Two–sided estimates for the approximation numbers of Hardy–type operators in L^∞ and L^1.

W. D. Evans, D. J. Harris, J. Lang

August 29, 2000

Abstract

In [1] and [2] upper and lower estimates and asymptotic results were obtained for the approximation numbers of the operator $T : L^p(\Re^+) \to L^p(\Re^+)$ defined by

$$(Tf)(x) := v(x) \int_0^\infty u(t)f(t)dt$$

when $1 < p < \infty$. Analogous results are given in this paper for the cases $p = 1, \infty$ not included in [1] and [2].

1 Introduction.

In [1] and [2] the operator $T : L^p(\Re^+) \to L^p(\Re^+)$ defined by

$$Tf(x) = v(x) \int_0^x u(t)f(t)dt \tag{1. 1}$$

was studied in the case $1 < p < \infty$, with u, v real–valued functions and $u \in L^{p'}_{loc}(\Re^+), v \in L^p(\Re^+), p' = p/(p-1)$. Estimates for the approximation numbers $a_n(T)$ of T were obtained in [1], but the procedure for extracting the upper and lower bounds from the results is rather cumbersome to apply. This deficiency was overcome in [2] where asymptotic bounds for the approximations numbers which are easy to check in practice, were determined. Specifically, it was proved that

$$\lim_{n\to\infty} na_n(T) = \frac{1}{\pi} \int_0^\infty |u(t)v(t)|dt \tag{1. 2}$$

when $p = 2$; and when $p \neq 2$

$$\frac{1}{4}\alpha_p \int_0^\infty |u(t)v(t)|dt \leq \liminf_{n\to\infty} na_n(T) \leq \limsup_{n\to\infty} na_n(T) \leq \alpha_p \int_0^\infty |u(t)v(t)|dt \tag{1. 3}$$

for some constant α_p depending on p. Further in [2], two sided estimates are given for the l^α and weak l^α norms of $\{a_n(T)\}$ when $\alpha > 1$: in the case $p = 2$, these results recover those in [5].

The analysis in [2] is no longer valid when $p = \infty$ or 1, and, indeed, the result itself has to be modified in the following way: when $p = \infty$, the function v in the integrals in (1. 2), (1. 3) is replaced by

$$v_s(t) = \lim_{\varepsilon \to 0_+} \|v\|_{L^\infty(t-\varepsilon, t+\varepsilon)},$$

while if $p = 1$, u is replaced by u_s. Three critical ingredients of the proof in [1] and [2] are no longer available in these cases. The first is that the operator P defined by the integral mean over an interval $I \subset \Re^+$, namely

$$Pf := \frac{1}{|I|} \int_I f dx,$$

where $|I|$ denotes the length of I, is such that the distance from T to the one–dimensional operators on $L^p(I)$ is comparable to $\|T - P|L^p(I) \to L^p(I)\|$. The second concerns the basic strategy which relies on a partition of $\Re^+$ into intervals I_k which are defined by means of a continuous set function $L(I)$ which, with $I = (c, d)$, is monotonic decreasing as c increases and increasing as d increases. In the L^∞ and L^1 cases the analogue of L is no longer continuous and an alternative function, and technique, have to be found. Finally, the fact that the step functions are not dense in L^∞ causes difficulties, and indeed, it is this which dictates the form of the result noted above.

It is just as easy to consider a general interval (a, b) instead of $\Re^+$, so that in this paper

$$Tf(x) := v(x) \int_a^x u(t)f(t)dt \ , \quad a < x < b; \tag{1. 4}$$

this simple extension will have a useful consequence when T is considered as an operator on L^1 as we can then simply translate the dual of the L^∞ result. Also, as was observed in [2], the condition on v assumed there, namely $v \in L^(\Re^+)$ can be weakened to $v \in L^p(x, \infty)$ for all $x > 0$, and we incorporate this fact in this paper.

