a point c_0 such that

$$\|u-u_{w_\gamma(j_0)}\|^2_{2,w_\gamma(j_0)} > \|u-u_{(d_0,c_0)}\|^2_{2,(d_0,c_0)} + \|u-u_{(c_0,e_0)}\|^2_{2,(c_0,e_0)},$$

and an interval $w_\gamma(j_1) = (d_1,c_1)$ and a point c_1 such that

$$\|u-u_{w_\gamma(j_1)}\|^2_{2,w_\gamma(j_1)} > \|u-u_{(d_1,c_1)}\|^2_{2,(d_1,c_1)} + \|u-u_{(c_1,e_1)}\|^2_{2,(c_1,e_1)}.$$

Now, if it is possible to have $j_0 = j_1$ and $c_0 = c_1$ we can get $\beta \in (0,\alpha)$ with $m_\beta = m_\alpha+1$. Otherwise, we can only conclude there is a $\beta \in (0,\alpha)$ for which m_β is one of $m_\alpha+1$ and $m_\alpha+2$. □

Lemma 3.7 *Let $-\infty \le a < b \le \infty$ and suppose that $u \in L^2(a,b)$ with $u' \in L^{2/3}(a,b) \cap C(a,b)$. For each small $h > 0$ define*

$$x_1 = -\frac{1}{h}, x_{i+1} := x_i + h \text{ for } i \in 1,\ldots,[2/h^2];$$

put $J_i = (a,b) \cap (x_i,x_{i+1})$, $i \in 1,\ldots,[2/h^2]$.

Then

$$\int_a^b |u'(t)|^{2/3}dt = \lim_{h\to 0}\sum_{i=1}^{[2/h^2]} |J_i| \max_{x\in J_i}|u'(x)|^{2/3}$$
$$= \lim_{h\to 0}\sum_{j=1}^{[2/h^2]} |J_i| \min_{x\in J_i}|u'(x)|^{2/3}.$$

Proof. Simply use the definition of the integral. □

We are now prepared to establish an important estimate for $\limsup_{\alpha\to 0_+} \alpha m_\alpha$.

Theorem 3.8 *Suppose $u, v \in L^2(a,b)$ and $u', v' \in L^{2/3}(a,b) \cap C(a,b)$. Then,*

$$\limsup_{\alpha \to 0_+} \alpha m_\alpha \leq \sqrt{2}(\|u'\|_{2/3,(a,b)} + \|v'\|_{2/3,(a,b)}).$$

Proof. The result is trivial if both u and v are constant so we assume at least one of them, say u, is not.

Given $\beta, 0 < \beta < \|u\|_{2,(a,b)}$, let $w_\beta(i) = (a_i, a_{i+1}),\ i = 1, 2, \ldots, n_\beta^u$, be a partition of (a,b) satisfying

$$\|u - u_{w_\beta(i)}\|_{2,w_\beta(i)}^2 = \beta^2, \qquad i = 1, 2, \ldots, n_\beta^u - 1,$$

and $\|u - u_{w_\beta(i)}\|_{2,w_\beta(i)}^2 \leq \beta^2$, $i = n_\beta^u$. Fix λ, $0 < \lambda < 1$, and define the $[\lambda n_\beta^u]$ points x_k by the rule that if (a,b) is bounded, then

$$x_k := a + \frac{b-a}{\lambda n_\beta^u} k, \qquad k = 1, 2, \ldots, [\lambda n_\beta^u];$$

if $(a,b) = (-\infty, \infty)$, then, with $h = (\frac{2}{\lambda n_\beta^u})^{1/2}$,

$$x_1 = -\frac{1}{h}, \quad x_{k+1} = x_k + h, \qquad k = 1, 2, \ldots, [\lambda n_\beta^u];$$

for other types of intervals proceed in the same sort of way.

Form the union of the points $a_1, a_2, \ldots, a_{n_\beta^u + 1}$ and $x_1, x_2, \ldots, x_{[\lambda n_\beta^u]}$, arrange them in ascending order and denote the resulting points by $b_j, j = 1, 2, \ldots, J(\beta) + 1$, so that $n_\beta^u \leq J(\beta) \leq n_\beta^u + [\lambda n_\beta^u]$. Put $I_j^\beta = (b_j, b_{j+1}), j = 1, 2, \ldots, J(\beta)$. We observe there are at least $n_\beta^u - [\lambda n_\beta^u]$ intervals I_j^β with

$$I_j^\beta = w_\beta(i)$$

for some i.

