

# A DIFFERENCE BETWEEN CONTINUOUS AND ABSOLUTELY CONTINUOUS NORMS IN BANACH FUNCTION SPACES

J. LANG AND A. NEKVINDA (PRAGUE)

ABSTRACT. On examples we show a difference between a continuous and absolutely continuous norm in Banach function spaces.

## 1. INTRODUCTION

This paper deals with a relation between the well-known term “absolutely continuous norm” in a Banach function space and “continuous norm” which was introduced by Q.Lai and L. Pick in [2]. Authors proved that the Hardy operator  $Tf(x) = \int_0^x f(t)dt$  is compact from a Banach function space  $(X, v)$  into  $L_\infty$  if and only if the function  $1/v$  has a continuous norm in the associate space  $(X', v)$ . In connection with this result there arose a problem whether the set of all functions with an absolutely continuous norm is equal to the set of all functions with a continuous norm in any Banach function space  $X$ .

In the third and fourth section an answer to this problem is given. In the first section a Banach function space is found in which there exists a function with a continuous and non-absolutely continuous norm. But in this space there is a function with a non-continuous norm. In the second section another space is found such that every function has a continuous norm and there is a function with a non-absolutely continuous norm.

## 2. PRELIMINARIES

This section has a preparatory character and is devoted to basic notation, definitions and assertions which are used in constructions of norms defining Banach function spaces. Let  $\Omega$  be a non-empty open subset of  $\mathbb{R}^m$  and let  $\mathcal{M}(\Omega)$  be the set of all measurable functions defined on  $\Omega$ . Denote by  $|E|$  the Lebesgue measure of any measurable subset  $E$  of  $\Omega$  and by  $\chi_E$  the characteristic function of  $E$ . Let the symbol  $|f|$  stand for the modulus of a function  $f, f \in \mathcal{M}(\Omega)$ .

**2.1. Definition.** *We say that a normed linear space  $(X, \|\cdot\|_X)$  is called a Banach function space if the following conditions are satisfied:*

- (2.1) *the norm  $\|f\|_X$  is defined for all  $f \in \mathcal{M}(\Omega)$  and  $f \in X$  if and only if  $\|f\|_X < \infty$ ;*
- (2.2)  *$\|f\|_X = \||f|\|_X$  for every  $f \in \mathcal{M}(\Omega)$ ;*
- (2.3) *if  $0 \leq f_n \nearrow f$  a.e. in  $\Omega$  then  $\|f_n\|_X \nearrow \|f\|_X$ ;*

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(2.4) if  $|E| < \infty, E \subset \Omega$ , then  $\chi_E \in X$ ;

(2.5) for every set  $E, |E| < \infty, E \subset \Omega$ , there exists

a positive constant  $C_E$  such that  $\int_E |f(x)| dx \leq C_E \|f\|_X$ .

Recall that the condition (2.3) immediately yields the following property:

(2.6) if  $0 \leq f \leq g$  then  $\|f\|_X \leq \|g\|_X$ .

To see this it suffices to set  $f_1 = f, f_n = g$  for  $n \geq 2$  in (2.3).

We will work in the following text only with bounded domains and therefore we will assume this property automatically.

**2.2. Definition.** Let  $X = (X(\Omega), \|\cdot\|)$  be a Banach function space and let  $f \in X$  be an arbitrary function. We say that the function  $f$  has an absolutely continuous norm in  $X$  if and only if for any sequence of open sets  $G_n, G_1 \supset G_2 \supset G_3 \dots, \bigcap_{n=1}^{\infty} G_n = \emptyset$ , the norms  $\|f\chi_{G_n}\|$  tend to zero for  $n \rightarrow \infty$ . Denote the set of all functions with an absolutely continuous norm by  $X_a$ .

**2.3. Definition.** Let the same assumptions as in Definition 2.2 be satisfied. Let  $B(x, \varepsilon)$  be a ball with centre  $x$  and radius  $\varepsilon$ . We say that  $f$  has a continuous norm in  $X$  if and only if  $\lim_{\varepsilon \rightarrow 0+} \|f\chi_{B(x, \varepsilon) \cap \Omega}\| = 0$  for every  $x \in \overline{\Omega}$ , where  $\overline{\Omega}$  stands for the closure of the domain  $\Omega$ . Denote the set of all functions with a continuous norm by  $X_c$ .

We will keep in the sequel a special notation  $u$  for the unit function, i.e. the symbol  $u$  will always be the function defined by  $u(x) = 1$  for all  $x \in \Omega$ .

**2.4. Remark.** It is clear that it suffices to show that the function  $u$  belongs to  $X$  and the imbedding  $X \hookrightarrow L_1(\Omega)$  holds in order to verify the conditions (2.4) and (2.5).

**2.5. Definition.** We say that a normed linear space  $(X, \|\cdot\|_X)$  is called a weak Banach function space if and only if the conditions (2.1), (2.2) and (2.3) are fulfilled.