Finally, to give some insight into the significance of the function v_s in the $L^\infty(a, b)$ case, we show that, with the operator in (1. 4) denoted by $T_{u,v}$, the following is possible:

$$\|T_{u,v}\| = \|T_{u,v_s}\| = \|T_{u,v} - T_{u,v-v_s}\|,$$

$$\int_a^b |u(t)v(t)|dt \neq \int_a^b |u(t)|v_s(t)dt$$

$$\limsup na_n(T_{u,v}) \asymp \limsup na_n(T_{u,v_s}),$$

$$\liminf na_n(T_{u,v}) \asymp \liminf na_n(T_{u,v_s}),$$

where the symbol $\asymp$ indicates that the quotient of the two sides is bounded above and below by positive constants. Analogous possibilities exist in the $L^1(a,b)$ case.

2 Preliminaries.

In most of the paper we shall be concerned with the operator T defined in (1. 4) as a map from $L^\infty(a,b)$ into itself. The assumptions made on u, v in this case are that, for all $x \in (a,b)$,

$$u \in L^1(a,x), \tag{2. 1}$$

and

$$v \in L^\infty(x,b). \tag{2. 2}$$

The results for T acting between $L^1(a,b)$ will follow on taking duals, and for this part of the paper alternative conditions to (2. 1) and (2. 2) will be required.

For $I = (c,d) \subseteq (a,b)$, define

$$J(I) \equiv J(c,d) := \sup_{x\in I}\left\{\int_c^x |u(t)|dt \; \|v\chi_{(x,d)}\|_\infty\right\}, \tag{2. 3}$$

where χ_S denotes the characteristic function of the set S, and $\|.\|_\infty$ denotes the norm on $L^\infty(a,b)$; we shall write $\|.\|_{p,I}$ for the usual norm on $L^p(I), 1 \le p \le \infty$, but use $\|.\|_p$ when $I = (a,b)$. It is easy to see that

$$J(I) = \operatorname*{esssup}_{x\in I}\left\{\int_c^x |u(t)|dt \; |v(x)|\right\}. \tag{2. 4}$$

We also have

Lemma 2.1 *Suppose that (2. 1) and (2. 2) are satisfied. Then the function $J(.,d)$ is continuous and non–increasing on (a,d), for any $d < b$.*

Proof. Given $x \in (a,b)$ and $\varepsilon > 0$, there exists $h = h(x,\varepsilon) \in \left(0, \min\{\frac{1}{2}(x+a), b-x\}\right)$ such that

$$\int_{x-h}^{x+h} |u(t)|dt < \min\left(\frac{\varepsilon}{\|v\|_{\infty,(\frac{x-a}{2},d)}}, \varepsilon\right).$$

Then

$$\begin{aligned} J(x,d) \quad &\le \quad J(x-h,d) = \max\Bigg\{\sup_{x-h<z<x}\left[\int_{x-h}^z |u(t)|dt \; \|v\|_{\infty,(z,d)}\right], \\ &\qquad \sup_{x<z<d}\left[\left(\int_{x-h}^x + \int_x^z\right)|u(t)|dt \; \|v\|_{\infty,(z,d)}\right]\Bigg\} \\ &\le \quad \max\{\varepsilon, \varepsilon + J(x,d)\} = \varepsilon + J(x,d) \end{aligned} \tag{2. 5}$$

and so $0 < J(x-h,a) - J(x,d) < \varepsilon$. Similarly $0 < J(x) - J(x+h) < \varepsilon$ and the continuity is established. It is obvious that $J(.,d)$ is non–increasing and hence the lemma is proved. □

The following result is known (see [4] and [6])

Proposition 2.2 *The operator T in (1. 4), with u, v satisfying (2. 1) and (2. 2), is bounded as a map from $L^\infty(a,b)$ into $L^\infty(a,b)$ if and only if $J(a,b) < \infty$. It is compact if and only if $\lim_{c\to a_+} J(a,c) = \lim_{d\to b_-} J(d,b) = 0$.*

