THE HARDY OPERATOR AND THE GAP BETWEEN L^∞ AND BMO

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ABSTRACT. We study boundedness and compactness properties of the Hardy integral operator $Tf(x) = \int_A^x f$ from a weighted Banach function space $X(v)$ into L_∞ and BMO. We give a new simple characterization of compactness of T from $X(v)$ into BMO. We construct examples of spaces $X(v)$ such that $T(X(v))$ is (a) bounded in L^∞ but not compact in BMO; (b) compact in BMO but not bounded in L^∞; (c) bounded in BMO but neither bounded in L^∞ nor compact in BMO; (d) bounded in L^∞, compact in BMO and yet not compact in L^∞. In order to obtain the last of the counterexamples we construct a new weighted Banach function space.

1. INTRODUCTION

Let $-\infty \le A < B \le \infty$ and let T be the Hardy operator

$$Tf(x) = \int_A^x f(t)\,\mathrm{d}t, \quad x \in (A,B),$$

defined for functions f on (A,B) for which the integral exists.

The spaces L^∞ and BMO are defined by

$$L^\infty = \big\{f;\ \|f\|_\infty = \operatorname*{ess\,sup}_{x\in(A,B)} |f(x)| < \infty\big\},$$

and

$$\text{BMO } = \Big\{f;\ \|f\|_{\text{BMO}} = \sup_{A<a<b<B} \frac{1}{b-a}\int_a^b |f(x) - f_{(a,b)}|\,\mathrm{d}x < \infty\Big\},$$

respectively, where

$$f_{(a,b)} = \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x.$$

Obviously, $L^\infty \subset$ BMO. The gap between L^∞ and BMO is of interest in harmonic analysis and has been studied by various authors (see for example [**4**], [**3**],

1991 *Mathematics Subject Classification.* 47G10, 46E30, 26D10.

The research of the second author was supported by the grant No. 201/94/1066 of the Grant Agency of the Czech Republic

[8], [1] and others). BMO has been used in many ways as a convenient substitute for L^∞, in particular as an alternative endpoint in interpolation of operators.

Let $X(v)$ be a weighted Banach function space (see the definitions in Section 2 below). Properties of the Hardy operator acting from weighted Banach function spaces into BMO are of special interest since the Hardy operator controls various important integral operators which are useful in harmonic analysis and PDE's. Such properties were recently investigated by Q. Lai and the second author in [5]. The main objective of the present paper is to study relations between the following four statements:

(1.1) T is bounded from $X(v)$ into L^∞;

(1.2) T is bounded from $X(v)$ into BMO;

(1.3) T is compact from $X(v)$ into L^∞;

(1.4) T is compact from $X(v)$ into BMO.

Since $L^\infty \subset$ BMO and compactness implies boundedness, the following diagram is trivially true.

1.1. Diagram.

(1.2); (1.4); (1.3); (1.1) ;

Our aim is to investigate whether or not other relations between the statements (1.1)–(1.4) are valid. More precisely, we are interested in the following problem.

1.2. Problem. *Is any of the following four statements true for every weighted Banach function space $X(v)$?*

(1.5) (1.4) $\Rightarrow$ (1.1);

(1.6) (1.1) $\Rightarrow$ (1.4);

(1.7) (1.2) $\Rightarrow$ ((1.1) *or* (1.4));

(1.8) ((1.1)&(1.4)) $\Rightarrow$ (1.3).

We shall construct a counterexample to each of (1.5)–(1.8). The results can be summarized as follows.

1.3. Theorem. (i) *There exists a weighted Banach function space $X(v)$ such that* (1.4) *is true and* (1.1) *is false.*

(ii) *There exists a weighted Banach function space $X(v)$ such that* (1.1) *is true and* (1.4) *is false.*

(iii) *There exists a weighted Banach function space $X(v)$ such that* (1.2) *is true and both* (1.1) *and* (1.4) *are false.*

(iv) *There exists a weighted Banach function space $X(v)$ such that both* (1.1) *and* (1.4) *are true and yet* (1.3) *is false.*

1.4. Corollary. *No other implications between the statements* (1.1)–(1.4) *than those illustrated on Diagram 1.1 (and their trivial consequences) are true. In particular, none of the four implications in Diagram 1.1 is reversible.*

Proof. Suppose, say, that (1.1) implies (1.3). Then (1.1) would imply (1.4), which contradicts Theorem 1.3 (ii). And so on. □

Theorem 1.3 provides some interesting new facts concerning the gap between L^∞ and BMO. For example, (i) shows that the image under the Hardy operator of a structure as rich as a weighted Banach function space can be compact in BMO without even being bounded in L^∞. In fact, we shall prove in Section 4 that every Lebesgue space L^p, $1 < p < \infty$, is such a space when accompanied with an appropriate weight. In (ii), whose proof is given in Section 5, we point out that the image under T of a weighted Banach function space can be bounded in L^∞ without being compact in BMO. We show that an appropriate weighted Lorentz space has this property. In Section 6 we prove (iii) by showing that the T–image of L^p, $1 < p < \infty$, equipped with the power weight x^{p-1} near zero, is bounded in BMO but is neither bounded in L^∞ nor compact in BMO. Finally, (iv) asserts that there is a weighted Banach function space whose T–image is bounded in L^∞, compact in BMO, but not compact in L^∞. In order to give such an example we construct a new Banach function space

$$Z = \{f \text{ defined on } (0,1); \|f\|_Z = \sup_{0<\alpha\le 1} \int_0^1 |f(x)|\alpha x^{\alpha-1}\mathrm{d}x < \infty\},$$

and show that its associate space Z', accompanied with the weight $v \equiv 1$ (that is, non-weighted), has the properties required. Together with some basic facts about Z, this is done in Section 7.