Now,

$$\sum_{j=1}^{J(\beta)} \|u - u_{I_j^\beta}\|_{2,I_j^\beta}^{2/3} \leq \sum_{j=1}^{J(\beta)} |I_j^\beta| \max_{x \in I_j^\beta} |u'(x)|^{2/3}.$$

Again, setting $N = \#(\{j : I_j^\beta = w_\beta(i) \ \text{ for some } i < n_\beta^u\})$, we have $N \geq n_\beta^u - [\lambda n_\beta^u] - 1$ and

$$\beta^{2/3}(n_\beta^u - [\lambda n_\beta^u] - 1) \leq \beta^{2/3} N \leq \sum_{j=1}^{J(\beta)} \|u - u_{I_j^\beta}\|_{2,I_j^\beta}^{2/3} \leq \sum_{j=1}^{J(\beta)} |I_j^\beta| \max_{x \in I_j^\beta} |u'(x)|^{2/3}.$$

Thus, by Lemma 3.7,

$$\limsup_{\beta \to 0_+} \beta^{2/3}(n_\beta^u - [\lambda n_\beta^u]) \leq \int_a^b |u'(x)|^{2/3} dx. \tag{3. 3}$$

Similarly, if v is also not constant, there exists, for $0 < \beta < \|v\|_{2,(a,b)}$, a partition $\{w'_\beta(i)\}_{i=1}^{n^v_\beta}$ such that

$$\begin{aligned} \|v - v_{w'_\beta(i)}\|^2_{2,w'_\beta(i)} &= \beta^2, \qquad i = 1, 2, \ldots, n^v_\beta - 1, \\ \|v - v_{w'_\beta(i)}\|^2_{2,w'_\beta(i)} &\le \beta^2, \qquad i = n^v_\beta, \end{aligned}$$

and

$$\limsup_{\beta \to 0_+} \beta^{2/3}(n^v_\beta - [\lambda n^v_\beta]) \le \int_a^b |v'(x)|^{2/3} dx. \tag{3. 4}$$

Put $\alpha^2 = \beta^2(n_\beta + [\lambda n_\beta])$, $0 < \beta < \min[\|u\|_{2,(a,b)}, \|v\|_{2,(a,b)}]$, where $n_\beta = n^u_\beta + n^v_\beta$ if v is not constant and $n_\beta = n^u_\beta$ if it is. Note that (3. 3) and (3. 4) imply $\alpha \to 0_+$ as $\beta \to 0_+$.

Taking the refinement of the partition $\{I^\beta_j\}_{j=1}^{J(\beta)}$ and the analogous one for v (if necessary) we get a partition of (a, b), of at most $n_\beta + [\lambda n_\beta]$ subintervals, whose corresponding step-functions u_α and v_α satisfy

$$\max[\|u - u_\alpha\|^2_{2,(a,b)}, \|v - v_\alpha\|^2_{2,(a,b)}] \le \beta^2 \max[n^u_\beta, n^v_\beta] \le \alpha^2.$$

This means

$$m_\alpha \le n_\beta + [\lambda n_\beta];$$

hence

$$\begin{aligned} \limsup_{\alpha \to 0_+} (\alpha m_\alpha)^{2/3} &\le \limsup_{\beta \to 0_+} \beta^{2/3}(n_\beta + [\lambda n_\beta]) \\ &\le \limsup_{\beta \to 0_+} \beta^{2/3}(n_\beta - [\lambda n_\beta]) \left(\frac{n_\beta + [\lambda n_\beta]}{n_\beta - [\lambda n_\beta]} \right) \\ &\le (\|u'\|^{2/3}_{2/3,(a,b)} + \|v'\|^{2/3}_{2/3,(a,b)}) \frac{(1 + \lambda)}{(1 - \lambda)}. \end{aligned}$$

Since λ may be chosen arbitrarily small we obtain

$$\limsup_{\alpha \to 0_+} \alpha m_\alpha \le \sqrt{2}(\|u'\|_{2/3,(a,b)} + \|v'\|_{2/3,(a,b)}),$$

as asserted. □

4 The Main theorem.

In this section we give the remainder estimate promised in the Introduction. To begin, we prove

