Behaviour of the approximation numbers of a Sobolev embedding in the one-dimensional case.

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Abstract

We consider the Sobolev embeddings

$$E_1 : W_0^{1,p}(a,b) \to L^p(a,b) \quad \text{and} \quad E_2 : L^{1,p}(a,b)/\{1\} \to L^p(a,b)/\{1\},$$

with $-\infty < a < b < \infty$ and $1 < p < \infty$. We show that the approximation numbers $a_n(E_i)$ of E_i have the property that

$$\lim_{n\to\infty} n a_n(E_i) = c_p(b-a) \qquad (i = 1, 2)$$

where c_p is a constant dependent only on p. Moreover we show the precise value of $a_n(E_1)$ and we study the unbounded Sobolev embedding $E_3 : L^{1,p}(a,b) \to L^p(a,b)$ and determine precisely how closely it may be approximated by n-dimensional linear maps.

Key words: Approximation numbers, Sobolev Embedding, Hardy-type operators, Integral operators
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1 Introduction.

Let Ω be a bounded subset of $\mathbf{R}^n$ with smooth boundary, let $1 < p < \infty$ and consider the embedding

$$E_1 : W_0^{1,p}(\Omega) \to L^p(\Omega)$$

where $W_0^{1,p}(\Omega)$ is the usual first-order Sobolev space of functions with zero trace. This space is a closed subspace of the Sobolev space $W^{1,p}(\Omega)$. It is well-known that E_1 is compact. More precise information about E_1 is available via its approximation numbers, for there are positive constants c_1 and c_2, depending only on p and Ω, such that the m-th approximation number $a_m(E_1)$ of E_1 satisfies

$$\frac{c_1}{m} \leq a_m(E_1) \leq \frac{c_2}{m}, \qquad m \in \mathbf{N} \tag{1}$$

We recall that, given any $m \in \mathbf{N}$, the m–th approximation number $a_m(T)$ of a bounded linear operator $T : X \to Y$, where X and Y are Banach spaces, is defined by $a_m(T) := \inf \|T - F|X \to Y\|$, where the infimum is taken over all bounded linear maps $F : X \to Y$ with rank less than m.

A measure of non-compactness of T is given by $\beta(T) := \inf \|T - P|X \to Y\|$, where the infimum is taken over all compact linear maps $P : X \to Y$. In our case we have $X = W_0^{1,p}(\Omega)$ and $Y = L^p(\Omega)$. Then since $L^p(\Omega)$ has the approximation property for $1 \leq p \leq \infty$, T is compact if and only if $a_m(T) \to 0$ as $m \to \infty$, and $\beta(T) = \lim_{n\to\infty} a_n(T)$.

Of course, this is a very special case of quite general results concerning the approximation numbers of embeddings between function spaces, for which we refer to (T) and (ET).

When $p = 2$ it is possible to sharpen (1) by using the familiar relation

$$a_m(E_1) = \frac{1}{\lambda_m^{1/2}}$$

between the approximation numbers of E_1 and the eigenvalues λ_m of the Dirichlet Laplacian. Since the behaviour of the eigenvalues is well-known, it follows that $\lim m a_m(E_1)$ exists; and even sharper statements about the asymptotic behaviour of $a_m(E_1)$ can be made. It is natural to ask whether or not $\lim m a_m(E_1)$ exists when p is not equal to 2.

In (EHL) a new technique was given for the study of the approximation numbers of the Hardy-type operator T on a tree Γ:

$$(Tf)(x) = v(x) \int_0^x f(t)u(t)dt, \qquad x \in \Gamma.$$

Using this it was shown that $T : L^p(\Gamma) \to L^p(\Gamma)$ has approximation numbers $a_m(T)$ for which $\lim m a_m(T)$ exists, when $1 \le p \le \infty$. This technique was improved and extended in (EKL), where in the case in which Γ is an interval and $p = 2$, remainder estimates were obtained. These results were extended in (L) to cover the cases $1 < p < \infty$.

In the present paper we obtain sharper information about $a_m(E_1)$ than was previously known. We deal only with the case in which $n = 1$ and Ω is a bounded interval in the line. The techniques of this paper are based on methods derived from (EHL), (EKL), (L), (Li2) and (DM). In more detail, for the Sobolev embeddings

$$E_1 : W_0^{1,p}(I) \to L^p(I)$$
$$E_2 : L^{1,p}(I)/\{1\} \to L^p(I)/\{1\},$$

where $I = (a, b)$, $-\infty < a < b < \infty$ and $L^{1,p}(I)$ is the space of all $u \in L^p_{loc}(I)$ with derivative $u' \in L^p(I)$, we show that there is a positive constant α_p such that

$$\lim_{m\to\infty} m a_m(E_i) = \alpha_p |I| \qquad \text{for } i = 1 \text{ or } 2.$$

Moreover, it turns out that for every $m \in \mathbf{N}$, there is a linear map P_m with $\operatorname{rank} P_m = m$ such that

$$\|E_2 - P_m\| = \alpha_p |I|/m \ge a_{m+1}(E_2) \ge \alpha_p |I|/(m + 1).$$

For embedding E_1 we have that for every $m \in \mathbf{N}$, there is a linear map B_m with $\operatorname{rank} B_m = m$ such that

$$\|E_1 - B_m\| = \alpha_p |I|/(m + 1) = a_{m+1}(E_1).$$

We also study the best approximation of the unbounded Sobolev embedding

$$E_3 : L^{1,p}(I) \to L^p(I)$$

by linear maps of finite rank. We show that for every $m \in \mathbf{N}$, there is a linear map R_m with $\operatorname{rank} R_m = m$ such that

$$\|E_3 - R_m\| = \alpha_p |I|/(m) = \inf\{\|E_3 - P\|; P \text{ linear map}, \operatorname{rank} P < m+1\}.$$

We also show that $\alpha_p = (\frac{1}{\lambda_{n,I}})^{1/p}$ where $\lambda_{n,I}$ is the first eigenvalue of a $p-$Laplacian eigenvalue problem.