A key to the proof of Theorem 1.3 is an appropriate characterization of each of the statements (1.1)–(1.4). In [**5**, Theorems 1,3,5,6] such a characterization was given in terms of norm continuity of certain functions. Whereas the necessary and sufficient conditions given in [**5**] for (1.1)–(1.3) are quite satisfactory, the characterization of (1.4) ([**5**, Theorem 6]) is too complicated. In Section 3 below we prove a new characterization of compactness of T into BMO (Theorem 3.3), which is simpler than the one from [**5**] and also better fitting in the mould of the other three. This criterion is then frequently used in subsequent sections to prove that (1.4) is either true or false, as needed.

2. Preliminaries

Let v be a *weight* on (A,B) (almost everywhere positive and finite function). We shall write $v(E)$ for $\int_E v$ and χ_E for the characteristic function of a set E. We shall write $a \lesssim b$ if there is a constant c such that $a \le cb$. If both $a \lesssim b$ and $b \lesssim a$ hold, we write $a \approx b$.

We say that a normed linear space $X(v)$ is a *weighted Banach function space* if it satisfies the following axioms (cf. [**7**], [**1**]):

(P1) the Banach function norm $\|\cdot\|_{X(v)}$ is defined for all measurable f on [A,B], and $f \in X(v)$ if and only if $\|f\|_{X(v)} < \infty$;

(P2) $0 \le f \le g$ a.e. implies $\|f\|_{X(v)} \le \|g\|_{X(v)}$;

(P3) $0 \le f_n \nearrow f$ a.e. implies $\|f_n\|_{X(v)} \nearrow \|f\|_{X(v)}$;

(P4) $v(E) < \infty$ implies $\chi_E \in X(v)$;

(P5) $v(E) < \infty$ implies $\left|\int_E fv\right| \le C_E \|f\|_{X(v)}$ for some positive constant C_E independent of f.

Let

$$f_v^*(t) = \inf\left\{\lambda > 0;\, v(\{x, |f(x)| > \lambda\}) \le t\right\}$$

be a *non-increasing rearrangement* of a function f with respect to the measure $v(x)\,\mathrm{d}x$.

We say that a weighted Banach function space $X(v)$ is *rearrangement–invariant* if $\|f\|_{X(v)} = \|g\|_{X(v)}$ whenever $f_v^* = g_v^*$.

The set

$$X'(v) = \Big\{f;\, \Big|\int_A^B fgv\Big| < \infty \text{ for every } g \in X(v)\Big\},$$

equipped with the norm

$$\|f\|_{X'(v)} = \sup\Big\{\Big|\int_A^B fgv\Big|;\, \|g\|_{X(v)} \le 1\Big\},$$

is called the *associate space* of $X(v)$. It is known ([**7**], [**1**]) that $X''(v) = X(v)$ and that $X'(v)$ is again a weighted Banach function space (with respect to the same weight).

We shall write $E_n \searrow \emptyset$ if E_n is a sequence of measurable subsets of $[A, B]$ such that $E_{n+1} \subset E_n$ and $\bigcap E_n = \emptyset$.

For $x \in [A, B]$ (note that we allow $x = \pm\infty$) and $\varepsilon > 0$ we define

$$\mathcal{U}(x, \varepsilon) = \begin{cases} (x - \varepsilon, x + \varepsilon) \cap [A, B] & \text{if} \quad -\infty < x < \infty \\ (\frac{1}{\varepsilon}, \infty] \cap [A, B] & \text{if} \quad x = \infty \\ [-\infty, -\frac{1}{\varepsilon}) \cap [A, B] & \text{if} \quad x = -\infty, \end{cases}$$

and the one–sided versions

$$\mathcal{U}^+(x, \varepsilon) = \mathcal{U}(x, \varepsilon) \cap [x, B], \qquad \mathcal{U}^-(x, \varepsilon) = \mathcal{U}(x, \varepsilon) \cap [A, x].$$

2.1. Definition. We say that $f \in X(v)$ has *absolutely continuous norm* in $X(v)$ if $\|f\chi_{E_n}\|_{X(v)} \to 0$ whenever $E_n \searrow \emptyset$. The subspace of $X(v)$ containing such functions is denoted $X_a(v)$. If $X(v) = X_a(v)$ we say that $X(v)$ has absolutely continuous norm.

We say that $f \in X(v)$ has *continuous norm* in $X(v)$ at $x \in [A, B]$ if $\|f\chi_{\mathcal{U}(x,\varepsilon)}\|_{X(v)} \to 0$ for $\varepsilon \to 0_+$. We say that a function f has continuous norm in $X(v)$ if it has continuous norm at every point $x \in [A, B]$. The subset of $X(v)$ containing such functions is denoted $X_c(v)$.