Theorem 4.1 *Let* $-\infty \leq a < b \leq \infty$*, let* $u, v \in L^2(a,b)$ *and suppose that* $u', v' \in L^{2/3}(a,b) \cap C(a,b)$ *and* $u^2, v^2 > 0$*. Then*

$$\limsup_{\varepsilon\to 0_+} \left| \frac{1}{\pi}\int_a^b |u(t)v(t)|dt - \varepsilon N(\varepsilon)\right| N^{1/2}(\varepsilon)$$

$$\leq 3\sqrt{2}(\|u'\|_{2/3,(a,b)} + \|v'\|_{2/3,(a,b)})\left(\|u\|_{2,(a,b)} + \|v\|_{2,(a,b)}\right) + \frac{3}{\pi}\|uv\|_{1,(a,b)}.$$

Proof. Let $\alpha > 0$. Then (see (3. 1) and (3. 2)) there are $m_\alpha \in \mathbf{N}$ and step-functions u_α, v_α such that

$$\|u_\alpha - u\|_{2,(a,b)} < \alpha, \qquad \|v_\alpha - v\|_{2,(a,b)} < \alpha;$$

and $\{w_\alpha(j)\}_{j=1}^{m_\alpha}$ is a corresponding family of non-overlapping intervals which cover (a,b). Plainly,

$$|\int_a^b (uv - u_\alpha v_\alpha)dt| \leq \alpha(\|u\|_{2,(a,b)} + \|v\|_{2,(a,b)} + \alpha). \tag{4. 1}$$

Let $\varepsilon > 0$ be small and let $\{I_i\}_{i=1}^{N(\varepsilon)}$ be the non-overlapping intervals which occur in the definition of $N(\varepsilon)$.

Put $J_1 = \{j; I_i \subset w_\alpha(j) \text{ for some } i\}$, $J_2 = \{j; w_\alpha(j) \subset I_i \text{ for some } i\}$, $J_3 = \{j; w_\alpha(j) \not\subset I_i \not\subset w_\alpha(j), \text{ for all } i\}$, $L_1 = \{i; I_i \subset w_\alpha(j) \text{ for some } j\}$ and $L_2 = \{i; \text{ for all } j, I_i \not\subset w_\alpha(j)\}$. Then see from Lemma 3.1 that

$$\begin{aligned}\frac{1}{\pi}\int_a^b u_\alpha v_\alpha dt &= \frac{1}{\pi}\left(\sum_{j\in J_1} + \sum_{j\in J_2} + \sum_{j\in J_3}\right)\xi_j\psi_j|w_\alpha(j)| \\ &\leq \sum_{i\in L_1} A(I_i, u_\alpha, v_\alpha) \\ &\quad +2\sum_{i\in L_2} A(I_i, u_\alpha, v_\alpha) \\ &\quad +\sum_{j\in J_3}\frac{1}{\pi}\xi_j\psi_j|w_\alpha(j)|. \end{aligned} \tag{4. 2}$$

Lemmas 3.2, 3.3 as well as the estimates

$$\begin{aligned}\frac{1}{\pi}\xi_j\psi_j|w_\alpha(j)| &\leq A(w_\alpha(j), u_\alpha, v_\alpha) \\ &\leq A(w_\alpha(j), u, v) + \|u - u_\alpha\|_{2,w_\alpha(j)}\|v - v_\alpha\|_{2,w_\alpha(j)} \\ &\quad +\|u\|_{2,w_\alpha(j)}\|v - v_\alpha\|_{2,w_\alpha(j)} \\ &\quad +\|u - u_\alpha\|_{2,w_\alpha(j)}\|v\|_{2,w_\alpha(j)}\end{aligned}$$

and $A(w_\alpha(j),u,v) \le A(I_i,u,v) \le \varepsilon$ for $w_\alpha(j) \subset I_i$ now show that the right-hand side of (4. 2) may be estimated from above by

$$\sum_{I_i \subset w_\alpha(j)} A(I_i,u,v) + 2 \sum_{I_i \not\subset w_\alpha(j)} A(I_i,u,v) + \varepsilon m_\alpha$$
$$+3 \sum_{i=1}^{N(\varepsilon)} (\|u-u_\alpha\|_{2,I_i}\|v-v_\alpha\|_{2,I_i} + \|u\|_{2,I_i}\|v-v_\alpha\|_{2,I_i}$$
$$+\|u-u_\alpha\|_{2,I_i}\|v\|_{2,I_i}) \qquad (4.\ 3)$$

