

On $L^{p(x)}$ norms

D. E. EDMUNDS, J. LANG AND A. NEKVINDA

ABSTRACT. The relation between a Banach function space X and its subspaces formed by (i) those functions with absolutely continuous norm, (ii) those with continuous norm, and (iii) the closure of the set of bounded functions are investigated in the case $X = L^{p(x)}$.

1. INTRODUCTION

J. Rákosník and O. Kováčik (1991) studied basic properties of the Lebesgue space $L^{p(x)}$ with variable p depending on a domain. We shall investigate in this paper properties of $L^{p(x)}$ as a Banach function space, especially in the case of unbounded $p(x)$.

Let Ω be a bounded measurable set in \mathbb{R}^N , by $\mathcal{M}(\Omega)$ denote the set of all measurable functions on Ω and let $p \in \mathcal{M}(\Omega)$ be a function on Ω with $1 \leq p(x) \leq \infty$ for all $x \in \Omega$. On the set of all functions $f \in \mathcal{M}(\Omega)$ define a modular

$$\rho_p(f) = \int_{\Omega \setminus \Omega_\infty} |f(x)|^{p(x)} dx + \text{ess sup}_{x \in \Omega_\infty} |f(x)|,$$

where $\Omega_\infty = \{x; x \in \Omega, p(x) = \infty\}$, and a norm by

$$\|f\|_{p(x)} = \inf \{ \lambda > 0; \rho_p(f/\lambda) \leq 1 \}.$$

The space $L^{p(x)}$ is the set of all functions $f \in \mathcal{M}(\Omega)$ with bounded norm.

By $|E|$ denote the Lebesgue measure of E , $E \subset \mathbb{R}^N$. Recall the definition of a Banach function space (written BFS).

We say that the linear space X , $X \subset \mathcal{M}(\Omega)$, is a BFS if there exists a functional $\|\cdot\| : \mathcal{M}(\Omega) \rightarrow \mathbb{R}$ with norm property, satisfying:

- (1.1) $f \in X$ if and only if $\|f\| < \infty$;
- (1.2) $\|f\| = \| |f| \|$ for all $f \in \mathcal{M}(\Omega)$;
- (1.3) if $0 \leq f_n \nearrow f$ then $\|f_n\| \nearrow \|f\|$;
- (1.4) if $E \subset \Omega$, $|E| < \infty$ then $\|\chi_E\| < \infty$;
- (1.5) for any $E \subset \Omega$ with $|E| < \infty$ there is a constant $c(E)$ with $\int_E |f(x)| dx \leq c(E)\|f\|$ for all $f \in X$.

1991 *Mathematics Subject Classification.* 46E30.

Key words and phrases. Banach function spaces, Absolutely continuous norms, Continuous norms, $L^{p(x)}$.

The second author was supported by Grant No. 201/96/0431 of the Grant Agency of the Czech Republic and NATO Postdoctoral Fellowship programme. The third author was supported by Grant No. 201/97/0744 of the Grant Agency of the Czech Republic and by NATO Grant CRG 970071.

Basic properties of these spaces are studied in Bennett and Sharpley (1988).

Given a BFS X we say that f has an absolutely continuous norm in X , written $f \in X_a$, if for every nonincreasing sequence of measurable sets $G_n \subset \Omega$ with $|G_n| \searrow 0$ we have $\|f\chi_{G_n}\| \searrow 0$.

Denote by X_b the closure of the set of all bounded functions in X .

L. Pick and Q. Lai (1993) introduced an interesting concept of a continuous norm which has a connection with compactness of Hardy operators from a weighted BFS (X, v) into L_∞ . Following them, we say that a function f has a continuous norm, written $f \in X_c$, if and only if for every point x in the closure of Ω we have $\lim_{\varepsilon \rightarrow 0} \|f\chi_{B(x, \varepsilon) \cap \Omega}\| = 0$, where $B(x, \varepsilon)$ is the ball with centre x and radius ε .

We list some useful Propositions.