Our conclusion appears to be the first result of this kind in the literature, apart from the special case $p = 2$. It remains to be seen whether or not this can be extended to higher dimensions.

2 Preliminaries and technical results.

Throughout the paper we shall assume that $-\infty < a < b < \infty$ and that $I = (a, b)$. We also assume that $1 < p < \infty$ and denote by $\|.\|_p$ or $\|.\|_{p,I}$ the usual norm on the Lebesgue space $L_p(I)$.

By the Sobolev space $W_0^{1,p}(I)$ we understand, as usual, the space of all functions $u \in L^p(I)$ with finite norm $\|u'\|_{p,I}$ and zero trace. We consider the embedding

$$E_1 : W_0^{1,p}(I) \to L^p(I) \tag{2}$$

and define the norm of E_1 by

$$\|E_1\| = \sup_{\|u'\|_{p,I}>0} \frac{\|u\|_{p,I}}{\|u'\|_{p,I}}. \tag{3}$$

Plainly $\|E_1\| < \infty$; moreover, it is well known (see, for example, (EE), Theorem V.4.18) that E_1 is compact.

We will consider in this paper also the approximation numbers for the embedding

$$E_2 : L^{1,p}(I)/\{1\} \to L^p(I)/\{1\},$$

where $L^{1,p}(I)$ is the space of all functions $u \in L^p_{loc}(I)$ with finite pseudonorm $\|u'\|_{p,I}$ which vanishes on the subspace of all constant functions. By $L^{1,p}/\{1\}$ we mean the factorization of the space $L^{1,p}(I)$ with respect to constant functions, equipped with the norm $\|u'\|_{p,I}$. Then we have $f \in L^{1,p}/\{1\}$ if and only if $\|f\|_{p,I} = \inf_{c\in\Re} \|f - c\|_{p,I}$. In a similar way $L^p(I)/\{1\}$ is defined. The norm of E_2 is defined by

$$\|E_2\| = \sup_{\|u'\|_p>0} \frac{\|u\|_p}{\|u'\|_p}.$$

It is obvious that $\|E_2\| = a_1(E_2) < \infty$ and also $\lim_{n\to\infty} a_n(E_2) = 0$.

We will also consider the unbounded embedding

$$E_3 : L^{1,p}(I) \to L^p(I).$$

Since $L^{1,p}(I)$ is defined by the pseudonorm $\|u'\|_{p,I}$ and E_3 is unbounded, we will study the best approximation of E_3 by linear maps of finite rank ($a_n(E_3)$ are not well defined).

Definition 2.1 *Let $J = (c, d) \subset I$. We define*

$$A_0(J) = \sup_{\|u'\|_{p,J}>0} \inf_{\alpha\in\Re} \frac{\|u - \alpha\|_{p,J}}{\|u'\|_{p,J}}.$$

Since every function in $W^{1,p}(J)$ is absolutely continuous, we can rewrite $A_0(J)$ as

$$A_0(J) = \sup_{\|u'\|_{p,J}>0} \inf_{\alpha\in\Re} \frac{\| \int_c^x u'(t)dt + u(c) - \alpha\|_{p,J}}{\|u'\|_{p,J}}.$$

From this we can see the connection between A_0 and the Hardy operator.

Lemma 2.2 *Let I_n be a decreasing sequence of subintervals of I with $|I_n| \to 0$ as $n \to \infty$. Then $\{A_0(I_n)\}$ is a decreasing sequence bounded above by $A_0(I)$ and with limit* 0.

Proof. In this proof we extend $u \in W^{1,p}(I_{n+1})$ outside I_{n+1} by a constant, i.e. $u' = 0$ outside I_{n+1}. From the definition of A_0 we have for $I_{i+1} \subset I_i$,

$$\begin{aligned} A_0^p(I_{i+1}) &= \sup_{\|u'\|_{p,I_{i+1}}>0} \inf_{\alpha\in\Re} \frac{\| \int_c^x u'(t)dt - \alpha\|^p_{p,I_{i+1}}}{\|u'\|^p_{p,I_{i+1}}} \\ &\le \sup_{\|u'\|_{p,I_{i+1}}>0} \inf_{\alpha\in\Re} \frac{\| \int_c^x u'(t)dt - \alpha\|^p_{p,I_i}}{\|u'\|^p_{p,I_i}} \\ &\le \sup_{\|u'\|_{p,I_i}>0} \inf_{\alpha\in\Re} \frac{\| \int_c^x u'(t)dt - \alpha\|^p_{p,I_i}}{\|u'\|^p_{p,I_i}} = A_0^p(I_i) \end{aligned}$$

and so $A_0(I_i) \geq A_0(I_{i+1})$. For $A_0(J)$ we have

$$\begin{aligned} A_0(J) &\leq \sup_{\|u'\|_{p,J}=1} \frac{\|\int_c^x u'(t)dt\|_{p,J}}{\|u'\|_{p,J}} \\ &= \sup_{\|u'\|_{p,J}=1} \left\| \int_c^x |u'(t)|dt \right\|_{p,J} \\ &\leq \sup_{\|u'\|_{p,J}=1} \left\| \left(\int_J |u'|^p \right)^{1/p} |J|^{1/p'} \right\|_{p,J} = |J|^{1/p'}. \end{aligned}$$