For the sake of simplicity we shall write ‘‘BFS’’ instead of ‘‘Banach function space’’ and ‘‘WBFS’’ instead of ‘‘Weak Banach function space’’.

In what follows we shall prove some assertions which make it possible to construct norms in Banach function spaces.

**2.6. Lemma.** Let  $\Omega_a, a \in I$ , be a system of non-empty open subsets of  $\Omega$  such that  $\Omega = \bigcup_{a \in I} \Omega_a \cup M$ , where  $|M| = 0$ . Let  $X_a = (X_a(\Omega_a), \|\cdot\|_a)$  be a system of WBFS defined on  $\Omega_a$ . Define a space  $X(\Omega)$  as the set of all functions  $f$  with a finite norm  $\|f\| = \sup_{a \in I} \|f_a\|_a$ , where  $f_a$  denotes the restriction of  $f$  onto the set  $\Omega_a$ . Then the space  $X = (X(\Omega), \|\cdot\|)$  is a WBFS.

*Proof.* Clearly, the expression  $\|f\|$  is defined for every  $f \in \mathcal{M}(\Omega)$  and defines a norm. The properties (2.1) and (2.2) are evident. To prove (2.3) assume  $0 \leq f_n \nearrow f$  a.e. in  $\Omega$ . It is not difficult to verify the inequality  $\|f_n\| \leq \|f\|$ . Suppose that

$$(2.7) \quad \|f_n\| = \sup_{a \in I} \|(f_n)_a\|_a \leq K < \sup_{a \in I} \|f_a\|_a = \|f\| \quad \text{for all } n.$$

Since  $\sup_{a \in I} \|f_a\|_a > K$  there exists  $b, b \in I$ , such that  $\|f_b\|_b > K$ . Now,  $X_b$  is a WBFS and using (2.3) it follows that there is a positive integer  $n_0$  such that  $\|(f_{n_0})_b\|_b > K$ , which gives a contradiction with (2.7). The lemma is proved.  $\square$

**2.7. Lemma.** Let  $\Omega_n$  be a sequence of non-empty open subsets of  $\Omega$  and  $\Omega = \bigcup_{n=1}^{\infty} \Omega_n \cup M$ , where  $|M| = 0$ . Let  $X_n = (X_n(\Omega_n), \|\cdot\|_n)$  be a corresponding system of WBFS. Define a norm  $\|f\| = \sum_{n=1}^{\infty} \|f_n\|_n$  for  $f_n$  being the restriction of  $f$  onto  $\Omega_n$ . Then the space  $X = (X(\Omega), \|\cdot\|)$  is a WBFS.

*Proof.* The conditions (2.1) and (2.2) are again trivial. To prove (2.3) it suffices to use the following well-known property of the space  $l_1$ . Let  $a^n = \{a_k^n\}_{k=1}^{\infty}, n = 1, 2, \dots$ , be a sequence of elements of  $l_1$ ,  $a_k^n \geq 0$  and  $a_k^n \nearrow a_k$  for  $n \rightarrow \infty$ . Denote by  $a$  the sequence  $\{a_k\}_{k=1}^{\infty}$ . Then  $\|a^k\|_{l_1} \nearrow \|a\|_{l_1}$ . Note that  $\|a\|_{l_1}$  can be equal to infinity. Now, let  $0 \leq f_k \nearrow f$  a.e. in  $\Omega$ . It is again easy to see that  $\|f_k\| \leq \|f\|$ . To complete the proof it suffices to set  $a_n^k = \|(f_k)_n\|_n$  and  $a_n = \|f_n\|_n$ .  $\square$

**2.8. Lemma.** Let  $(X(\Omega), \|\cdot\|_X)$  be a WBFS such that  $u \in X$ . Define a norm by  $\|f\|_Y = \|f\|_X + \int_{\Omega} |f|$ . Then the space  $(Y(\Omega), \|\cdot\|_Y)$  is a BFS.

*Proof.* The proof follows immediately from Remark 2.4.  $\square$

### 3. THE FIRST CONSTRUCTION

In this part we give a construction of a norm defining a BFS  $X$  such that  $X_a \subsetneq X_c \subsetneq X$ . We will consider in this section  $\Omega = (0, 1)$ . To construct this norm we will use the idea of the construction of the Cantor set which we will denote by  $\mathfrak{C}$ . Let  $I = (1/3, 2/3)$ ,  $I_0 = (1/3^2, 2/3^2)$ ,  $I_1 = (7/3^2, 8/3^2)$ ,  $I_{00} = (1/3^3, 2/3^3)$ ,  $I_{01} = (7/3^3, 8/3^3)$ ,  $I_{10} = (19/3^3, 20/3^3)$ ,  $I_{11} = (25/3^3, 26/3^3)$  and so on. Denote further the ‘‘complementary’’ intervals  $J = [0, 1]$ ,  $J_0 = [0, 1/3]$ ,  $J_1 = [2/3, 1]$ ,  $J_{00} = [0, 1/3^2]$ ,  $J_{01} = [2/3^2, 3/3^2]$ ,  $J_{10} = [6/3^2, 7/3^2]$ ,  $J_{11} = [8/3^2, 1]$  and so on.