We say that functions from a family $\mathcal{F} \subset X(v)$ have *uniformly continuous norms* in $X(v)$ if for every $\delta > 0$ and $x \in [A, B]$ there is an $\varepsilon > 0$ such that for every $f \in \mathcal{F}$ $\|f\chi_{\mathcal{U}(x,\varepsilon)}\|_{X(v)} < \delta$. □

2.2. Remark. Absolute continuity of norm is a classical notion in theory of Banach function spaces ([**7**], [**1**]). As far as we know, pointwise continuity of norm was first introduced in [**5**] and proved a crucial tool to balance compactness theorems (the attempts to give necessary conditions in terms of absolute continuity of norm have failed). For a similar purpose it was used in [**2**]. Obviously, $X_a(v) \subset X_c(v) \subset X(v)$. The first author and A. Nekvinda ([**6**]) have recently constructed an example of a weighted Banach function space $X(v)$ for which both these inclusions are proper. □

Given a weight v and an interval $I = [a, b] \subset (A, B)$, we define the functions (cf. [**5**])

$$G_I(x) = \chi_I(x) \frac{(b-x)(x-a)}{(b-a)^2} \frac{1}{v(x)}.$$

Observe that for every I and x

$$\sup_I G_I(x) \le \frac{1}{4v(x)}. \tag{2.1}$$

The functions G_I were introduced in [**5**], and the following representation formula for the BMO norm of Tf was proved:

$$\frac{1}{2}\|Tf\|_{\mathrm{BMO}} \le \sup_{I \subset (A,B)} \int_A^B G_I f v \le \|Tf\|_{\mathrm{BMO}}, \qquad f \ge 0. \tag{2.2}$$

In [**5**], the statements (1.1)–(1.4) were characterized in the following way.

2.3. Theorem. (i) *The operator T is bounded from $X(v)$ into L^∞ if and only if $\frac{1}{v} \in X'(v)$.*

(ii) *The operator T is bounded from $X(v)$ into BMO if and only if the set $\{G_I\}_{I \subset (A,B)}$ is bounded in $X'(v)$.*

(iii) *The operator T is compact from $X(v)$ into L^∞ if and only if $\frac{1}{v} \in X'_c(v)$.*

(iv) *The operator T is compact from $X(v)$ into BMO if and only if the set $\{G_I\}_{I \subset (A,B)}$ is bounded in $X'(v)$, $\frac{\chi_I}{v} \in X'_c(v)$ for every $I \subset (0,\infty)$, and for every sequence of intervals $J_n = (\alpha_n, \beta_n)$ such that either $\beta_n < \alpha_{n+1}$ for all n or $\beta_{n+1} < \alpha_n$ for all n we have*

$$\lim_{n\to\infty} \sup_{I \subset (A,B)} \|G_I \chi_{J_n}\|_{X'(v)} = 0. \qquad \square$$

2.4. Examples. (i) *Weighted Lebesgue spaces*, defined by

$$L^p(v) = \Big\{ f;\ \|f\|_{p(v)} = \Big(\int_A^B |f(x)|^p v(x)\,\mathrm{d}x \Big)^{1/p} < \infty \Big\}, \quad 1 \le p < \infty,$$

are Banach function spaces, and so is L^∞. BMO is not a Banach function space.

Let p' be defined by $\frac{1}{p'} + \frac{1}{p} = 1$. Then we have

$$\big(L^p(v)\big)' = L^{p'}(v), \qquad 1 \le p \le \infty.$$

For $1 \le p < \infty$ we have

$$L^p_a(v) = L^p_c(v) = L^p(v).$$

On the other hand,

$$L_a^\infty = L_c^\infty = \{0\}.$$

(ii) Define *weighted Lorentz spaces* by

$$L^{p,q}(v) = \{f;\ \|f\|_{p,q(v)} < \infty\},$$

where

$$\|f\|_{p,q(v)} = \|t^{\frac{1}{p}-\frac{1}{q}} f_v^*(t)\|_q, \quad 1 \le p, q \le \infty.$$

Then (cf. [**1**]), $\|f\|_{p,q(v)}$ is equivalent to a norm with respect to which $L^{p,q}(v)$ is a weighted Banach function space if and only if either $1 < p < \infty$ or $p = q = 1$ or $p = q = \infty$. We have

$$\big(L^{p,q}(v)\big)' = L^{p',q'}(v).$$

Let further $q < \infty$. Then

$$L_a^{p,q}(v) = L_c^{p,q}(v) = L^{p,q}(v). \qquad \square$$

For more examples of $X(v)$ such that $\{0\} \subsetneqq X_a(v) \subsetneqq X(v)$ (Orlicz spaces etc.) we refer the reader to [**7**].

3. A new characterization of (1.4)

By (2.1), if $x \in [A, B]$ and

$$\lim_{\varepsilon \to 0+} \left\| \frac{\chi_{\mathcal{U}(x,\varepsilon)}}{v} \right\|_{X'(v)} = 0, \tag{3.1}$$

then also

$$\lim_{\varepsilon \to 0+} \sup_I \|\chi_{\mathcal{U}(x,\varepsilon)} G_I\|_{X'(v)} = 0. \tag{3.2}$$

As we shall see (Theorem 4.1 below), the converse is not true in general. Instead, the following implication holds: If (3.2) is true and (3.1) is false, then necessarily x is one of the endpoints A, B. This fact, stated in the next lemma, is interesting on its own, but also will be used in the proof of the sufficiency part of the main theorem of this section. Moreover, it gives us a hint where to look for counterexamples.

3.1. Lemma. *Let $A < x < B$. Then* (3.1) *is equivalent to* (3.2).

Proof. Assume that (3.2) holds. For $\varepsilon > 0$ define $C = x - 2\varepsilon$, $D = x + 2\varepsilon$, and $I_\varepsilon = (C, D)$. Take ε small enough that $A < C < D < B$. Observe that $(D - C)^2 \le 16(D - t)(t - C)$ for $t \in \mathcal{U}(x, \varepsilon)$. Hence

$$\begin{aligned}\left\| \frac{\chi_{\mathcal{U}(x,\varepsilon)}}{v} \right\|_{X'(v)} &= \left\| \chi_{\mathcal{U}(x,\varepsilon)}(t) \frac{(D-t)(t-C)}{(D-C)^2 v(t)} \frac{(D-C)^2}{(D-t)(t-C)} \right\|_{X'(v)} \\ &\le 16 \|\chi_{\mathcal{U}(x,\varepsilon)} G_{I_\varepsilon}\|_{X'(v)},\end{aligned}$$

and (3.1) follows. (Of course a better constant, namely 4, can be obtained, but that is beside the point.)