In [1], the analogue of the function J in (2. 3) could have been used to construct the partition of (a,b) into the intervals I_i which feature so prominently in the analysis; see the Remark at the end of §4 in [1]. However, in the L^∞ case, for the reason given in the Introduction, we need to use directly the function

$$A(I) := \begin{cases} \sup_{f\in L^\infty(I), f\neq 0} \inf_{\alpha\in\Re} \|Tf - \alpha v\,\|_{\infty,I}/\|f\|_{\infty,I} & \text{if } v(I) > 0, \\ 0 & \text{if } v(I) = 0. \end{cases} \tag{2. 6}$$

where $v(I) := \int_I v(t)dt$. If v is continuous, it can be shown that $A(.,b)$ is continuous, but in general, this is not so. For, consider the example

$$\begin{aligned} v(x) &= \begin{cases} 1, & x \in (0,1)\cup(2,\infty), \\ 0, & \text{otherwise,} \end{cases} \\ u(x) &= \chi_{(1,2)}(x). \end{aligned}$$

with $(a,b) = (0,\infty)$. Then $A(x,\infty) = 0$ for $x > 1$, but for $x < 1$,

$$\begin{aligned} A(x,\infty) &\geq \inf_{\alpha\in\Re} \| \left[\int_0^y u\; dt - \alpha\right] v(y)\|_{\infty,(x,\infty)} \\ &= \inf_{\alpha\in\Re} \max\{|\alpha|, |1-\alpha|\} \\ &= \frac{1}{2}. \end{aligned}$$

It is of interest to note that if (2. 1) and (2. 2) are satisfied and $v \notin L^\infty(a,b)$, then, since $\int_a^x u(t)f(t) \to 0$ as $x \to a_+$ for every $f \in L^\infty(a,b)$, we must have that, if $\alpha \neq 0$, $\|Tf - \alpha v\|_{\infty,(a,c)} = \infty$ for $c \in (a,b]$. Hence, with $I = (a,c)$

$$\begin{aligned} A(a,c) &= \sup_{\|f\|_{\infty,I}=1} \|Tf\|_{\infty,I} \\ &= \operatorname*{esssup}_{x\in I} |v(x)| \int_a^x |u(t)|dt \\ &= J(a,c) \end{aligned}$$

by (2. 4).

We now define, for any interval $I \subseteq (a,b)$ and $\varepsilon > 0$,

$$M(I,\varepsilon) := \inf\{n : I = \cup_{i=1}^n I_i, A(I_i) \leq \varepsilon\}. \tag{2. 7}$$

Observe that if $\overline{I} \subset (a,b)$, then $M(I,\varepsilon) < \infty$. For, since $J(c,d) \leq \|u\|_{1,(c,d)} \|v\|_{\infty,I}$ for any $(c,d) \subset I$ and $\|.\|_1$ is absolutely continuous, then the number

$$N(I,\varepsilon) := \inf \{n : I = \cup_{i=1}^n I_i, J(I_i) \leq \varepsilon\} \tag{2. 8}$$

is finite, and

$$\begin{aligned} A(I) &\leq \sup_{f \in L^\infty(I), f \neq 0} \frac{\|Tf\|_{\infty,I}}{\|f\|_{\infty,I}} \\ &\leq \sup_{f \in L^\infty(I), f \neq 0} \operatorname*{esssup}_I \frac{\left[|v(x)| \int_a^x |u(t)||f(t)|dt\right]}{\|f\|_{\infty,I}} \\ &\leq J(I) \end{aligned} \tag{2. 9}$$

by (2. 4); thus $M(I,\varepsilon) \leq N(I,\varepsilon) < \infty$. If $I = (a,b)$, we still have $M(I,\varepsilon) < \infty$ if

$$\lim_{x \to b_-} J(x,b) = \lim_{x \to a_+} J(a,x) = 0$$

since $N(I,\varepsilon) < \infty$ and (2. 9) remains valid.

Lemma 2.3 *Suppose that (2. 1) and (2. 2) are satisfied and let $M(I,\varepsilon) = m < \infty$ for $I \subseteq (a,b)$ and $\varepsilon > 0$. Then we have:*

(i) *if $m = 2n$, there exist intervals $J_i, i = 1,2,\dots,n$ such that $I = \cup_{i=1}^n J_i$ and $A(J_i) > \varepsilon$;*

(ii) *if $m = 2n+1$, there exist intervals $J_i, i = 1,2,\dots,n+1$ such that $I = \cup_{i=1}^{n+1} J_i, A(J_i) > \varepsilon, i = 1,\dots,n$ and $A(J_{n+1}) \leq \varepsilon$.*

Proof. From the definition of $M(I,\varepsilon)$ in (2.7) there exist I_i, $i = 1,2,\dots,m$ such that $A(I_i) \leq \varepsilon$ and $A(I_i \cup I_{i+1}) > \varepsilon$. Now set $J_1 = I_1 \cup I_2, J_2 = I_3 \cup I_4, \dots,$ with $J_{n+1} = I_m$ in case (ii). □

The final preliminary result is the following critical lemma which will yield a one–dimensional approximation to T on I.