Proposition 1.1. *Let X be a BFS. Then:*

- (i) X_a is a closed subspace of X and $X_a \subseteq X_b$ (see Bennett and Sharpley (1988));
- (ii) X_c is a closed subspace of X and $X_a \subseteq X_c$ (see Lang and Nekvinda (1997));
- (iii) X is separable if and only if $X_a = X$ (see Bennett and Sharpley (1988)).

Proposition 1.2. (see Kováčik and Rákosník (1991)) *If $p(x) \in L^\infty(\Omega)$ then $C(\Omega) \cap L^{p(x)}$ is dense in $L^{p(x)}$.*

It is not difficult to prove the following proposition.

Proposition 1.3. *The space $L^{p(x)}$ is a BFS.*

As an immediate consequence of Propositions 1.1, 1.2 and 1.3 we have the following theorem.

Theorem 1.4. *Let $p(x) \in L^\infty(\Omega)$ and set $X = L^{p(x)}$. Then $X_a = X_b = X_c = X$.*

In what follows we investigate the relation between X_a, X_b, X_c and X when $X = L^{p(x)}$. For a general BFS the situation is very complicated, as the following proposition shows.

Proposition 1.5. (see Lang and Nekvinda (1996)) *There exists a BFS X with $\{0\} = X_a \subsetneq X_c = X$.*

PROPERTIES OF $L^{p(x)}$

First we investigate the relation between X_a and X_c when $X = L^{p(x)}$.

Lemma 2.1. *Assume $|\Omega_\infty| = 0$. Let $f \in \mathcal{M}(\Omega)$ and suppose that there exists a positive β such that $\int_\Omega \left(\frac{f(x)}{\beta}\right)^{p(x)} dx = \infty$. Then f does not have a continuous norm in $L^{p(x)}$.*

Proof. Since Ω is bounded then there exists a cube Q in \mathbb{R}^N with side-length $l(Q)$ such that $\Omega \subset Q$. Divide this cube into 2^n cubes $\{Q_i\}_{i=1}^{2^n}$ with $l(Q_i) = \frac{1}{2}l(Q)$. Since

$$\infty = \int_\Omega \left(\frac{f(x)}{\beta}\right)^{p(x)} dx = \sum_{i=1}^{2^n} \int_{\Omega \cap Q_i} \left(\frac{f(x)}{\beta}\right)^{p(x)} dx$$

we can find an index j with

$$\int_{Q_j} \left(\frac{f(x)}{\beta}\right)^{p(x)} dx = \infty.$$

Set $Q^1 = Q_j$. Again divide Q^1 into 2^n cubes. In the same way we find $Q^2 \subset Q^1$ such that

$$\int_{\Omega \cap Q^2} \left(\frac{f(x)}{\beta} \right)^{p(x)} dx = \infty.$$

Using this construction we obtain a decreasing sequence of $\{Q^n\}_{n=1}^\infty$ of cubes with $l(Q^n) = \frac{1}{2^n} l(Q)$ and

$$\int_{\Omega \cap Q^n} \left(\frac{f(x)}{\beta} \right)^{p(x)} dx = \infty \quad \text{for every } n.$$

Let $x \in \bigcap_{n=1}^\infty \overline{Q^n}$. Taking any ball with centre x and radius ε we have $Q^n \subset B(x, \varepsilon)$ for sufficiently large n . Then

$$\begin{aligned} \int_{\Omega \cap B(x, \varepsilon)} \left(\frac{f(x)}{\beta} \right)^{p(x)} dx &\geq \int_{\Omega \cap Q^n(x, \varepsilon)} \left(\frac{f(x)}{\beta} \right)^{p(x)} dx \\ &= \int_{\Omega} \left(\frac{f \chi_{Q^n(x, \varepsilon)}(x)}{\beta} \right)^{p(x)} dx = \infty. \end{aligned}$$

This fact immediately yields

$$\|f \chi_{\Omega \cap Q^n(x, \varepsilon)}\| > \beta,$$

which proves the lemma. \square

Lemma 2.2. *Let $|\Omega_\infty| = 0$, and $X = L^{p(x)}$. Then $X_c \subseteq X_a$.*