From this observation it follows that $A_0(I_n) \to 0$ as $I_n \to 0$. □

Lemma 2.3 *Let $J = (x, y) \subset I$. Then $A_0((x, y))$ is a continuous function of x and y.*

Proof. Let us suppose that there are $x, y \in I$ and $\varepsilon > 0$ such that $A_0(x, y + h_n) - A_0(x, y) > \varepsilon$ for some sequence $\{h_n\}$ with $0 < h_n \searrow 0$. Then we have that there is $\varepsilon_1 > 0$ such that $A_0^p(x, y + h_n) - A_0^p(x, y) > \varepsilon_1$ for any $n \in \mathbf{N}$. But for all $h > 0$,

$$\begin{aligned} A_0^p(x, y+h) - A_0^p(x, y) = &\sup_{\|u'\|_{p,(x,y+h)}>0} \inf_{\alpha \in \mathbf{R}} \frac{\| \int_x^t u' - \alpha\|^p_{p,(x,y+h)}}{\|u'\|^p_{p,(x,y+h)}} \\ &- \sup_{\|u'\|_{p,(x,y)}>0} \inf_{\alpha \in \mathbf{R}} \frac{\| \int_x^t u' - \alpha\|^p_{p,(x,y)}}{\|u'\|^p_{p,(x,y)}} \\ \leq &\sup_{\|u'\|_{p,(x,y+h)}>0} (\inf_{\alpha \in \mathbf{R}} \frac{\| \int_x^t u' - \alpha\|^p_{p,(x,y+h)}}{\|u'\|^p_{p,(x,y+h)}} \\ &- \inf_{\alpha \in \mathbf{R}} \frac{\| \int_x^t u' - \alpha\|^p_{p,(x,y)}}{\|u'\|^p_{p,(x,y+h)}}) \\ \leq &\sup_{\|u'\|_{p,(x,y+h)}>0} (\inf_{\alpha \in \mathbf{R}} \frac{\| \int_x^t u' - \alpha\|^p_{p,(x,y)}}{\|u'\|^p_{p,(x,y+h)}} \\ &+ \inf_{\alpha \in \mathbf{R}} \frac{\| \int_x^t u' - \alpha\|^p_{p,(y,h)}}{\|u'\|^p_{p,(x,y+h)}} - \inf_{\alpha \in \mathbf{R}} \frac{\| \int_x^t u' - \alpha\|^p_{p,(x,y)}}{\|u'\|^p_{p,(x,y+h)}}) \\ \leq &\sup_{\|u'\|_{p,(x,y+h)}>0} \inf_{\alpha \in \mathbf{R}} \frac{\| \int_x^t u' - \alpha\|^p_{p,(x,y)}}{|u'|^p_{p,(x,y+h)}} \\ \leq &\sup_{\|u'\|_{p,(y,y+h)}>0} \frac{\| \int_y^t u'\|^p_{p,(y,y+h)}}{\|u'\|^p_{p,(y,y+h)}} \\ \leq &|(y, y+h)|^{p/p'} \leq h^{p/p'}, \end{aligned}$$

and we have a contradiction. Hence $A_0(x, y+h) \to A_0(x, y)$ as $h \to 0$. Similarly we find that $A_0(x+h, y) \to A_0(x, y)$ as $h \to 0$ and the result follows. □

Lemma 2.4 *Let $J = (c, d) \subset I$. Then there is a function $f \in W^{1,p}(J)$ such that*

$$A_0(J) = \frac{\|f\|_{p,J}}{\|f'\|_{p,J}} = \inf_{\alpha \in \mathbf{R}} \frac{\|f - \alpha\|_{p,J}}{\|f'\|_{p,J}}.$$

Proof: It is possible to find a sequence $\{f_n\}_{n=1}^{\infty}$ of functions in $W^{1,p}(J)$ such that for each n in $\mathbf{N}$,

$$\frac{\|f_n\|_{p,J}}{\|f_n'\|_{p,J}} + 1/n = \inf_{\alpha \in \mathbf{R}} \frac{\|f_n - \alpha\|_{p,J}}{\|f_n'\|_{p,J}} + 1/n > A_0(J)$$

and $\|f_n\|_{W^{1,p}(J)} = 1$. Since E is compact, it follows that there exists a subsequence of $\{f_n\}$, again denoted by $\{f_n\}$ for convenience, which converges weakly in $W^{1,p}(J)$, to f, say, and this subsequence converges strongly to f in $L^p(J)$. By a standard compactness argument we get that f_n converges strongly to f in $W^{1,p}(J)$ and then

$$A_0(J) = \frac{\|f\|_{p,J}}{\|f'\|_{p,J}} = \inf_{\alpha \in \mathbf{R}} \frac{\|f - \alpha\|_{p,J}}{\|f'\|_{p,J}}.$$

□

Lemma 2.5 *Let $J = (c, d) \subset I$ and let f be as in the previous lemma. Then $f(x) = 0$ only for $x = (c+d)/2$, f is monotone and $f'(c_+) = f'(d_-) = 0$.*