Lemma 2.4 *There exists $\omega_I \in \{L^\infty(I)\}^*$ such that $\omega_I(1) = 1$, $\|\omega_I\|_{\{L^\infty(I)\}^*} = 1$ and, for all $f \in L^\infty(I)$*

$$\inf_{\alpha \in \Re} \|(f-\alpha)v\|_{\infty,I} \leq \|(f - \omega_I(f))v\|_{\infty,I} \leq 2 \inf_{\alpha \in \Re} \|(f-\alpha)v\|_{\infty,I} \tag{2. 10}$$

Proof. For $0 < \gamma < \|v\|_{\infty,I}$ and $A_\gamma := \{x : v(x) > \gamma\}$, define $\omega_\gamma \in \{L^\infty(I)\}^*$ by

$$\omega_\gamma(f) := \frac{1}{|A_\gamma|} \int_{A_\gamma} f(x)dx, \ f \in L^\infty(I).$$

Then $\omega_\gamma(1) = 1, \|\omega_\gamma\|_{\{L^\infty(I)\}^*} = 1$ and

$$|\omega_\gamma(f)| \leq \frac{1}{\gamma} \|fv\|_{\infty,I}. \tag{2. 11}$$

The set $W := \{W_\beta : 0 < \beta < \|v\|_{\infty,I}\}$, where $W_\beta = \{\omega_\gamma : \gamma > \beta\}$, is a filter base whose members W_β are subsets of the unit ball in $\{L^\infty(I)\}^*$. Hence, by the weak* compactness of this unit ball, W has an adherent point, ω_I say. It follows that $\omega_I(1) = 1, \|\omega_I\|_{\{L^\infty(I)\}^*} = 1$ and, from (2.11), for all $\beta \in (0, \|v\|_{\infty,I})$,

$$|\omega_I(f)| \le \frac{1}{\beta}\|fv\|_{\infty,I}, \qquad f \in L^\infty(I).$$

Consequently, for any $\delta \in \Re$,

$$\begin{aligned} \inf_{\alpha\in\Re} \|(f-\alpha)v\|_{\infty,I} &\le \|(f-\omega_I(f))v\|_{\infty,I} \\ &\le \|(f-\delta)v\ \|_{\infty,I} + \|\omega_I(f-\delta)v\|_{\infty,I} \\ &\le \|(f-\delta)v\|_{\infty,I}\left\{1+\frac{\|v\|_{\infty,I}}{\beta}\right\}. \end{aligned}$$

Since $\delta \in \Re$ and $\beta \in (0, \|v\|_{\infty,I})$ are arbitrary, the lemma follows. □

3 Bounds for the approximation numbers.

We recall that, given any $m \in \mathbf{N}$, the m th approximation number of a bounded operator $T, a_m(T)$, is defined by

$$a_m(T) := \inf \|T - F\|,$$

where the infimum is taken over all bounded linear maps $F : L^\infty(a,b) \to L^\infty(a,b)$ with rank less than m. General information on approximation numbers may be found in [3]. Since $L^\infty(a,b)$ has the approximation property, T is compact if and only if $a_m(T) \to 0$ as $m \to \infty$.

The first two lemmas of this section give estimates for $a_m(T)$ which are the analogues of those obtained in [1]. Hereafter, until §7, we shall always assume (2. 1) and (2. 2).