To proceed further, note that $A(I_i,u,v) \le \varepsilon$,

$$\#\{i; I_i \subset w_\alpha(j) \text{ for some } j\} \le N(\varepsilon)$$

and

$$\#\{i, \text{ for all } j, I_i \not\subset w_\alpha(j)\} \le m_\alpha.$$

It follows that

$$\begin{aligned}
\frac{1}{\pi}\int_a^b u_\alpha v_\alpha \quad &\le \quad N(\varepsilon)\varepsilon + 3m_\alpha\varepsilon \\
&+3\sum_{i=1}^{N(\varepsilon)} (\|u-u_\alpha\|_{2,I_i}\|v-v_\alpha\|_{2,I_i} + \|u\|_{2,I_i}\|v-v_\alpha\|_{2,I_i} \\
&\qquad +\|u-u_\alpha\|_{2,I_i}\|v\|_{2,I_i}) \\
&\le \quad N(\varepsilon)\varepsilon + 3m_\alpha\varepsilon + 2\alpha^2 + 2\alpha(\|u\|_{2,(a,b)} + \|v\|_{2,(a,b)}). \qquad (4.\ 4)
\end{aligned}$$

On the other hand, since $A(I_i,u,v) = \varepsilon$ for $1 \le i \le N(\varepsilon)-1$ and $N(\varepsilon) - 2m_\alpha \le \#\{i; I_i \subset w_\alpha(j) \text{ for some } j\}$, we see that

$$\begin{aligned}
(N(\varepsilon)-2m_\alpha-1)\varepsilon \quad &\le \quad \sum_{I_i \subset w_\alpha(j)} A(I_i,u,v) \\
&= \quad \sum_{I_i \subset w_\alpha(j)} A(I_i,u_\alpha,v_\alpha) + \sum_{I_i \subset w_\alpha(j)} [A(I_i,u,v) - A(I_i,u_\alpha,v_\alpha)] \\
&\le \quad \sum_{I_i \subset w_\alpha(j)} \frac{1}{\pi}|I_i||\xi_j||\psi_j| \\
&\quad + \sum_{I_i \subset w_\alpha(j)} (\|u-u_\alpha\|_{2,I_i}\|v-v_\alpha\|_{2,I_i} \\
&\qquad +\|u\|_{2,I_i}\|v-v_\alpha\|_{2,I_i} + \|u-u_\alpha\|_{2,I_i}\|v\|_{2,I_i}) \\
&\le \quad \frac{1}{\pi}\int_a^b |u_\alpha v_\alpha| dt + \alpha^2 + \alpha(\|u\|_{2,(a,b)} + \|v\|_{2,(a,b)}) \\
&\le \quad \frac{1}{\pi}\int_a^b |uv| dt + 2\alpha^2 + 2\alpha(\|u\|_{2,(a,b)} + \|v\|_{2,(a,b)}) \qquad (4.\ 5)
\end{aligned}$$

the final inequality following from (4. 1). Together with (4. 4) and (4. 1) this shows that

$$\begin{aligned}\varepsilon(N(\varepsilon) &- 2m_\alpha - 1) - 2\alpha^2 - 2\alpha(\|u\|_{2,(a,b)} + \|v\|_{2,(a,b)}) \\ &\leq \frac{1}{\pi}\int_a^b |uv|dt \qquad (4.\ 6)\\ &\leq \varepsilon(N(\varepsilon) + 3m_\alpha) + 3\alpha^2 + 3\alpha(\|u\|_{2,(a,b)} + \|v\|_{2,(a,b)}).\end{aligned}$$

From Lemma 3.4 together with repeated applications of Lemma 3.6 we can see that for any small $\varepsilon > 0$, we can find $\alpha > 0$ such that $m_\alpha \geq [N^{1/2}(\varepsilon)] \geq m_\alpha - 2$. Then (4. 6) gives

$$\begin{aligned}N^{1/2}(\varepsilon)|\frac{1}{\pi}\int_a^b |uv|dt - N(\varepsilon)\varepsilon| &\leq 3N(\varepsilon)\varepsilon + 3\alpha^2(N^{1/2}(\varepsilon) - 1)\\ &\quad +3\alpha(\|u\|_{2,(a,b)} + \|v\|_{2,(a,b)})N^{1/2}(\varepsilon).\end{aligned}$$