As mentioned above, the converse implication follows from (2.1). The proof is complete. $\square$

3.2. Lemma. *The following statements are equivalent.*

(i) $\{G_I\}$ *do not have uniformly continuous norms in* $X'(v)$.

(ii) *There is an* $x \in [A,B]$, $\delta > 0$, *a sequence of disjoint closed intervals* $[\alpha_n,\beta_n] \subset (A,B)$ *such that either* $\alpha_n \searrow x$ *or* $\beta_n \nearrow x$, *and a sequence* $\{G_{I_n}\} \subset \{G_I\}$ *such that*

$$\|G_{I_n}\chi_{(\alpha_n,\beta_n)}\|_{X'(v)} \geq \delta, \quad n \in \mathbb{N}. \tag{3.3}$$

Proof. (ii) $\Rightarrow$ (i) is obvious.

Assume (i). Then there is an $x \in [A,B]$, $\delta > 0$, and intervals $\{I_\varepsilon\}_{\varepsilon>0}$ such that for $G_\varepsilon := G_{I_\varepsilon}$ we have

$$\|G_\varepsilon\chi_{\mathcal{U}(x,\varepsilon)}\|_{X'(v)} \geq \delta > 0 \qquad \text{for every} \quad \varepsilon > 0.$$

Hence either $\|G_\varepsilon\chi_{\mathcal{U}^+(x,\varepsilon)}\|_{X'(v)} \geq \frac{\delta}{2}$ for every $\varepsilon > 0$ or $\|G_\varepsilon\chi_{\mathcal{U}^-(x,\varepsilon)}\|_{X'(v)} \geq \frac{\delta}{2}$ for every $\varepsilon > 0$. Assume the former; the latter would be handled in the same way (in particular, we assume $x < B$). By the sharpness of Hölder's inequality, there are functions $\{f_\varepsilon\}_{\varepsilon>0}$ in the unit ball of $X(v)$, supported on $\mathcal{U}^+(x,\varepsilon)$, and such that $\int f_\varepsilon G_\varepsilon v \geq \frac{\delta}{3}$. Take arbitrary $\beta_1 \in (x,B)$ and define ε_1 by $[x,\beta_1) = \mathcal{U}^+(x,\varepsilon_1)$. By absolute continuity of integral, there is an $\alpha_1 \in (x,\beta_1)$ such that $\int_{\alpha_1}^{\beta_1} f_{\varepsilon_1}G_{\varepsilon_1}v \geq \frac{\delta}{4}$. If α_{n-1} is fixed, take arbitrary $\beta_n \in (x,\alpha_{n-1})$ and repeat the process. We obtain a sequence of disjoint intervals $[\alpha_n,\beta_n]$, $\alpha_n \searrow x$, such that $\int_{\alpha_n}^{\beta_n} f_{\varepsilon_n}G_{\varepsilon_n}v \geq \frac{\delta}{4}$, where ε_n is defined by $[x,\beta_n) = \mathcal{U}^+(x,\varepsilon_n)$. Since $\|f_{\varepsilon_n}\|_{X(v)} \leq 1$, Hölder's inequality yields (3.3) (with δ replaced by $\frac{\delta}{4}$). The proof is complete. □

3.3. Theorem. *The operator* T *is compact from* $X(v)$ *into BMO if and only if the functions* $\{G_I\}_{I\subset(A,B)}$ *have uniformly continuous norms in* $X'(v)$.

Proof. Sufficiency. Let $A < r < R < B$. We write

$$T = T_r + P + T_R,$$

where $T_rf = T(\chi_{[A,r)}f)$, $T_Rf = T(\chi_{(R,B]}f)$, and $Pf = T(\chi_{[r,R]}f)$. Then by (2.2) (using also that $G_Iv \geq 0$)

$$\begin{aligned} \|T_r\|_{X(v)\to\,\mathrm{BMO}} &\lesssim \sup_I \sup\Big\{\int_A^B f\chi_{[A,r)}G_Iv;\ f\geq 0,\ \|f\|_{X(v)}\leq 1\Big\} \\ &= \sup_I \|G_I\chi_{[A,r)}\|_{X'(v)}, \end{aligned}$$

and analogously

$$\|T_R\|_{X(v)\to\mathrm{BMO}} \lesssim \sup_I \|G_I\chi_{(R,B]}\|_{X'(v)}.$$

Let $\varepsilon > 0$. Then there exist r, R sufficiently close to A, B, respectively, that $\|T_r\| + \|T_R\| < \varepsilon$. This follows from the uniform continuity of norms of $\{G_I\}$ at the endpoints A, B.

By assumption, G_I have uniformly continuous norms in $X'(v)$, whence, by Lemma 3.1, $\frac{\chi_{[r,R]}}{v} \in X'_c(v)$ for any fixed r, R such that $A < r < R < B$. Therefore, using Theorem 2.3 (iii) and observing that $Pf = 0$ on $(0,r)$ and Pf is constant on

$(R, B]$, we see that P is a compact operator from $X(v)$ into L^∞, and the more so it is compact from $X(v)$ into BMO. Thus, T is a limit of compact operators in the operator norm and therefore it is itself compact.