Proof: Let f be from the previous lemma. Let $f_+(x) = \max\{f(x), 0\}$ and $f_-(x) = \max\{-f(x), 0\}$; then $\|f_+\|_{p,J}^p = \|f_-\|_{p,J}^p$, $f = f_+ - f_-$ and $|\{x : f(x) = 0\}| = 0$. Since we know that for any $g \in W^{1,p}(J)$, $g \geq 0$ we have $\|g'\|_{p,J} \geq \|(g^*)'\|_{p,(0,|J|)}$ (where g^* is the non-increasing rearrangament of the function g). Then we have that

$$\frac{\|f_+^*\|_{p,(0,|J|)}^p + \|f_-^*\|_{p,(0,|J|)}^p}{\|(f_+^*)'\|_{p,(0,|J|)}^p + \|(f_-^*)'\|_{p,(0,|J|)}^p} = A_0^p(J).$$

Now define $r = |\{x : f(x) > 0\} \cap J|$ and $g(x) = f_+^*(c+r-x)$ for $c \leq x \leq c+r$ and $g(x) = -f_-^*(c+r+x)$ for $c+r \leq x \leq d$. Then

$$\frac{\|g\|_{p,J}}{\|g'\|_{p,J}} = A_0(J),$$

and $\|g_+\|^p_{p,J} = \|g_-\|^p_{p,J}$.

From all this we can see that we have found a function g such that: g is monotone, $g(c+r) = 0$ where $c < c+r < d$ and $(\|g\|_{p,J}/\|g'\|_{p,J}) = A_0(J)$.

Now we show that $g((c+d)/2) = 0$ (i.e. $r = (c+d)/2$). Put $J_1 = (c, c+r)$ and $J_2 = (c+r, d)$; then we have

$$\frac{\|g\|^p_{p,J_1} + \|g\|^p_{p,J_2}}{\|g'\|^p_{p,J_1} + \|g'\|^p_{p,J_2}} = A_0^p(J). \tag{4}$$

Since $A_0(J) = |J| A_0((0,1))$, we see that

$$\frac{\|g\|^p_{p,J_1}}{\|g'\|^p_{p,J_1}} \leq A_0^p((0,1))|J_1|^p 2^p.$$

For if not then we can define $h(x) = g(x)$ on $(c, c+r)$ and $h(x) = -g(-x + 2(r+c))$ on $(c+r, c+2r)$ and we have that $\inf_{\alpha\in\Re} \|h-\alpha\|_{p,(c,c+2r)} = \|h\|_{p,(c,c+2r)}$ and

$$\frac{\|h\|^p_{p,(c,c+2r)}}{\|h'\|^p_{p,(c,c+2r)}} > A_0^p((c, c+2r)),$$

which is a contradiction with the definition of A_0. Similarly we have

$$\frac{\|g\|^p_{p,J_2}}{\|g'\|^p_{p,J_2}} \leq A_0^p((0,1))|J_2|^p 2^p.$$

Observe that (4) holds if and only if

$$\frac{\|g\|^p_{p,J_1}}{\|g'\|^p_{p,J_1}} = \frac{\|g\|^p_{p,J_2}}{\|g'\|^p_{p,J_2}} = A_0^p(J)$$

(do not forget that $\|g\|^p_{p,J_1} = \|g\|^p_{p,J_2}$). This means that $c+r = (c+d)/2$ and moreover we can suppose that $g(x) = -g(-x + (c+d))$ (i.e. $g(x)$ is odd with respect to $(c+d)/2$).

Next we show that $g'(c) = g'(d) = 0$. Note that $g(c) = -g(d) \geq 0$. Suppose that $g'(c) = -g'(d) < 0$; then there are a number $z > 0$ and a sequence of numbers $\{x_n\}_{n=1}^{\infty}$ such that $x_n > c$, $x_n \to c$ and

$$\frac{g(c) - g(x_n)}{c - x_n} < z < 0$$

(i.e. $\int_c^{x_n} g'(t)dt < (x_n - c)z$). A similar procedure can be carried out in the neighbourhood of d.

Then we have $|z|(x_n - c) < \int_c^{x_n} |g'(t)|dt \le (\int_c^{x_n} |g'(t)|^p dt)^{1/p}(x_n - c)^{1/p'}$. And also we have

$$A_0^p(J) = \frac{\int_{x_n}^d |g|^p + \int_c^{x_n} |g|^p}{\int_{x_n}^d |g'|^p + \int_c^{x_n} |g'|^p} \le \frac{\int_{x_n}^d |g|^p + (x_n - c)|g(c)|^p}{\int_{x_n}^d |g'|^p + (x_n - c)|z|^p}$$

Since $A_0(J) > 0$ and $|z| > 0$, plainly

$$|g(c)|^p < |z|^p A_0^p(J) + |g(c)|^p$$

and there exists $n_1 \in \mathbf{N}$ such that for any $n > n_1$ we have

$$(x_n - c)|g(c)|^p < (x_n - c)|z|^p \frac{\int_{x_n}^d |g|^p}{\int_{x_n}^d |g'|^p} + (x_n - c)|g(c) - z(x_n - c)|^p$$

and then

$$\begin{aligned}
(\int_{x_n}^{d} |g|^p)(\int_{x_n}^{d} |g'|^p) + (x_n - c)|g(c)|^p(\int_{x_n}^{d} |g'|^p) <& \\
& (\int_{x_n}^{d} |g|^p)(\int_{x_n}^{d} |g'|^p) + (x_n - c)|z|^p(\int_{x_n}^{d} |g|^p) \\
& + (x_n - c)|g(c) - z(x_n - c)|^p(\int_{x_n}^{d} |g'|^p) \\
& + |z|^p|g(c) - z(c - x_n)|^p(c - x_n)^2.
\end{aligned}$$