Proof It suffices to prove the implication $f \notin X_a \Rightarrow f \notin X_c$. Assume that there are a number $\alpha > 0$ and a sequence of sets $G_n, n = 1, 2, \dots$, with $G_{n+1} \subseteq G_n$, such that $|G_n| \rightarrow 0$ and

$$\|f \chi_{G_n}\|_{p(x)} = \inf \left\{ \lambda; \int_{G_n} \left(\frac{f(x)}{\lambda} \right)^{p(x)} \leq 1 \right\} \geq \alpha > 0, \quad \text{for every } n.$$

Set $\beta = \alpha/2$. Then we have

$$(1) \quad \int_{G_n} \left(\frac{f(x)}{\beta} \right)^{p(x)} dx > 1 \quad \text{for every } n.$$

Assume for a moment that

$$\int_{G_m} \left(\frac{f(x)}{\beta} \right)^{p(x)} dx \leq K < \infty \quad \text{for a fixed } m.$$

Since $G_{n+1} \subset G_n$ and $|G_n| \rightarrow 0$ we have, according to Lebesgue's dominated convergence theorem,

$$\lim_{n \rightarrow \infty} \int_{G_n} \left(\frac{f(x)}{\beta} \right)^{p(x)} dx = 0$$

which contradicts (1). Hence

$$\int_{G_n} \left(\frac{f(x)}{\beta} \right)^{p(x)} dx = \infty$$

for every n and because of Lemma 2.1 it follows that $f \notin X_c$. The proof is complete. \square

Theorem 2.3. *Let $X = L^{p(x)}$. Then $X_a = X_c$.*

Proof. Proposition 1.1 gives $X_a \subseteq X_c$. To prove the second inclusion take a function $f \in X_c$. Clearly $f(x) = 0$ for $x \in \Omega_\infty$. By Lemma 2.2 we know that $f \in X_a$. \square

We next investigate the space X_b for $X = L^{p(x)}$.

Theorem 2.4. *Assume that $p(x) \in L^\infty(\Omega \setminus \Omega_\infty)$. Then $X_b = X$.*

Proof. The result is an immediate consequence of Theorem 1.4 and the trivial fact that $L^\infty(\Omega_\infty) = X_b(\Omega_\infty)$. \square

Theorem 2.5. *Let $|\Omega_\infty| = 0$ and assume that $\text{ess sup } p(x) = \infty$. Set $X = L^{p(x)}$. Then $X_b \subsetneq X$.*

Proof. Set $\Omega_k = \{x \in \Omega; k \leq p(x) < k+1\}$. Take positive numbers c_k such that

$$\begin{aligned} \int_{\Omega_k} c_k^{p(x)} dx &= \frac{1}{2^k} & \text{if } |\Omega_k| > 0, \\ c_k &= 0 & \text{if } |\Omega_k| = 0. \end{aligned}$$

Define numbers $d_k := \max(c_k, 2k)$, $k \in \mathbb{N}$. Then there are sets $\Omega'_k, \Omega''_k \subset \Omega_k$, with

$$\begin{aligned} \int_{\Omega'_k} d_k^{p(x)} dx &= \frac{1}{2^k} & \text{if } |\Omega'_k| > 0, \\ d_k &= 0 & \text{if } |\Omega'_k| = 0. \end{aligned}$$

Set $f(x) = \sum_{k=1}^{\infty} d_k \chi_{\Omega'_k}(x)$. Note that $f(x) = 0$ on Ω_∞ ; thus

$$\rho_p(f) = \int_{\Omega \setminus \Omega_\infty} f(x)^{p(x)} dx.$$

An easy calculation gives

$$\int_{\Omega \setminus \Omega_\infty} f(x)^{p(x)} dx = \sum_{k=1}^{\infty} \int_{\Omega'_k} d_k^{p(x)} \leq \sum_{k=1}^{\infty} \frac{1}{2^k} \leq 1,$$

and so $\|f\|_p \leq 1$.