Necessity. Assume that $\{G_I\}$ do not have uniformly continuous norms in $X'(v)$. By Lemma 3.2, there is an $x \in [A, B]$, a $\delta > 0$, a sequence of disjoint closed intervals $[\alpha_n, \beta_n] \subset (A, B)$ such that $\alpha_n \searrow x$, say (the case $\beta_n \nearrow x$ would be handled analogously), and a sequence $\{G_{I_n}\} \subset \{G_I\}$ such that (3.3) holds. Hence there are non-negative functions $\{f_n\}$ supported in $[\alpha_n, \beta_n]$, such that $\|f_n\|_{X(v)} \le 1$, and

$$\int_{\alpha_n}^{\beta_n} f_n G_{I_n} v \ge \frac{\delta}{2}, \qquad n \in \mathbb{N}. \tag{3.4}$$

Assume first that $x = A$. Let $I_n = [a_n, b_n]$. Then we choose a subsequence in the following way: Put $n(1) = 1$. For $n(k-1)$ fixed, take $n(k) > n(k-1)$ so that $\beta_{n(k)} < a_{n(k-1)}$. Then for every $k < s$ we have $\operatorname{supp} f_s \cap I_{n(k)} = \emptyset$, and by (3.4) and (2.2),

$$\frac{\delta}{2} \le \int_{\alpha_{n(k)}}^{\beta_{n(k)}} f_{n(k)} G_{I_{n(k)}} v = \int_A^B G_{I_{n(k)}} (f_{n(k)} - f_{n(s)}) v \le \|T f_{n(k)} - T f_{n(s)}\|_{\mathrm{BMO}},$$

whence T is not compact.

Now assume that $x > A$. Then the above method is useless as there is no way to guarantee that $(\alpha_{n(k)}, \beta_{n(k)})$ do not meet $I_{n(s)}$ for $k < s$. Instead, we use the technique from the proof of Lemma 3.1.

Take $n(1)$ so that $2x - \beta_{n(1)} > A$. Assuming that $n(k-1)$ is fixed, take $n(k) > n(k-1)$ so that

$$2\beta_{n(k)} - x < \alpha_{n(k-1)}.$$

We define

$$C_k = 2\alpha_{n(k)} - \beta_{n(k)}, \qquad D_k = 2\beta_{n(k)} - \alpha_{n(k)}, \qquad k \in \mathbb{N}.$$

Then we have for every k

$$[C_k, D_k] \subset (A, \alpha_{n(k-1)}) \tag{3.5}$$

and

$$D_k - \beta_{n(k)} = \beta_{n(k)} - \alpha_{n(k)} = \alpha_{n(k)} - C_k.$$

In particular,

$$\frac{(D_k - C_k)^2}{(D_k - t)(t - C_k)} \le 9, \qquad t \in (\alpha_{n(k)}, \beta_{n(k)}). \tag{3.6}$$

It follows from (3.5) that for $k, s \in \mathbb{N}$, $k < s$,

$$(C_s, D_s) \cap \operatorname{supp} f_{n(k)} = \emptyset, \tag{3.7}$$

Thus, for $k < s$ we have by (3.4), (2.1), (3.6), (3.7), and (2.2)

$$\frac{\delta}{2} \le \int_{\alpha_{n(s)}}^{\beta_{n(s)}} f_{n(s)} G_{I_{n(s)}} v \le \frac{1}{4} \int_{\alpha_{n(s)}}^{\beta_{n(s)}} f_{n(s)} \le \frac{9}{4} \int_{\alpha_{n(s)}}^{\beta_{n(s)}} f_{n(s)} G_{(C_s,D_s)} v$$
$$\le \frac{9}{4} \int_A^B (f_{n(s)} - f_{n(k)}) G_{(C_s,D_s)} v \le \frac{9}{4} \|T f_{n(s)} - T f_{n(k)}\|_{\mathrm{BMO}}.$$

In other words, T is not compact. The proof is complete. □

4. A counterexample to (1.5)

4.1. Theorem. *Put $A = 0$, $B = \frac{1}{4}$. Let $1 < p < \infty$, and define for $n = 2, 3, \dots$*

$$\alpha_n = 2^{-n} e^{-1/n}, \qquad \beta_n = 2^{-n}, \qquad J_n = (\alpha_n, \beta_n), \qquad R_n = (\beta_{n+1}, \alpha_n),$$

and

$$v(x) = \sum_{n=2}^{\infty} \Big(x^{p-1} \chi_{J_n}(x) + \chi_{R_n}(x) \Big), \quad x \in \Big(0, \frac{1}{4}\Big).$$

Put $X(v) = L^p(v)$. Then (1.4) *is true and* (1.1) *is false.*

Proof. Since $v^{1-p'}(x) = \frac{1}{x}$ on $\bigcup J_n$, we have

$$\Big\| \frac{1}{v} \Big\|_{p'(v)}^{p'} = \int_0^{1/4} v^{1-p'}(x)\, \mathrm{d}x \ge \sum_{n=2}^{\infty} \int_{\alpha_n}^{\beta_n} \frac{\mathrm{d}x}{x} = \sum_{n=2}^{\infty} \frac{1}{n} = \infty.$$

Thus $\frac{1}{v} \notin L^{p'}(v)$, and by Theorem 2.3 (i), (1.1) is false. To prove (1.4) we have to show that $\{G_I\}$ have uniformly continuous norms in $L^{p'}(v)$.