Let the symbol  $\mathcal{K}$  stand for the set of all finite sequences containing only the numbers 0 and 1, including the empty sequence. We will call the elements of  $\mathcal{K}$  multiindices. For a given multiindex  $\alpha, \alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathcal{K}$  we introduce the length of  $\alpha$  (write  $|\alpha|$ ) as the number of all members of the sequence  $\alpha$ , i.e.  $|\alpha| = n$ . Let us define a partial ordering on  $\mathcal{K}$ . We say that  $\alpha \preceq \beta$  for  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k), \beta = (\beta_1, \beta_2, \dots, \beta_n)$  if and only if  $|\alpha| \leq |\beta|$ , i.e.  $k \leq n$ , and  $\alpha_i = \beta_i$  for  $i = 1, 2, \dots, k$ . The reader will notice that this partial ordering on  $\mathcal{K}$  is in fact the lexicographic ordering defined on the tree  $(\mathcal{K}, \preceq)$ , which is well-known for instance in the set theory. In the following we will use the notation  $\alpha \not\preceq \beta$  if  $\alpha \preceq \beta$  does not hold and  $\alpha \not\preceq \beta$  if  $\beta \not\preceq \alpha$ . The symbol  $\alpha \approx \beta$  is used in the case  $\alpha \not\preceq \beta$  and  $\alpha \not\preceq \beta$ , i.e. if there is no relation between  $\alpha$  and  $\beta$ . Recall that  $\alpha = \beta$  if and only if  $\alpha \preceq \beta$  and  $\alpha \succeq \beta$ , and  $\alpha \neq \beta$  if  $\alpha = \beta$  is not satisfied.

It is not difficult to see the following properties:

- (3.1) for  $|\alpha| = n$  we have  $|I_{\alpha}| = 1/3^{n+1}$  and  $|J_{\alpha}| = 1/3^n$ ;
- (3.2)  $\alpha = \beta$  if and only if  $I_{\alpha} = I_{\beta}$  and  $\alpha \neq \beta$  if and only if  $I_{\alpha} \cap I_{\beta} = \emptyset$ ;
- (3.3)  $\alpha \preceq \beta$  if and only if  $J_{\beta} \subset J_{\alpha}$  and  $\alpha \approx \beta$  if and only if  $J_{\alpha} \cap J_{\beta} = \emptyset$ ;
- (3.4)  $\alpha \preceq \beta$  if and only if  $I_{\beta} \subset J_{\alpha}$  and  $\alpha \not\preceq \beta$  if and only if  $I_{\beta} \cap J_{\alpha} = \emptyset$ ;
- (3.5) for a given  $\alpha \in \mathcal{K}$  and an integer  $k, k > |\alpha|$ , the number of all multiindices  $\beta$  such that  $\beta \succeq \alpha$  and  $|\beta| = k$  is equal to  $2^{k-|\alpha|}$ ;
- (3.6) for every  $x \in \mathfrak{C}$  there exists a unique sequence  $\{\alpha_n\}_{n=0}^{\infty}$  such that  $\alpha_0 \preceq \alpha_1 \preceq \alpha_2 \preceq \alpha_3 \preceq \dots, |\alpha_n| = n$  and  $\{x\} = \bigcap_{n=0}^{\infty} J_{\alpha_n}$ .

Let us now introduce a norm. Put

$$\|f\|_X = \sup_{n \geq 0} \frac{3^{n+1}}{2^n} \sum_{|\alpha|=n} \int_{I_{\alpha}} |f(x)| dx + \int_0^1 |f(x)| dx.$$

Denote by  $\|f\|$  the first summand in  $\|f\|_X$ , i.e.  $\|f\|_X = \|f\| + \int_0^1 |f|$ . First we shall prove that the norm  $\|f\|_X$  defines a *BFS*.

**3.1. Lemma.** *The space  $X = (X(0,1), \|\cdot\|_X)$  is a *BFS*.*

*Proof.* Obviously, Lemma 2.6 and Lemma 2.7 guarantee that  $\|f\|$  defines a *WBFS*. According to Lemma 2.8 it suffices to show that  $\|u\| < \infty$ . Recall that  $u$  is the unit function on  $(0,1)$ . Using (3.1) and (3.5) we obtain

$$\|u\| = \sup_{n \geq 0} \frac{3^{n+1}}{2^n} \sum_{|\alpha|=n} \frac{1}{3^{|\alpha|+1}} = 1 < \infty$$