Let g be a function on Ω with $|g| \leq n$. Define

$$f_n(x) = \begin{cases} f(x), & 0 \leq f(x) \leq n, \\ n, & n < f(x). \end{cases}$$

Then $|f(x) - g(x)| \geq |f(x) - f_n(x)|$, which gives $\|f - g\|_{p(x)} \geq \|f - f_n\|_{p(x)}$. Note that $f - f_n = 0$ on Ω_∞ . Then

$$\begin{aligned} \int_{\Omega \setminus \Omega_\infty} (4|f(x) - f_n(x)|)^{p(x)} dx &\geq \sum_{k=n}^{\infty} 4^k \int_{\Omega'_k} (d_k - n)^{p(x)} dx \\ &\geq \sum_{k=n}^{\infty} 4^k \int_{\Omega'_k} \left(\frac{d_k}{2}\right)^{p(x)} dx \geq \sum_{k=n}^{\infty} \frac{4^k}{2^{k+1}} \int_{\Omega'_k} (d_k)^{p(x)} dx. \end{aligned}$$

According to the assumption on $p(x)$ we have $\int_{\Omega'_k} d_k^{p(x)} dx = 2^{-k}$ for an infinite number of k 's and so

$$\sum_{k=n}^{\infty} \frac{4^k}{2^{k+1}} \int_{\Omega'_k} d_k^{p(x)} dx = \infty.$$

Hence for every g with $|g| \leq n$, $\|f - g\|_{p(x)} > \frac{1}{4}$. Then $f \notin X_b$. The proof is complete. \square

Theorem 2.6. *Assume that $|\Omega_\infty| = 0$ and that $\text{ess sup } p(x) = \infty$. Define $\Omega_k = \{x \in \Omega; k \leq p(x) < k+1\}$, for all $k \in \mathbb{N}$. Put $X = L^{p(x)}$. Then $X_a = X_b$ if and only if*

$$(2) \quad \sum_{k=1}^{\infty} A^k |\Omega_k| < \infty \quad \text{for all } A > 1.$$

Proof. Assume (2). Let f be a bounded function with $|f| \leq K$.

Take $\Omega \supset G_1 \supset G_2 \supset \dots, |G_n| \rightarrow 0$. Let λ be an arbitrary fixed number, $0 < \lambda < K$. Then for every n we have

$$\begin{aligned} \int_{G_n} \left(\frac{f(x)}{\lambda} \right)^{p(x)} dx &\leq \int_{G_n} \left(\frac{K}{\lambda} \right)^{p(x)} dx = \sum_{k=1}^{\infty} \int_{G_n \cap \Omega_k} \left(\frac{K}{\lambda} \right)^{p(x)} dx \\ &\leq \sum_{k=1}^{\infty} \left(\frac{K}{\lambda} \right)^{k+1} |G_n \cap \Omega_k|. \end{aligned}$$

Since $\frac{K}{\lambda} \leq 1$ we see, because of (2), that there exists k_0 such that

$$\sum_{k=k_0+1}^{\infty} \left(\frac{K}{\lambda} \right)^{k+1} |\Omega_k| \leq \frac{1}{2}.$$

Since $|G_n| \rightarrow 0$ we can find n_0 such that for $n \geq n_0$,

$$\sum_{k=1}^{\infty} \left(\frac{K}{\lambda} \right)^{k+1} |\Omega_k \cap G_n| \leq \frac{1}{2}.$$

Summation shows that for any λ with $0 < \lambda \leq K$ there exists n_0 such that

$$\int_{G_n} \left(\frac{f(x)}{\lambda} \right)^{p(x)} dx \leq 1 \quad \text{for all } n \geq n_0.$$

It follows that for any $\lambda > 0$ there exists n_0 with

$$\|f \chi_{G_n}\| \leq \lambda$$

for $n \geq n_0$ and, consequently,

$$\lim \|f \chi_{G_n}\| = 0.$$

Hence $f \in X_a$.

Now assume that $f \in X_b$. Then there is a sequence of bounded functions f_n with $\|f - f_n\| \rightarrow 0$. We have proved that $f_n \in X_a$. Since, according to Proposition 1.1, X_a is closed, it follows that $f \in X_a$.