Let $x > 0$. Then for $0 < \varepsilon < \frac{x}{2}$ we have by (2.1), using also $v^{1-p'}(t) = 1 < \frac{1}{t}$ on $\bigcup R_n$,

$$\|G_I \chi_{\mathcal{U}(x,\varepsilon)}\|_{p'(v)}^{p'} < 4^{-p'} \int_{\mathcal{U}(x,\varepsilon)} \frac{\mathrm{d}y}{y} \lesssim \frac{|\mathcal{U}(x,\varepsilon)|}{x},$$

which clearly tends to zero if $\varepsilon \to 0_+$.

It remains to show that G_I have uniformly continuous norms at zero. That is, we claim that

$$\lim_{\varepsilon \to 0_+} \sup_{I \subset (0, \frac{1}{4})} \|G_I \chi_{(0,\varepsilon)}\|_{p'(v)} = 0.$$

Fix $\varepsilon > 0$ and $I = [a, b] \subset (0, \frac{1}{4})$. Then

$$\|G_I \chi_{(0,\varepsilon)}\|_{p'(v)}^{p'} \le \sum_{n=2}^{\infty} \int_{J_n \cap (0,\varepsilon)} \chi_I(x) \Big(\frac{b-x}{b-a}\Big)^{p'} \Big(\frac{x-a}{b-a}\Big)^{p'-1} \Big(1 - \frac{a}{x}\Big) \frac{\mathrm{d}x}{b-a}$$
$$+ \int_0^{\varepsilon} \chi_I(x) \Big(\frac{b-x}{b-a}\Big)^{p'} \Big(\frac{x-a}{b-a}\Big)^{p'} \mathrm{d}x = A_1(\varepsilon, I) + A_2(\varepsilon, I).$$

Obviously, for any I,

$$A_2(\varepsilon, I) \le 4^{-p'}\varepsilon,$$

whence

$$\lim_{\varepsilon\to 0_+} \sup_I A_2(\varepsilon, I) = 0.$$

We define for $s = 2, 3, \dots$

$$B_s(I) = \sum_{n=s}^{\infty} \int_{J_n} \chi_I(x) \Big(\frac{b-x}{b-a}\Big)^{p'} \Big(\frac{x-a}{b-a}\Big)^{p'-1} \Big(1 - \frac{a}{x}\Big) \frac{\mathrm{d}x}{b-a}.$$

We have to prove

$$\lim_{\varepsilon\to 0_+} \sup_I A_1(\varepsilon, I) = 0. \tag{4.1}$$

It is easy to see that (4.1) is equivalent to

$$\lim_{s\to\infty} \sup_I B_s(I) = 0. \tag{4.2}$$

Therefore, the proof will be complete once we show (4.2).

Using the change of variables $t = \frac{b-x}{b-a}$ we get for $n = 2, 3, \dots$

$$\int_{J_n} \chi_I(x) \Big(\frac{b-x}{b-a}\Big)^{p'} \Big(\frac{x-a}{b-a}\Big)^{p'-1} \Big(1 - \frac{a}{x}\Big) \frac{\mathrm{d}x}{b-a} = \int_{\gamma_n}^{\delta_n} \chi_{(0,1)}(t) t^{p'} (1-t)^{p'-1} F(t)\,\mathrm{d}t,$$

where

$$F(t) = (1-t)\Big(\frac{b}{b-a} - t\Big)^{-1},$$

and

$$\gamma_n = \frac{b-\beta_n}{b-a}, \quad \delta_n = \frac{b-\alpha_n}{b-a}.$$

Note that since (α_n, β_n) are disjoint, so are (γ_n, δ_n).

Fix $s \ge 2$. We can, and will, assume that there exists at least one $n \ge s$ such that $I \cap J_n \ne \emptyset$ (otherwise there is nothing to prove). Observe that

$$\beta_n - \alpha_n \approx n^{-1} 2^{-n}, \qquad n \in \mathbb{N}. \tag{4.3}$$

We shall distinguish three cases.

1) Assume first that there is *exactly one* $n \ge s$ such that $I \cap J_n \ne \emptyset$ (hence $b > \alpha_n$), and this n satisfies

$$b < (b-a)n^{1/2}. \tag{4.4}$$

Then, using the definition of γ_n, δ_n, (4.4), $b > \alpha_n$, (4.3), $e^{-1/n} \approx 1$, and $n \ge s$, we see that

$$\delta_n - \gamma_n = \frac{\beta_n - \alpha_n}{b-a} < \frac{\beta_n - \alpha_n}{b} n^{1/2} < \frac{\beta_n - \alpha_n}{\alpha_n} n^{1/2} \approx n^{-1/2} \le s^{-1/2}. \tag{4.5}$$

Since $F(t) \le 1$, we have

$$B_s(I) \le \int_{\gamma_n}^{\delta_n} \chi_{(0,1)}(t) t^{p'} (1-t)^{p'-1} \mathrm{d}t. \tag{4.6}$$

The integrand of (4.6) is summable as the integral is majorized by $\mathcal{B}(p', p'+1)$, where $\mathcal{B}$ is the Beta function. Therefore, (4.5) implies (4.2).

2) Assume that there is *exactly one* $n \ge s$ such that $I \cap J_n \ne \emptyset$, and this n satisfies $b \ge (b-a)n^{1/2}$. Then $F(t) \le n^{-1/2}$ for every $t \in (0,1)$. Thus,

$$B_s(I) \le n^{-1/2} \int_{\gamma_n}^{\delta_n} \chi_{(0,1)}(t) t^{p'} (1-t)^{p'-1} \mathrm{d}t \le s^{-1/2} \mathcal{B}(p', p'+1)$$

for every I, and (4.2) follows.