, which completes the proof.  $\square$

**3.2. Theorem.** *The function  $u$  does not have an absolutely continuous norm in  $X$ .*

*Proof.* Take a sequence of open sets  $G_N = \cup_{|\alpha| \geq N} I_\alpha$ . Obviously,  $G_N$  is a decreasing sequence of sets and the intersection of all  $G_N$  is empty. Let us calculate  $\|u\chi_{G_N}\|_X$ . The statements (3.1) and (3.5) give

$$\|u\chi_{G_N}\| = \sup_{k \geq N} \frac{3^{k+1}}{2^k} \sum_{|\alpha|=k} |I_\alpha| + |G_N| \geq \sup_{k \geq N} \frac{1}{2^k} \sum_{|\alpha|=k} 1 = 1$$

for every  $N$ , and the proof is complete.  $\square$

**3.3. Lemma.** *Let  $\alpha_0 \preceq \alpha_1 \preceq \alpha_2 \preceq \dots$  be a sequence of multiindices,  $|\alpha_N| = N$ . Then*

$$\lim_{N \rightarrow \infty} \|u\chi_{J_{\alpha_N}}\|_X = 0.$$

*Proof.* Evidently, it suffices to estimate only the norms  $\|u\chi_{J_{\alpha_N}}\|$ . According to (3.4) we have

$$J_{\alpha_N} = \cup_{\beta \succeq \alpha_N} \overline{I_\beta} = \cup_{k=|\alpha_N|}^{\infty} \cup_{\substack{|\beta|=k \\ \beta \succeq \alpha_N}} \overline{I_\beta}.$$

Now, (3.1) and (3.5) immediately yield

$$\|u\chi_{J_{\alpha_N}}\| = \sup_{k \geq N} \frac{3^{k+1}}{2^k} \sum_{\substack{\beta \succeq \alpha_N \\ |\beta|=k}} |I_\beta| = \sup_{k \geq N} \frac{3^{k+1}}{2^k} \frac{2^{k-N}}{3^{k+1}} = \frac{1}{2^N}.$$

The last expression tends to zero for  $N$  increasing to infinity, which proves the lemma.  $\square$

**3.4. Theorem.** *The function  $u$  has a continuous norm in  $X$ .*

*Proof.* Fix  $x \in [0,1]$ . We have two cases: either  $x \in \cup_{|\alpha|=0}^{\infty} I_\alpha$  or  $x \in \mathfrak{C}$ . Let us investigate the first case. Since  $I_\alpha, \alpha \in \mathcal{K}$ , are open sets, there exists a multiindex  $\beta$  and a real number  $\epsilon_0 > 0$

such that  $(x - \epsilon_0, x + \epsilon_0) \subset I_\beta$ . Obviously, it suffices to estimate  $\|u\chi_{(x-\epsilon, x+\epsilon)}\|$  only for  $\epsilon < \epsilon_0$ . By an elementary calculation we obtain

$$\|u\chi_{(x-\epsilon, x+\epsilon)}\|_X = \frac{3^{|\beta|+1}}{2^{|\beta|}} \int_{x-\epsilon}^{x+\epsilon} |u| + \int_{x-\epsilon}^{x+\epsilon} |u|$$

which tends to zero for  $\epsilon \rightarrow 0_+$  due to absolute continuity of the Lebesgue integral.

Let us consider the case  $x \in \mathfrak{C}$ . According to (3.6) there exists a unique sequence  $J_{\alpha_N}$ ,  $|J_{\alpha_N}| = 1/3^N$ , such that  $\bigcap_{N=0}^{\infty} J_{\alpha_N} = \{x\}$ . Let us write  $\mathfrak{C} = \mathfrak{C}_1 \cup \mathfrak{C}_2 \cup \mathfrak{C}_3$ , where  $\mathfrak{C}_1$  is defined as the set of all  $x \in \mathfrak{C}$  such that  $x$  belongs for every  $N$  to the interior of  $J_{\alpha_N}$ , the set  $\mathfrak{C}_2$  contains all boundary points of all intervals  $I_\alpha$ ,  $\alpha \in \mathcal{K}$ , and  $\mathfrak{C}_3$  contains only 0 and 1.

Take  $x \in \mathfrak{C}_2$ . Without loss of generality we can assume that  $x$  is a left boundary point of an interval  $I_\beta$  (if  $x$  was a right boundary point the proof would be analogous). Clearly,

$$\|u\chi_{(x-\epsilon, x+\epsilon)}\|_X \leq \|\chi_{(x-\epsilon, x)}\| + \|\chi_{(x, x+\epsilon)}\|.$$

Lemma 3.3 guarantees that  $\|\chi_{(x-\epsilon, x)}\|$  tends to zero for  $\epsilon \rightarrow 0_+$ . The estimate of the second summand is analogous to the previous case and follows again from absolute continuity of the Lebesgue integral.

Finally, the use of Lemma 3.3 immediately completes the proof of this theorem for  $x \in \mathfrak{C}_1 \cup \mathfrak{C}_3$ .  $\square$

We promised at the beginning of this section to construct a space  $X$  such that  $X_a \neq X_c \neq X$ . To satisfy this we must construct a function  $g$  such that  $g$  does not have a continuous norm. It is not difficult, as the following example shows.