From this it follows that for any $n > n_1$,

$$\frac{\int_{x_n}^d |g|^p + (x_n - c)|g(c)|^p}{\int_{x_n}^d |g'|^p + (x_n - c)|z|^p} < \frac{\int_{x_n}^d |g|^p + (x_n - c)|g(x_n)|^p}{\int_{x_n}^d |g'|^p}.$$

But this means that for $l_n = \chi_{(x_n,d)}g + \chi_{(c,x_n)}g(x_n)$ we have:

$$A_0^p(J) < \frac{\int_c^d |l_n|^p}{\int_c^d |l_n'|^p} \qquad \text{for any } n > n_1.$$

In view of the antisymmetry of g we define a function $r_n(x) = \chi_{(c,d+c-x_n)} g(x) + \chi_{(d+c-x_n,d)} g(d+c-x_n)$, and have

$$A_0^p(J) < \frac{\int_c^d |r_n|^p}{\int_c^d |r_n'|^p} \qquad \text{for any } n > n_1.$$

Finally we define $k_n(x) = \chi_{(x_n,d+c-x_n)} g(x) + \chi_{(d+c-x_n,d)} g(d+c-x_n) + \chi_{(c,x_n)} g(x_n)$. Then for n large enough we have

$$A_0(J) < \inf_{c \in \mathbf{R}} \frac{\|k_n - c\|_{p,J}}{\|k_n'\|_{p,J}}.$$

But this contradicts the definition of $A_0(J)$: hence $g'(c) = g'(d) = 0$. □

Now we recall the p-Laplacian eigenvalue problem, which is defined, for $1 < p < \infty$, $\lambda > 0$ and $T > 0$ by

$$(|u'|^{p-2}u')' + \lambda |u|^{p-2}u = 0, \text{ on } (0,T),$$

$$u'(0) = 0, \qquad u'(T) = 0.$$

The set of eigenvalues of this problem is given by

$$\lambda_n(T) = \lambda_n := \left(\frac{2n\pi_p}{T}\right)^p \frac{1}{p'p^{p-1}} \text{ for each } n \in \mathbf{N}.$$

The corresponding eigenfunctions are $u_0(t) = c$, $c \in \mathbf{R} \setminus \{0\}$ and

$$u_n(t) = \frac{T}{n\pi_p} \sin_p \left(\frac{n\pi_p}{T}(t - \frac{T}{2n}) \right).$$

Here for $p > 1$ we put $p' = \frac{p}{p-1}$ and $\pi_p = 2B(\frac{1}{p}, \frac{1}{p'}) = \pi / \sin(\pi/p)$, where B denotes the beta function. Moreover $\sin_p(.)$ can be defined as the unique (global) solution to the initial–value problem

$$(|u'|^{p-2}u')' + \frac{2^p}{p'p^{p-1}} |u|^{p-2}u = 0$$

$$u(0) = 0, \qquad u'(0) = 1.$$

Also $\sin_p$ can be expressed in terms of hypergeometric functions, see ((AS), p.263),

$$\arcsin_p(s) = ps^{1/p}F(\frac{1}{p}, \frac{1}{p}, 1 + \frac{1}{p}; s),$$

or

$$\arcsin_p(s) = B(\frac{1}{p}, \frac{1}{p'}, (\frac{2s}{p})^p)$$

where $F(a, b, c; s)$ denotes the hypergeometric function and B is the incomplete beta function

$$B(1/q, 1/p', x) = \int_0^x z^{1/q-1}(1-z)^{-1/p}dz,$$

see (AS).

Moreover, for $s \in [0, p/2]$ we have

$$\arcsin_p(s) = \frac{p}{2}\int_0^{\frac{2s}{p}} \frac{dt}{(1-t^p)^{1/p}},$$

(note that this integral converges for all $s \in [0, p/2]$).

We note that in this paper we are using the definition of π_p and $\sin_p$ functions from the paper (DM) which is slightly different from the definition of π_p and the $\sin_p$ function used in (Li1) and (Li2). See also (PEM).

Note that as $\arcsin_p : [0, p/2] \to [0, \pi_p/2]$ is strictly increasing then its inverse function $\sin_p : [0, \pi_p/2] \to [0, p/2]$ is also strictly increasing.

We extended $\sin_p$ from $[0, \pi_p/2]$ to all $\mathbf{R}$ as a $2\pi_p$ periodic function by the usual way as in the $p = 2$ case (i.e. from sin).

For later use let us define $\cos_p(t) := \sin_p'(t)$. We have that

$$\left(\frac{p}{2}\right)^p |\cos_p(t)|^p + |\sin_p(t)|^p = 1 \text{ for all } t \in \mathbf{R},$$

and

$$\pi_p = \pi_{p'}.$$

From (DM) we have

$$\int_0^T |\sin_p(\frac{n\pi_p}{T}t)|^p dt = \frac{Tp'p^p}{2^p(p'+p)}$$

and

$$\int_0^T |\frac{d}{dt}\sin_p(\frac{n\pi_p}{T}t)|^p dt = \frac{n^p\pi_p^p p}{T^{p-1}(p'+p)}.$$

See (Li2) for more information about $\sin_p(.)$ and $\cos_p(.)$ functions.