3) Finally, assume that there are *at least two* n*'s such that* $n \ge s$ *and* $J_n \cap I \ne \emptyset$, that is, the set $Y = \{n \ge s; I \cap J_n \ne \emptyset\}$ has at least two elements. Then $Y = [M, N] \cap \mathbb{N}$, where $M, N \in \mathbb{N}$ and $s \le M \le N-1$. We have

$$b - a \ge \alpha_M - \beta_N \ge 2^{-M}\Big(e^{-\frac{1}{2}} - \frac{1}{2}\Big). \tag{4.7}$$

Further,

$$B_s(I) = \sum_{n=M}^{N} \int_{\gamma_n}^{\delta_n} \chi_{(0,1)}(t) t^{p'} (1-t)^{p'-1} F(t)\, \mathrm{d}t. \tag{4.8}$$

By the definition of γ_n, δ_n, (4.7) and (4.3),

$$\begin{aligned}\sum_{n=M}^{N} (\delta_n - \gamma_n) &= \sum_{n=M}^{N} \frac{\beta_n - \alpha_n}{b-a} \lesssim 2^M \sum_{n=M}^{N} (\beta_n - \alpha_n) \\ &\approx 2^M \sum_{n=M}^{N} n^{-1} 2^{-n} \le \frac{2^M}{M} \sum_{n=M}^{\infty} 2^{-n} \approx \frac{1}{M} \le \frac{1}{s}.\end{aligned}$$

Since the integrand in (4.8) is summable, (4.2) follows. The proof is complete. □

5. A counterexample to (1.6).

5.1. Theorem. *Put* $A = 0$, $B = \infty$. *Let* $1 < p < \infty$, $v(x) = x^{p-1}$, *and*

$$X(v) = L^{p,1}(v).$$

Then (1.1) *is true and* (1.4) *is false.*

Proof. First, observe that $X'(v) = L^{p',\infty}(v)$ and

$$\Big\|\frac{1}{v}\Big\|_{p',\infty(v)}^{p'} = \sup_{\lambda>0} \lambda^{p'} v\Big(\Big\{\frac{1}{v} > \lambda\Big\}\Big) = \sup_{\lambda>0} \lambda^{p'} \int_0^{\lambda^{1/(1-p)}} x^{p-1} \mathrm{d}x = \frac{1}{p} < \infty.$$

Hence $\frac{1}{v} \in X'(v)$, and, by Theorem 2.3 (i), (1.1) is true.

Now we claim that (1.4) is false. Given $\varepsilon > 0$, put $I_\varepsilon = (\varepsilon, 2\varepsilon)$. For a fixed $t \in \left(0, \frac{1}{2}\right)$ we define

$$I_\varepsilon^t = \big((1+t)\varepsilon, (2-t)\varepsilon\big).$$

Then $\emptyset \neq I_\varepsilon^t \subsetneq I_\varepsilon$, and

$$\frac{(2\varepsilon - x)(x-\varepsilon)}{\varepsilon^2} \geq t^2, \quad x \in I_\varepsilon^t.$$

Therefore, $G_{I_\varepsilon} \geq t^2 (v(x))^{-1}$ for $x \in I_\varepsilon^t$. Thus, for $\lambda > 0$,

$$\{x \in (0,\infty); G_{I_\varepsilon}(x) > \lambda\} \supset \Big\{x \in I_\varepsilon^t, \frac{1}{v(x)} > t^{-2}\lambda\Big\} = I_\varepsilon^t \cap \big(0, (t^{-2}\lambda)^{1/(1-p)}\big).$$

Since $(2-t)\varepsilon < (t^{-2}\lambda)^{1/(1-p)}$ for $0 < \lambda < t^2\big((2-t)\varepsilon\big)^{1-p}$, we have

$$\|G_{I_\varepsilon}\chi_{(0,2\varepsilon)}\|_{p',\infty(v)}^{p'} \geq \sup_{0<\lambda<t^2((2-t)\varepsilon)^{1-p}} \lambda^{p'} v(I_\varepsilon^t) = \frac{t^{2p'}}{p}\Big(1 - \Big(\frac{1+t}{2-t}\Big)^p\Big) > 0.$$

Hence $\|G_{I_\varepsilon}\chi_{(0,2\varepsilon)}\|_{p',\infty(v)}$ is bounded away from zero when $\varepsilon \to 0_+$. In other words, $\{G_I\}$ do not have uniformly continuous norms in $L^{p',\infty}(v)$ at 0. By Theorem 3.3, (1.4) is false. The proof is complete. □

6. A counterexample to (1.7)

6.1. Theorem. *Put $A = 0$, $B = \infty$. Let $1 < p < \infty$, $v(x) = x^{p-1}$, $x \in (0,\infty)$, and put $X(v) = L^p(v)$. Then (1.2) is true but both (1.1) and (1.4) are false.*

Proof. That (1.2) is true and (1.1) is false follows from [**5**, Example 1]. We have only to show that (1.4) is false. Recall that $X'(v) = L^{p'}(v)$.

For $\varepsilon > 0$ put $I_\varepsilon = (\varepsilon, 2\varepsilon)$. Then, changing variables $x \mapsto \varepsilon y$, we get

$$\|G_{I_\varepsilon}\chi_{(0,2\varepsilon)}\|_{p'(v)}^{p'} = \int_\varepsilon^{2\varepsilon} \Big(\frac{(2\varepsilon - x)(x-\varepsilon)}{\varepsilon^2}\Big)^{p'} \frac{\mathrm{d}x}{x} = \int_1^2 (2-y)^{p'}(y-1)^{p'} \frac{\mathrm{d}y}{y} > 0.$$

Since the last integral is a positive number independent of ε, $\{G_I\}$ do not have uniformly continuous norms in $L^{p'}(v)$ at 0. By Theorem 3.3, (1.4) is false. The proof is complete. □

7. A counterexample to (1.8)

We define

$$Z = \Big\{f; \|f\|_Z = \sup_{0<\alpha\leq 1} \int_0^1 |f(x)|\, \alpha x^{\alpha-1} \mathrm{d}x < \infty\Big\}. \tag{7.1}$$

The following properties of Z are readily verified.