**3.5. Theorem.** *There exists a function  $g$  such that  $g \notin X_c$ .*

*Proof.* Take a sequence of multiindices  $\alpha_0 = \emptyset, \alpha_1 = 0, \alpha_2 = 00, \alpha_3 = 000, \dots$ . Define a function  $g(x) = 2^N$  for  $x \in I_{\alpha_N}$  and  $g(x) = 0$  otherwise. It is easy to verify that  $g \in X$  and  $\|g\chi_{(0, 1/3^N)}\| \geq 1$  for all  $N$ . This implies the assertion of the theorem.  $\square$

#### 4. THE SECOND CONSTRUCTION

In this section we shall construct a BFS  $Y$  such that  $\{0\} \subsetneq Y_a \subsetneq Y_c = Y$ , i.e. every function in  $Y$  has a continuous norm and there is a function with a non-absolutely continuous norm.

A certain very simple idea how to construct this space  $Y$  would be based on the result of the previous section. It would be possible to take the space  $X$  and define  $Y = X_c$ . The following three assertions show that this idea cannot be used.

Remark that we shall suppose in these three assertions that  $\Omega$  is a non-empty open bounded subset of  $\mathbb{R}^m$ .

**4.1 Lemma.** *Let  $X$  be a BFS. Then  $X_a$  is a closed subspace of  $X$ .*

*Proof.* The proof can be found in Theorem 3.8. in [1].  $\square$

**4.2 Lemma.** *Let  $X$  be a BFS. Then  $X_c$  is a closed subspace of  $X$ .*

*Proof.* Let  $f_n \in X_c$  and  $f_n \rightarrow f$  in the topology of the space  $X$ . Let us estimate  $\|f\chi_{B(x,\epsilon)\cap\Omega}\|$ . Using the triangle inequality and (2.6) we obtain for any  $n$

$$\|f\chi_{B(x,\epsilon)\cap\Omega}\| \leq \|(f - f_n)\chi_{B(x,\epsilon)\cap\Omega}\| + \|f_n\chi_{B(x,\epsilon)\cap\Omega}\| \leq \|f - f_n\| + \|f_n\chi_{B(x,\epsilon)\cap\Omega}\|.$$

Now, given an  $\eta > 0$  we can find  $n_0$  such that  $\|f - f_{n_0}\| < \eta/2$ . Since  $f_{n_0} \in X_c$  we have for sufficiently small  $\epsilon$  the estimate  $\|f_{n_0}\chi_{B(x,\epsilon)\cap\Omega}\| < \eta/2$ , which completes the proof.  $\square$

Remark that Lemmas 4.1 and 4.2 imply that  $X_a$  is a closed subspace of  $X_c$ .

**4.3. Theorem.** *Let  $X$  be a BFS and let  $Y$  be a closed subspace of  $X$  such that  $Y \neq X$ . Then there exists no norm which would turn the space  $Y$  to a BFS.*

*Proof.* Assume that there is such a norm. We will write  $X = (X, \|\cdot\|_X)$  and  $Y = (Y, \|\cdot\|_Y)$ . Theorem 1.8 of Chapter I in [1] implies that  $Y$  is imbedded into  $X$ , i.e. that there exists a positive real constant  $c_1$  such that

$$\|f\|_X \leq c_1 \|f\|_Y$$

for all  $f \in Y$ . Consider the identical mapping from  $(Y, \|\cdot\|_Y)$  onto  $(Y, \|\cdot\|_X)$ . Since  $I$  is a continuous one-to-one linear mapping, the inverse mapping is continuous, too. This yields the existence of a positive real constant  $c_2$  such that

$$\|f\|_Y \leq c_2 \|f\|_X$$

for all  $f \in Y$ . This implies the equivalence of the norms  $\|\cdot\|_X$  and  $\|\cdot\|_Y$  on  $Y$ .

Now, take a function  $f \in X \setminus Y$ , i.e.  $\|f\|_X < \infty$  and  $\|f\|_Y = \infty$ . The property (2.2) gives  $|f| \in X \setminus Y$ . The measure theory yields the existence of a sequence of simple functions  $f_n$  such that  $0 \leq f_n \nearrow |f|$ . Recall that any simple function  $f$  can be expressed as a finite sum of characteristic functions of sets  $E_1, E_2, \dots, E_n$  such that  $|E_i| < \infty$  and  $E_i \cap E_j = \emptyset$  for  $i \neq j$ . Using this fact and (2.4) we know that  $f_n \in Y$  for all  $n$  which together with (2.2) and (2.3) implies  $\|f_n\|_X \nearrow \|f\|_X$  and  $\|f_n\|_Y \nearrow \|f\|_Y$ .