Definition 2.6 *Given $J = [c,d] \subset \mathbf{R}$ we denote by $u_{n,J}(t)$ the n-th eigenfunction of the p-Laplacian eigenvalue problem on J and by $\lambda_{n,J}$ the corresponding n-th eigenvalue.*

Note that

$$u_{0,J} = C,$$

$$u_{n,J}(t) = \frac{|J|}{n\pi_p}\sin_p\left(\frac{n\pi_p}{|J|}(t - \frac{|J|}{2n} - c)\right), \qquad \text{for } n \geq 1$$

and

$$\lambda_{n,J} = \left(\frac{2n\pi_p}{|J|}\right)^p \frac{1}{p'p^{p-1}}, \qquad \text{for each } n \in \mathbf{N},$$

where $\pi_p = \pi/\sin(\pi/p)$. It is simple to see that for any $n \in \mathbf{N}$, $\{u_{i,J}\}_{i=1}^n$ is a linearly independent set.

Lemma 2.7 *Let $J = (c,d) \subset I$. Then*

$$A_0(J) = \frac{\|u_{1,J}\|_{p,J}}{\|u'_{1,J}\|_{p,J}} = \inf_{\alpha\in\mathbf{R}} \frac{\|u_{1,J}-\alpha\|_{p,J}}{\|u'_{1,J}\|_{p,J}} = \left(\frac{1}{\lambda_{1,J}}\right)^{1/p}.$$

Proof: We can see that

$$A_0(J) = \sup_{u\in K(J)} \frac{\|u_{1,J}\|_{p,J}}{\|u'_{1,J}\|_{p,J}}$$

where $K(J) = \{f; 0 < \|f'\|_{p,J} < \infty, \inf_\alpha \|f-\alpha\|_{p,J} = \|f\|_{p,J}\}$. After taking the Fréchet derivative of $A_0^p(J)$ we can see that this lemma follows from the previous observation about eigenfunction and eigenvalues for the p-Laplacian problem with Neumann boundary value conditions together with Lemma 4 (more can be found in (DKN)) $\square$

3 The Main Theorem.

Definition 3.1 *Let $\varepsilon > 0$ and $I = (a, b) \subset \mathbf{R}$. We define*

$$N(\varepsilon, I) = \inf\{n; I = \cup_{i=1}^{n} I_i, A_0(I_i) \leq \varepsilon, |I_i \cap I_j| = 0 \text{ for } i \neq j\}.$$

From our previous observation that $A_0(J) = \left(\frac{1}{\lambda_{1,J}}\right)^{1/p} = (p' p^{p-1})^{1/p} \frac{|J|}{2\pi_p}$ we have:

Observation 3.2 *i) Given any $\varepsilon > 0$ we have $N(\varepsilon, I) < \infty$.*

ii) Let $\varepsilon > 0$. Then there is a covering set of intervals (that is, a set of non-overlapping intervals)

$\{I_i\}_{i=1}^{N(\varepsilon)}$ such that $A_0(I_i) = \varepsilon$ for $i = 1, ..., N(\varepsilon)$ and $A_0(I_{N(\varepsilon,I)}) \leq \varepsilon$.

iii) For any $n \in \mathbf{N}$ there exist $\varepsilon > 0$, such that $n = N(\varepsilon, I)$ and corresponding covering sets $\{I_i\}_{i=1}^{N(\varepsilon,I)}$ for which $A_0(I_i) = \varepsilon$ for $i = 1, ...N(\varepsilon, I)$.

Moreover we can see:

Observation 3.3 *Let $n \in \mathbf{N}$ and $\varepsilon \in \left[\frac{|I|}{2n\pi_p}(p' p^{p-1})^{1/p}, \frac{|I|}{2(n-1)\pi_p}(p' p^{p-1})^{1/p}\right)$. Then $N(\varepsilon, I) = n$.*

From this observation we obtain the following two lemmas as in (EEH2).

Lemma 3.4 *Let $n \in \mathbf{N}$. Then*

$$a_n(E_1) \leq \frac{|I|}{2n\pi_p}(p' p^{p-1})^{1/p}$$

and

$$a_n(E_2) \leq \frac{|I|}{2n\pi_p}(p' p^{p-1})^{1/p}$$

and

$$\inf \|E_3 - P_{n+1}\| \le \frac{|I|}{2n\pi_p}(p'p^{p-1})^{1/p}$$

where the infimum is taken over all linear maps $P_{n+1} : L^{1,p}(I) \to L^p(I)$ *with* rank *less than* $n+1$.

Proof: Let $\{I_i\}_1^n$ be the partition from Observation 4 with $\varepsilon = \frac{|I|}{2n\pi_p}(p'p^{p-1})^{1/p}$. Set $Pf = \sum_{i=1}^n P_i f$ where

$$P_i f(x) := \chi_{I_i}(x)(f((a_i + b_i)/2)), \text{ where } I_i = (a_i, b_i).$$

We can see that $P_i f$ is a linear map from $L^{1,p}(I_i)$ into $L^p(I_i)$ (not necessarily bounded) and it is a bounded linear map from $L^{1,p}(I_i)/\{1\}$ into $L^p(I_i)$ with rank less or equal to 1. Then $\operatorname{rank} P \le n$ and P is a linear map from $L^{1,p}(I)$ into $L^p(I)$ and it is a linear map from $L^{1,p}(I)/\{1\}$ into $L^p(I)$. From (Li1) and Lemma 5 we have that $A_0(I_i) = \sup_{\|u'\|_{p,I_i}>0} \frac{\|u-P_i u\|_{p,I_i}}{\|u'\|_{p,I_i}}$. Then we have:

$$\begin{aligned}
\|(E_3 - P)f\|_{p,I}^p &= \sum_{i=1}^n \|(E_3 - P)f\|_{p,I_i}^p \\
&= \sum_{i=1}^n \|(f(.) - f((a_i + b_i)/2)\|_{p,I_i}^p \\
&\le \sum_{i=1}^n A_0^p(I_i)\|f'\|_{p,I_i}^p \\
&\le \sum_{i=1}^n \frac{|I|}{2n\pi_p}(p'p^{p-1})\|f'\|_{p,I_i}^p \\
&\le \frac{|I|}{2n\pi_p}(p'p^{p-1})\|f'\|_{p,I}^p.
\end{aligned}$$

(Note that $\|f - f((a+b)/2)\|_{p,I}/\|f'\|_{p,I} < \infty$ for any $f \in L^{1,p}(I)$.) From this follows the third inequality for E_3.