7.1. Lemma. (i) *Z is a Banach function space (non-weighted, that is, $v \equiv 1$).*

(ii) *The embeddings*

$$L^\infty \hookrightarrow Z \hookrightarrow L^1$$

hold and both have norm equal to 1.

(iii) *The norm of a characteristic function of an interval $(a,b) \subset (0,1)$ in Z is given by*

$$\|\chi_{(a,b)}\|_Z = \begin{cases} 1 & \text{if } a = 0; \\ 1-a & \text{if } b = 1; \\ b-a & \text{if } 0 < a < b < 1 \text{ and } a\log a \le b\log b; \\ a^\beta\left(\frac{\log a}{\log b} - 1\right) & \text{if } 0 < a < b < 1 \text{ and } a\log a > b\log b, \end{cases} \tag{7.2}$$

where $\beta = \log_{(b/a)}\left(\frac{\log a}{\log b}\right)$.

(iv) *Z is not a rearrangement–invariant space.*

(v) *Z does not have absolutely continuous norm.*

(vi) *If f is a positive non-increasing function, then $\|f\|_Z = \|f\|_{L^\infty}$.*

Proof. It is easy to verify axioms (P1)–(P5) of a Banach function space, thus proving (i). To prove (ii), note that

$$\|f\|_1 \le \|f\|_Z \le \|f\|_\infty \sup_{0<\alpha\le 1} \int_0^1 \alpha x^{\alpha-1}\mathrm{d}x = \|f\|_\infty,$$

and that the function $\chi_{(0,1)}$ has norm equal to 1 in all the three spaces L^1, L^∞, Z. Differentiating $(b^\alpha - a^\alpha)$ with respect to α, we get (iii) which in turn implies both (iv) (characteristic functions of intervals of the same length have different norms) and (v) ($\|\chi_{(0,\varepsilon)}\|_Z = 1$ for any ε). As for (vi), observe that for $\varepsilon \in (0,1)$ we have by (P2), the monotonicity of f, and (7.2)

$$\|f\|_Z \ge \|f\chi_{(0,\varepsilon)}\|_Z \ge f(\varepsilon)\|\chi_{(0,\varepsilon)}\|_Z = f(\varepsilon).$$

On letting $\varepsilon \to 0_+$ we obtain $\|f\|_Z \ge \|f\|_\infty$; the converse inequality has been established in the proof of (ii). The proof is complete. □

7.2. Theorem. *Put $A = 0$, $B = 1$, and $X = Z'$ (recall that $v \equiv 1$). Then both* (1.1) *and* (1.4) *are true but* (1.3) *is false.*

Proof. Evidently, $\frac{1}{v} = \chi_{(0,1)} \in Z = X'$, and, by (7.2), $\frac{1}{v} \notin X'_c$. By Theorem 2.3 (i), (iii), (1.1) is true and (1.3) is false.

Now we claim that (1.4) is true. It is easy to show that $\{G_I\}_{I\subset(0,1)}$ have uniformly continuous norms in Z at any point $x > 0$. To see this, note that for $\varepsilon < \frac{x}{2}$ we have by (7.1) and (2.1)

$$\|G_I\chi_{\mathcal{U}(x,\varepsilon)}\|_Z = \sup_{0<\alpha\le 1} \int_{\mathcal{U}(x,\varepsilon)} G_I(t)\alpha t^{\alpha-1}\mathrm{d}t \le \int_{\mathcal{U}(x,\varepsilon)} G_I(t)\frac{\mathrm{d}t}{t} \le \frac{\varepsilon}{x},$$

which tends to zero uniformly in I as $\varepsilon \to 0_+$.

It remains to show that G_I have uniformly continuous norms in Z at zero. Let $I = [a, b]$, $0 < a < b < 1$. Observe that for $a \le t \le b$

$$\frac{(b-t)(t-a)}{(b-a)^2} \le \frac{t-a}{b-a} \le \frac{t}{b}.$$

Therefore, for $0 < \varepsilon < e^{-1}$,

$$\begin{aligned} \|G_I \chi_{(0,\varepsilon)}\|_Z &\le \Big\| \chi_I(t)\chi_{(0,\varepsilon)}(t)\frac{t}{b}\Big\|_Z = \frac{1}{b}\sup_{0<\alpha\le 1} \int\limits_{I\cap(0,\varepsilon)} \alpha t^\alpha \mathrm{d}t \\ &\le \frac{1}{b}\sup_{0<\alpha\le 1} \int\limits_0^{\min(b,\varepsilon)} \alpha t^\alpha \mathrm{d}t = \frac{1}{b}\sup_{0<\alpha\le 1} \frac{\alpha}{\alpha+1}\big(\min(b,\varepsilon)\big)^{\alpha+1} \\ &\le \sup_{0<\alpha\le 1} \alpha\varepsilon^\alpha = \frac{1}{e\log(1/\varepsilon)}, \end{aligned}$$

which tends to zero uniformly in I as $\varepsilon \to 0_+$. Hence $\{G_I\}$ have uniformly continuous norms in Z and by Theorem 3.3, (1.4) is true. The proof is complete. □

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