Thus  $\|f\|_X = \infty$  and  $\|f\|_Y < \infty$ , which is a contradiction with the equivalence of norms and the proof is complete.  $\square$

It is seen from this theorem that we must construct the space  $Y$  in another way. We shall define a norm again on the interval  $(0, 1)$  and keep the notation of  $I_\alpha$  and  $J_\alpha$ ,  $\alpha \in \mathcal{K}$ , from the previous section. For any  $f \in \mathcal{M}(0, 1)$  let us define a norm

$$\|f\| = \sum_{k=0}^{\infty} \max_{|\alpha|=k} \sup_{n \geq k} \frac{3^{n+1}}{2^n} \sum_{\substack{|\beta|=n \\ \beta \succeq \alpha}} \int_{I_\beta} |f(x)| dx$$

and the norm in a space  $Y$  by

$$\|f\|_Y = \|f\| + \int_0^1 |f(x)| dx.$$

**4.4. Lemma.** *The space  $Y = (Y, \|\cdot\|_Y)$  is a BFS.*

*Proof.* Lemmas 2.6 and 2.7 guarantee that the norm  $\|\cdot\|$  defines a WBS. According to Lemma 2.8 it suffices to show  $\|u\| < \infty$ . Recall again that  $u$  denotes the unit function. The statements (3.1) and (3.5) give

$$\begin{aligned} \|u\| &= \sum_{k=0}^{\infty} \max_{|\alpha|=k} \sup_{n \geq k} \frac{3^{n+1}}{2^n} \sum_{\substack{|\beta|=n \\ \beta \succeq \alpha}} \frac{1}{3^{|\beta|+1}} \\ &= \sum_{k=0}^{\infty} \max_{|\alpha|=k} \sup_{n \geq k} \frac{1}{2^{|\alpha|}} = \sum_{k=0}^{\infty} \frac{1}{2^k} < \infty, \end{aligned}$$

which completes the proof.  $\square$

**4.5. Theorem.** *The function  $u$  does not belong to  $Y_a$ .*

*Proof.* Define  $G_N = \cup_{|\alpha| \geq N} I_\alpha$  as in the proof of Theorem 3.2. Recall that we can write  $G_N$  in the following way:

$$G_N = \cup_{k=N}^{\infty} \cup_{|\alpha|=k} I_\alpha.$$

Fix  $N$  and estimate  $\|u\chi_{G_N}\|_Y$ . Omitting the  $L_1$ -norm and taking from the norm  $\|u\chi_{G_N}\|$  only the first summand of the whole series we obtain

$$\begin{aligned} \|u\chi_{G_N}\|_Y &\geq \sup_{n \geq 0} \frac{3^{n+1}}{2^n} \sum_{|\beta|=n} |I_\beta \cap (\cup_{k=N}^{\infty} \cup_{|\alpha|=k} I_\alpha)| \\ &\geq \frac{3^{N+1}}{2^N} \sum_{|\beta|=N} |I_\beta| = 1 \end{aligned}$$

for every  $N$ , which completes the proof.  $\square$

Now, we shall prove that every function from  $Y$  has a continuous norm in  $Y$ , i.e.  $Y_c = Y$ . In the proof the following lemma plays a key role.

**4.6 Lemma.** *Let  $\|f\| < \infty$  and let  $\{\alpha_N\}_{N=1}^{\infty}$  be an increasing sequence of multiindices,  $|\alpha_N| = N$ . Then*

$$\lim_{N \rightarrow \infty} \|f\chi_{J_{\alpha_N}}\| = 0.$$

*Proof.* Assume that the assertion of this lemma does not hold. Then (2.6) and (3.3) imply that  $\|f\chi_{J_{\alpha_N}}\|$  is non-increasing and there exists a positive real number  $c$  such that for every  $N$  the inequality

$$(4.1) \quad \|f\chi_{J_{\alpha_N}}\| \geq c$$

holds. Now, fix  $N$  and calculate  $\|f\chi_{J_{\alpha_N}}\|$ . According to the definition of  $\|f\|$  we have

$$\begin{aligned}
(4.2) \quad \|f\chi_{J_{\alpha_N}}\| &= \sum_{k=0}^{\infty} \max_{|\alpha|=k} \sup_{n \geq k} \frac{3^{n+1}}{2^n} \sum_{\substack{|\beta|=n \\ \beta \succeq \alpha}} \int_{I_\beta \cap J_{\alpha_N}} |f| \\
&= \sum_{k=0}^{N-1} \max_{|\alpha|=k} \sup_{n \geq k} \frac{3^{n+1}}{2^n} \sum_{\substack{|\beta|=n \\ \beta \succeq \alpha}} \int_{I_\beta \cap J_{\alpha_N}} |f| \\
&\quad + \sum_{k=N}^{\infty} \max_{|\alpha|=k} \sup_{n \geq k} \frac{3^{n+1}}{2^n} \sum_{\substack{|\beta|=n \\ \beta \succeq \alpha}} \int_{I_\beta \cap J_{\alpha_N}} |f| \\
&= A_1(N) + A_2(N).
\end{aligned}$$