To prove the second implication assume that there exists $A > 1$ such that

$$\sum_{k=1}^{\infty} A^k |\Omega_k| = \infty.$$

Taking $f = 1$ on Ω and $G_n = \sum_{k=n}^{\infty} \Omega_k$ we have

$$\begin{aligned} \int_{\Omega} \left(\frac{f \chi_{G_n}}{1/A} \right)^{p(x)} dx &= \sum_{k=1}^{\infty} \int_{\Omega_k} A^{p(x)} \chi_{G_n}(x) dx \\ &\geq \sum_{k=n}^{\infty} A^n |\Omega_k| = \infty \end{aligned}$$

This implies that $\|f \chi_{G_n}\|_{p(x)} > 1/A$ for any n . Since $|G_n| \rightarrow 0, G_1 \supset G_2 \supset \dots$, it follows that $f \notin X_a$. Of course, $f \in X_b$. The proof is complete. \square

Corollary 2.7. *Under the same assumptions as in Theorem 2.6, $X_a = X_b$ if and only if*

$$\int_0^{|\Omega|} A^{p^*(t)} dt < \infty \quad \text{for all } A > 1.$$

Here $p^*(t)$ is the non-increasing rearrangement of $p(x)$.

Proof. Since $\int_0^{|\Omega|} A^{p^*(t)} dt = \int_{\Omega} A^{p(x)} dx$ the result follows easily. \square

To illustrate Corollary 2.7 the following examples may be useful.

Examples. (i) Let $N = 1, \Omega = (0, e^{-1}), p^*(x) = x^{\alpha}, \alpha < 0$. Then

$$\int_0^{e^{-1}} A^{p^*(x)} dx = \int_0^{e^{-1}} A^{x^{\alpha}} dx = \int_0^{e^{-1}} \sum_{n=0}^{\infty} \frac{(x^{\alpha} \log A)^n}{n!} dx = \infty$$

for all $\alpha < 0$ and all $A > 1$. Thus $X_a \subsetneq X_b$.

(ii) Let $N = 1, \Omega = (0, e^{-1}), p^*(x) = (\log x^{-1})^{\alpha}, \alpha \geq 0$. Then

$$\int_0^{e^{-1}} A^{p^*(x)} dx = \int_0^{e^{-1}} e^{(\log x^{-1})^{\alpha} \log A} dx = \int_1^{\infty} e^{y^{\alpha} \log A - y} dy < \infty$$

for all $\alpha \in (0, 1)$ and all $A > 1$. Hence $X_a = X_b$.

Let $A = e$; then

$$\int_0^{e^{-1}} A^{p^*(x)} dx = \int_1^{\infty} e^{y^{\alpha} \log A - y} dy = \infty$$

for all $\alpha \in [1, \infty)$. Hence $X_a \subsetneq X_b$.

REFERENCES

- Kováčik, O. and Rákosník, J. 1991, *On spaces $L^{p(x)}$ and $W^{k,p(x)}$* , Czech. Math. J. **41**, 592–618.
- Bennett, C. and Sharpley, R. 1987, *Interpolation of operators*, Academic Press, New York.
- Lai, Q. and Pick, L. 1993, *The Hardy operator, L_∞ and BMO*, J. London Math. Soc. **48**, 167–177.
- Lang, J. and Nekvinda, A. 1997, *A difference between the continuous and the absolutely continuous norms in Banach function spaces*, Czechoslovak Math. Journal **47**, 221–232.
- Lang, J. and Nekvinda, A. 1996, *Maximal difference between continuous norm and absolutely continuous norm in Banach function spaces*, Preprint 4/96 Faculty of Civil Engineering, CTU, Prague.

D. E. Edmunds, Centre for Mathematical Analysis and Its Applications, University of Sussex, Falmer, Brighton BN1 9QH, U.K.

E-mail address: `d.e.edmunds@sussex.ac.uk`

J. Lang, Matematický ústav AV ČR, Žitná 25, 115 67 Praha 1, Czech Republic

E-mail address: `lang@math.cas.cz`

A. Nekvinda, Stavební fakulta, ČVUT, Thákurova 7, 166 29 Praha 6, Czech Republic

E-mail address: `nekvinda@fsv.cvut.cz`