The proof of the inequality for E_2 is the same, just use that $\operatorname{rank} P \le n-1$ since the target space is $L^p(I)/\{1\}$.

For the first inequality for $a_n(E_1)$ we have to define a new partition of I. Let $\{I_i\}_1^n$ by the partition from Observation 4 with $\varepsilon = \frac{|I|}{2n\pi_p}(p'p^{p-1})^{1/p}$. Put $J_i = (a_i + |I_i|/2, b_i + |I_i|/2)$ for $i = 1, \ldots, n-1$ and $J_0 = (a, a + |I_1|/2)$, $J_n = (a_n + |I_n|/2, b)$ where $I_i = (a_i, b_i)$. Define $\{c_i\}_0^n$ and $\{d_i\}_0^n$ by $J_i = (c_i, d_i)$.

Set $Gf = \sum_{i=0}^n G_i f$ where $G_i f(x) := \chi_{J_i}(x)(f((c_i+d_i)/2))$, for $i = 1, \ldots, n-1$, $G_0 f(x) := f(a) = 0$ and $G_n f(x) := f(b) = 0$ where $I_i = (a_i, b_i)$. Then rank $G \le n_1$ and G is a bounded linear map from $W_0^{1,p}(I)$ into $L^p(I)$. Since $A_0^p(I_i) = A_0^p(J_j)$ then as before we have for $f \in W_0^{1,p}(I)$:

$$\begin{aligned}
\|(E_1 - G)f\|_{p,I}^p &= \sum_{i=0}^{n} \|(E-P)f\|_{p,J_i}^p \\
&= \sum_{i=1}^{n-1} \|(f(.) - f((a_i+b_i)/2)\|_{p,J_i}^p + \|f\|_{p,J_0}^p + \|f\|_{p,J_n}^p \\
&\le \sum_{i=1}^{n-1} A_0^p(J_i)\|f'\|_{p,J_i}^p + A_0^p(I_1)\|f'\|_{p,I_0}^p + A_0^p(I_n)\|f'\|_{p,J_n}^p \\
&\le \sum_{i=0}^{n} \frac{|I|}{2n\pi_p}(p'p^{p-1})\|f'\|_{p,I_i}^p \\
&\le \frac{|I|}{2n\pi_p}(p'p^{p-1})\|f'\|_{p,I}^p.
\end{aligned}$$

From this follows the first inequality for $a_n(E_1)$.

□

From the proof of Lemma 6 we can see that for any n there exists K_n, an n-dimensional linear subspace of L^p, such that for any $f \in L^{1,p}(I)/\{1\}$ (or from any $f \in L^{1,p}(I)$) we have

$$\inf_{g\in K_n} \|f-g\|_p^p \le \frac{|I|}{2n\pi_p}(p'p^{p-1})\|f'\|_p^p.$$

Moreover, for any n there exists R_{n-1}, an $n-1$ dimensional linear subspace of L^p, such that for any $f \in W_0^{1,p}(I)$ we have

$$\inf_{g\in R_{n-1}} \|f-g\|_p^p \le \frac{|I|}{2n\pi_p}(p'p^{p-1})\|f'\|_p^p.$$

Lemma 3.5 *Let* $n \in \mathbf{N}$. *Then*

$$a_n(E_1) \ge \frac{|I|}{2n\pi_p}(p'p^{p-1})^{1/p}$$

and

$$a_n(E_2) \ge \frac{|I|}{2n\pi_p}(p'p^{p-1})^{1/p}.$$

and

$$\inf \|E_3 - P_{n+1}\| \geq \frac{|I|}{2n\pi_p}(p'p^{p-1})^{1/p}$$

where the infimum is taken over all linear maps $P_{n+1} : L^{1,p}(I) \to L^p(I)$ *with* rank *less than* $n+1$.

Proof: First we prove the second inequality for E_2. Let $\{I_i\}_1^n$ be the partition from Observation 4 with $\varepsilon = \frac{|I|}{2n\pi_p}(p'p^{p-1})^{1/p}$.

From the definition of $A_0(I_i)$ we know that for $i = 1, ..., n$ there exists $\varphi_i \in W^{1,p}(I_i)$, $\|\varphi_i'\|_{p,I_i} = 1$ such that

$$\inf_{\alpha \in \mathbf{R}} \|\varphi_i - \alpha\|_{p,I_i} = A_0(I_i) = \varepsilon.$$

We extend each φ_i to I by taking $\varphi_i' = 0$ outside I_i and define $\phi_i = \varphi_i + c_i$ where $c_i \in \mathbf{R}$ is such that $\phi_i \in L^{1,p}/\{1\}$.