Let us first estimate  $A_1(N)$ . Let  $k, 0 \leq k \leq N-1$ , be a fixed integer. Suppose for a moment that  $|\alpha| = k, \alpha \approx \alpha_k$ . Since  $\beta \succeq \alpha$  and  $\alpha_N \succeq \alpha_k$  then  $\beta \approx \alpha_N$  and according to (3.4) we have  $I_\beta \cap J_{\alpha_N} = \emptyset$ . This yields

$$\max_{\substack{|\alpha|=k \\ \alpha \approx \alpha_k}} \sup_{n \geq k} \frac{3^{n+1}}{2^n} \sum_{\substack{|\beta|=n \\ \beta \succeq \alpha}} \int_{I_\beta \cap J_{\alpha_N}} |f| = 0$$

and we can rewrite  $A_1(N)$  in the form

$$A_1(N) = \sum_{k=0}^{N-1} \sup_{n \geq k} \frac{3^{n+1}}{2^n} \sum_{\substack{|\beta|=n \\ \beta \succeq \alpha_k}} \int_{I_\beta \cap J_{\alpha_N}} |f|.$$

Fix  $n$ . If  $k \leq n \leq N-1$  then the inequalities  $|\beta| \leq N-1 < N = |\alpha_N|$  imply  $\beta \not\succeq \alpha_N$  and the statement (3.4) gives  $I_\beta \cap J_{\alpha_N} = \emptyset$ . Then we can write

$$A_1(N) = \sum_{k=0}^{N-1} \sup_{n \geq N} \frac{3^{n+1}}{2^n} \sum_{\substack{|\beta|=n \\ \beta \succeq \alpha_k}} \int_{I_\beta \cap J_{\alpha_N}} |f|.$$

Let  $n \geq N$ . Let  $|\beta| = n$  such that  $\alpha_N \not\succeq \beta$ . Then the property (3.4) again yields  $I_\beta \cap J_{\alpha_N} = \emptyset$  and it follows that the last term can be non-zero if and only if  $\beta \succeq \alpha_N$ . But in this case the property (3.4) gives  $I_\beta \cap J_{\alpha_N} = I_\beta$ . Now, we can finally write

$$\begin{aligned}
(4.3) \quad A_1(N) &= \sum_{k=0}^{N-1} \sup_{n \geq N} \frac{3^{n+1}}{2^n} \sum_{\substack{|\beta|=n \\ \beta \succeq \alpha_N}} \int_{I_\beta} |f| \\
&= N \sup_{n \geq N} \frac{3^{n+1}}{2^n} \sum_{\substack{|\beta|=n \\ \beta \succeq \alpha_N}} \int_{I_\beta} |f|.
\end{aligned}$$

Let us estimate  $A_2(N)$ . Increasing the integration domain from  $I_\beta \cap J_{\alpha_N}$  to  $I_\beta$  we obtain

$$A_2(N) \leq \sum_{k=N}^{\infty} \left( \max_{|\alpha|=k} \sup_{n \geq k} \frac{3^{n+1}}{2^n} \sum_{\substack{|\beta|=n \\ \beta \succeq \alpha}} \int_{I_\beta} |f| \right).$$

Since the last expression is the rest of the series

$$\|f\| = \sum_{k=0}^{\infty} \left( \max_{|\alpha|=k} \sup_{n \geq k} \frac{3^{n+1}}{2^n} \sum_{\substack{|\beta|=n \\ \beta \succeq \alpha}} \int_{I_\beta} |f| \right)$$

and this series is convergent according to the assumption,  $A_2(N)$  tends to zero for  $N$  increasing to infinity. This together with (4.1) and (4.2) guarantees the existence of  $N_0$  such that  $A_1(N) \geq c/2$  for  $N \geq N_0$ . The equality (4.3) gives for  $N \geq N_0$  an estimate

$$\sup_{n \geq N} \frac{3^{n+1}}{2^n} \sum_{\substack{|\beta|=n \\ \beta \succeq \alpha_N}} \int_{I_\beta} |f| \geq \frac{c}{2} \frac{1}{N}.$$

The use of the last inequality and the fact that  $|\alpha_N| = N$  immediately yield

$$\begin{aligned} \|f\| &= \sum_{k=0}^{\infty} \left( \max_{|\alpha|=k} \sup_{n \geq k} \frac{3^{n+1}}{2^n} \sum_{\substack{|\beta|=n \\ \beta \succeq \alpha}} \int_{I_\beta} |f| \right) \\ &\geq \sum_{k=N_0}^{\infty} \left( \sup_{n \geq k} \frac{3^{n+1}}{2^n} \sum_{\substack{|\beta|=n \\ \beta \succeq \alpha_k}} \int_{I_\beta} |f| \right) \\ &\geq \frac{c}{2} \sum_{k=N_0}^{\infty} \frac{1}{k} = \infty, \end{aligned}$$

which is a contradiction with the assumption, and the proof is complete.  $\square$

**4.7. Theorem.** *Every function from  $Y$  has a continuous norm in  $Y$ .*

*Proof.* Fix  $x \in [0, 1]$ . It is clear that either  $x \in \cup_{|\alpha|=0}^{\infty} I_\alpha$  or  $x \in \mathfrak{C}$ .