Let $\varepsilon \to 0_+$; then $m_\alpha \leq N^{1/2}(\varepsilon) + 2 \to \infty$ and so $\alpha \to 0_+$. Hence

$$\begin{aligned}&\limsup_{\varepsilon\to 0_+} N^{1/2}(\varepsilon)|\frac{1}{\pi}\int_a^b |uv|dt - N(\varepsilon)\varepsilon|\\ &\leq 3\limsup_{\varepsilon\to 0_+} N(\varepsilon)\varepsilon + 3\limsup_{\varepsilon\to 0_+} \alpha^2 N^{1/2}(\varepsilon)\\ &\quad +3\limsup_{\varepsilon\to 0_+} \alpha N^{1/2}(\varepsilon)(\|u\|_{2,(a,b)} + \|v\|_{2,(a,b)}).\end{aligned}$$

Since $\lim_{\varepsilon\to 0} \varepsilon N(\varepsilon) = \frac{1}{\pi}\int_a^b |uv|dt$, by Lemma 2.7, we finally see, with the help of Lemma 3.8, that

$$\begin{aligned}\limsup_{\varepsilon\to 0_+} \quad & N^{1/2}(\varepsilon)|\frac{1}{\pi}\int_a^b |uv| - N(\varepsilon)\varepsilon|\\ \leq \quad & \frac{3}{\pi}\int_a^b |uv|dt + 3\sqrt{2}(\|u'\|_{2/3,(a,b)} + \|v'\|_{2/3,(a,b)})(\|u\|_{2,(a,b)} + \|v\|_{2,(a,b)}),\end{aligned}$$

as required. □

Armed with this result it is now easy to give the promised remainder estimate for the approximation numbers of $T : L^2(a,b) \to L^2(a,b)$ given by (1. 1).

Theorem 4.2 *Let $-\infty \leq a < b \leq \infty$, suppose that $u, v \in L^2(a,b)$ and let $u', v' \in L^{2/3}(a,b) \cap C(a,b)$ and $u^2, v^2 > 0$. Then*

$$\begin{aligned}\limsup_{n\to\infty} n^{1/2}\left|\frac{1}{\pi}\int_a^b |uv|dt - na_n\right| &\leq \frac{3}{\pi}\int_a^b |uv|dt\\ &+3\sqrt{2}(\|u'\|_{2/3,(a,b)} + \|v'|)\|_{2/3,(a,b)})(\|u\|_{2,(a,b)} + \|v\|_{2,(a,b)}).\end{aligned}$$

Proof. Simply use Theorem 4.1, Lemma 2.6, Lemma 2.5 and the fact that

$$\lim_{n\to\infty} n^{1/2} a_n(T) = 0.$$

□

If the interval (a, b) is bounded, it follows immediately from Hölder's inequality that Theorem 4.2 gives rise to

Theorem 4.3 *Let $-\infty < a < b < \infty$ and suppose that $u', v' \in C([a, b])$. Then*

$$\limsup_{n\to\infty} n^{1/2} |\frac{1}{\pi}\int_a^b |uv| dt - n a_n|$$

$$\leq \frac{3}{\pi}\int_a^b |uv| dt + 3\sqrt{2}(\|u'\|_{2,(a,b)} + \|v'\|_{2,(a,b)})(b-a)(\|u\|_{2,(a,b)} + \|v\|_{2,(a,b)}).$$

From the following observation we can see that any optimal exponent from Theorem 4.2 has to belong to $[1/2, 1]$.

Observation 4.4 *Let $-\infty \leq a < b \leq \infty$.*

(i) Let $\alpha < 1/2$. Then for every $u, v \in L^2(a, b)$ with $u', v' \in L^{2/3}(a, b) \cap C([a, b])$ we have

$$\limsup_{n\to\infty} n^{\alpha} \left| \frac{1}{\pi}\int_a^b |uv| dt - n a_n(T) \right| = 0.$$

(ii) Let $\alpha > 1$. Then there exist a and b, and functions u and v satisfying the conditions of Theorem 4.2 on the interval defined by a and b, such that

$$\limsup_{n\to\infty} n^{\alpha} \left| \frac{1}{\pi}\int_a^b |uv| dt - n a_n(T) \right| = \infty.$$

Proof. (i) follows from (4. 6) on putting $m_\alpha = [N^\alpha(\varepsilon)]$ or $[N^\alpha(\varepsilon)] + 1$.

(ii) Take $(a, b) = (0, 1)$ and $u = 1$, $v = 1 + x$. Then from (4. 6), with $m_\alpha = [N^\alpha(\varepsilon)]$ a lower bound results which is unbounded as $\varepsilon \to 0$ and the result follows. □

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