Let $P : L^{1,p}(I)/\{1\} \to L^p(I)/\{1\}$ be a bounded linear operator with rank$(P) < n$. Then there are constants $\lambda_1, ..., \lambda_n$, not all zero, such that

$$P\phi = 0, \qquad \phi = \sum_{i=1}^n \lambda_i \phi_i.$$

Note that $\phi \in L^p(I)/\{1\}$. Then, noting that the following summation is over $\lambda_i \neq 0$,

$$\begin{aligned}\|E_2\phi - P\phi\|_{p,I}^p = \|E_2\phi\|_{p,I}^p &= \sum_{i=1}^n \|\phi\|_{p,I_i}^p \\ &\geq \sum_{i=1}^n \inf_\alpha \|\phi - \alpha\|_{p,I_i}^p \geq \sum_{i=1}^n \inf_\alpha \|\phi_i - \alpha\|_{p,I_i}^p |\lambda_i|^p \\ &\geq \varepsilon^p \sum_{i=1}^n \|\phi_i'\|_{p,I_i}^p |\lambda_i|^p \geq \varepsilon^p \|\phi'\|_{p,I}^p.\end{aligned}$$

Then we have that $\|E_2 - P\|_{p,I} \geq \varepsilon$, so that $a_n(E_2) \geq \varepsilon$.

We prove the inequality for E_3 in the same way as for E_2. Let $P : L^{1,p}(I) \to L^p(I)$ be a linear operator with rank$(P) < n+1$. Let we have the system of functions $\{\phi_i\}_{i=1}^n$ considered previously and put $\phi_{n+1} = 1$; then we have $n+1$ linearly independent functions from $L^{1,p}(I)$ (note that $W^{1,p}(I)/\{1\} \subset L^{1,p}(I)$).

Then there are constants $\lambda_1, ..., \lambda_{n+1}$, not all zero, such that

$$P\phi = 0, \qquad \phi = \sum_{i=1}^{n+1} \lambda_i \phi_i.$$

Then, noting that the following summation is over $\lambda_i \neq 0$ we have

$$\begin{aligned}
\|E_3\phi - P\phi\|_{p,I}^p = \|E_3\phi\|_{p,I}^p &= \sum_{i=1}^{n+1} \|\phi\|_{p,I_i}^p \\
&\geq \sum_{i=1}^{n+1} \inf_{\alpha} \|\phi - \alpha\|_{p,I_i}^p \geq \sum_{i=1}^{n+1} \inf_{\alpha} \|\phi_i - \alpha\|_{p,I_i}^p |\lambda_i|^p \\
&\geq \varepsilon^p \sum_{i=1}^{n+1} \|\phi_i'\|_{p,I_i}^p |\lambda_i|^p \geq \varepsilon^p \|\phi'\|_{p,I}^p.
\end{aligned}$$

Hence $\|E_3 - P\|_{p,I} \geq \varepsilon$ and then the third inequality for E_3 is satisfied.

Now we prove the inequality for $a_n(E_1)$. Take $u_{n,I}$ the n-th eigenfunction of the $p-$Laplacian eigenvalue problem on I with Neumann boundary condition. Let $\{I_i\}_1^n$ be the partition from Observation 4 with $\varepsilon = \frac{|I|}{2n\pi_p}(p'p^{p-1})^{1/p}$. Then we define $\phi_i = u_{n,I}\chi_{I_i}$ and $\phi_i \in W_0^{1,p}(I_i)$ and $\|\phi_i\|_{p,I}/\|\phi_i'\|_{p,I} = A_0(I_i)$. Let $P : L^{1,p}(I) \to L^p(I)$ be a linear operator with rank$(P) < n$. Then there are constants $\lambda_1, ..., \lambda_n$, not all zero, such that

$$P\phi = 0, \qquad \phi = \sum_{i=1}^{n} \lambda_i \phi_i.$$

Noting that the following summation is over $\lambda_i \neq 0$ we have

$$\begin{aligned}
\|E_1\phi - P\phi\|_{p,I}^p = \|E_1\phi\|_{p,I}^p &= \sum_{i=1}^{n} \|\phi\|_{p,I_i}^p \\
&\geq \sum_{i=1}^{n} \|\phi\|_{p,I_i}^p \geq \sum_{i=1}^{n} \|\phi_i\|_{p,I_i}^p |\lambda_i|^p \\
&\geq \varepsilon^p \sum_{i=1}^{n+1} \|\phi_i'\|_{p,I_i}^p |\lambda_i|^p \geq \varepsilon^p \|\phi'\|_{p,I}^p.
\end{aligned}$$

Thus $\|E_1 - P\|_{p,I} \geq \varepsilon$ and so the third inequality for $a_n(E_1)$ is satisfied. □

The previous two lemmas give us:

Theorem 3.6 *If $|I| < \infty$, then*

$$a_n(E_1) = a_n(E_2) = \frac{|I|}{2n\pi_p}(p' p^{p-1})^{1/p} = \frac{1}{\lambda_n^{1/p}(I)}$$

and

$$\inf \|E_3 - P_{n+1}\| = \frac{|I|}{2n\pi_p}(p' p^{p-1})^{1/p} = \frac{1}{\lambda_n^{1/p}(I)}$$

where the infimum is taken over all linear maps $P_{n+1} : L^{1,p}(I) \to L^p(I)$ with rank less than $n + 1$.

Thus

$$\lim_{n\to\infty} a_n(E_1)n = \lim_{n\to\infty} a_n(E_2)n = \frac{|I|}{2\pi_p}(p' p^{p-1})^{1/p},$$

where $\pi_p = \pi/\sin(\pi/p)$.

Final note: After finishing the paper we learned about the paper (EHS) in which similar results were proved on trees. Also we would like to mention the paper (BS) in which the authors carefully studied problems related to Lemma 2.7.

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