Let us investigate the first case, i.e., there exists a multiindex  $\gamma$  such that  $x \in I_\gamma$ . Since  $I_\gamma$  is open there is a positive real number  $\epsilon_0$  such that for all  $\epsilon, 0 < \epsilon < \epsilon_0$ , we have  $I_\epsilon = (x - \epsilon, x + \epsilon) \subset I_\gamma$ . Let us calculate the norm  $\|f\chi_{I_\epsilon}\|_Y$ . By virtue of absolute continuity of the Lebesgue integral it suffices to prove that  $A_\epsilon = \|f\chi_{I_\epsilon}\| \rightarrow 0$  for  $\epsilon \rightarrow 0_+$ .

Clearly,

$$(4.4) \quad A_\epsilon = \sum_{k=0}^{\infty} \max_{|\alpha|=k} \sup_{n \geq k} \frac{3^{n+1}}{2^n} \sum_{\substack{|\beta|=n \\ \beta \succeq \alpha}} \int_{I_\beta \cap I_\epsilon} |f|.$$

Let us now fix  $k$ . If  $k > |\gamma|$  then  $\beta \neq \gamma$  and by virtue of (3.2) we have  $I_\beta \cap I_\gamma = \emptyset$ . Since  $I_\epsilon \subset I_\gamma$  we immediately obtain  $I_\beta \cap I_\epsilon = \emptyset$ . Let us consider the case  $0 \leq k \leq |\gamma|$ . Let us fix  $\alpha$ ,  $|\alpha| = k$ . Provided  $\alpha \approx \gamma$  and due to  $\beta \succeq \alpha$  we have  $\beta \approx \gamma$  and consequently,  $I_\beta \cap I_\epsilon = \emptyset$ . We have just shown that the maximum in (4.4) is attained for  $\alpha$  such that  $|\alpha| = k$ ,  $\alpha \preceq \gamma$ . Such an  $\alpha$  is determined by  $k$  definitely and this enables us to write  $\alpha = \alpha(k)$ . Thus, it is possible to rewrite the norm  $\|f\chi_{I_\epsilon}\|$  in the following way:

$$A_\epsilon = \sum_{k=0}^{|\gamma|} \sup_{n \geq k} \frac{3^{n+1}}{2^n} \sum_{\substack{|\beta|=n \\ \beta \succeq \alpha(k)}} \int_{I_\beta \cap I_\epsilon} |f|$$

where  $\alpha(k)$  satisfies  $\alpha(k) \preceq \gamma$  and  $|\alpha(k)| = k$ .

Obviously, if  $n \neq |\gamma|$ , then  $\beta \neq \gamma$  and (3.2) gives  $I_\beta \cap I_\epsilon = \emptyset$ . Thus, the supremum is attained for  $n = |\gamma|$ , that is,

$$A_\epsilon = \sum_{k=0}^{|\gamma|} \frac{3^{|\gamma|+1}}{2^{|\gamma|}} \sum_{\substack{|\beta|=|\gamma| \\ \beta \succeq \alpha(k)}} \int_{I_\beta \cap I_\epsilon} |f|.$$

The conditions  $|\beta| = |\gamma|$ ,  $\beta \succeq \alpha(k)$  and  $\gamma \succeq \alpha(k)$  entail  $\beta = \gamma$  and as a consequence we obtain  $I_\epsilon \subset I_\beta$ , which yields

$$A_\epsilon = \sum_{k=0}^{|\gamma|} \frac{3^{|\gamma|+1}}{2^{|\gamma|}} \int_{I_\epsilon} |f| = (|\gamma| + 1) \frac{3^{|\gamma|+1}}{2^{|\gamma|}} \int_{I_\epsilon} |f|.$$

The absolute continuity of the Lebesgue integral guarantees  $A_\epsilon \rightarrow 0$  for  $\epsilon \rightarrow 0_+$ .

Let us consider the case  $x \in \mathfrak{C}$ . It is clear from absolute continuity of the Lebesgue integral that we can omit the second summand in the norm  $\|f\|_Y$  and to investigate only the member  $\|f\|$ .

The estimate of  $\|f\chi_{I_\epsilon}\|$  follows immediately from Lemma 4.6.  $\square$

We have constructed the space  $Y$  and proved that  $Y_a \subsetneq Y_c = Y$ . To show  $\{0\} \subsetneq Y_a$  it suffices to take the characteristic function of the interval  $I = (1/3, 2/3)$ .

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J. LANG, INSTITUTE OF MATHEMATICS, CZECH ACAD. SCI., ŽITNÁ 25, 115 67 PRAGUE 1, CZECH REPUBLIC

*E-mail address:* lang@karlin.mff.cuni.cz

A. NEKVINDA, FACULTY OF CIVIL ENGINEERING, CZECH TECHNICAL UNIVERSITY, THÁKUROVA 7, 166 29 PRAHA 6, CZECH REPUBLIC