Lemma 3.1 *Suppose that $T : L^\infty(a,b) \to L^\infty(a,b)$ is bounded. Let $\varepsilon > 0$ and suppose that there exist $N \in \mathbf{N}$ and numbers $c_k, k = 0,1,\dots,N$, with $a = c_0 < c_1 < \dots < c_N = b$, such that $A(I_k) \le \varepsilon$ for $k = 0,1,\dots,N-1$, where $I_k = (c_k, c_{k+1})$. Then $a_{N+1}(T) \le 2\varepsilon$.*

Proof. Let $f \in L^\infty(a,b)$ be such that $\|f\|_\infty = 1$, and write

$$Pf := \sum_{i=0}^{N-1} P_{I_k} f$$

where the P_{I_k} are the one–dimensional operators

$$P_{I_k} f(x) := \chi_{I_k}(x) v(x) \hat{\omega_{I_k}}\left(\int_a^x uf dt\right), \quad k = 0,1,\dots N-1$$

where

$$\omega_{I_k}\left(\int_a^x ufdt\right) = \int_a^{c_k} ufdt + \omega_{I_k}\left(\int_{c_k}^x ufdt\right).$$

with $\omega_{I_k} \in \{L^\infty(I_k)\}^*$ the functionals in Lemma 2.4.

It is obvious that $P_k, k = 1, \ldots, N-2$ are bounded. With $k = 0$ or $N-1$ we have on $I = (a, c_1)$ or (c_N, b)

$$\left|v(x)\omega_I\left(\int_{c_k}^x ufdt\right)\right| \le \|\omega_I\|_{\{L^\infty(I)\}^*}\ |v(x)| \int_{c_k}^x |u(t)|dt\ \|f\|_{\infty,I}$$

and hence P is bounded in view of Proposition 2.2 and (2. 4). We have

$$\begin{aligned}
\|Tf - Pf\|_\infty &= \sup_{k\in\{0,1,\ldots,N-1\}} \|Tf - P_{I_k}f\|_{\infty,I_k} \\
&= \sup_{k\in\{0,1,\ldots,N-1\}} \|v(x)\left[\int_{c_k}^x ufdt - \omega_{I_k}\left(\int_{c_k}^x ufdt\right)\right]\|_{\infty,I_k} \\
&\le 2 \sup_{k\in\{0,1,\ldots,N-1\}} A(I_k)\|f\|_{\infty,I_k} \le 2\varepsilon\|f\|_{\infty,I}.
\end{aligned}$$

by Lemma 2.4. Since $\operatorname{rank} P \le N$, the lemma follows. □

Lemma 3.2 *Suppose that $T : L^\infty(a,b) \to L^\infty(a,b)$ is bounded. Let $\varepsilon > 0$ and suppose that there exist $N \in \mathbf{N}$ and numbers $d_k, k = 0, 1, \ldots, K$ with $a = d_0 < d_1 < \ldots < d_K \le b$ such that $A(I_k) \ge \varepsilon$ for $k = 0, 1, \ldots, K-1$, where $I_k = (d_k, d_{k+1})$. Then $a_K(T) \ge \varepsilon$.*

Proof. Let $\lambda \in (0,1)$. From the definition of $A(I_k)$ we see that there exists $\phi_k \in L^\infty(I_k)$ with $\|\phi_k\|_{\infty,I_k} = 1$ and such that

$$\inf_{\alpha\in\Re} \|T\phi_k - \alpha v\ \|_{\infty,I_k} > \lambda A(I_k) \ge \lambda\varepsilon. \tag{3. 1}$$

Set $\phi_k(x) = 0$ for $x \notin I_k$. Let $P : L^\infty(a,b) \to L^\infty(a,b)$ be bounded and $\operatorname{rank} P \le K-1$. Then, there are constants $\lambda_0, \ldots, \lambda_{n-1}$ not all zero, such that

$$P\left(\sum_{k=0}^{K-1} \lambda_k\phi_k\right) = 0.$$

Put $\phi = \sum_{k=0}^{K-1} \lambda_k\phi_k$. Then

$$\begin{aligned}
\|T\phi - P\phi\|_\infty &= \|T\phi\|_\infty \\
&\ge \sup_{k\in\{0,1,\ldots,K-1\}} \|v(x)\left(\int_{c_k}^x \lambda_k\phi_k(t)u(t)dt + \int_a^{c_k} \phi(t)u(t)dt\right)\|_{\infty,I_k} \\
&= \sup_{k\in\{0,1,\ldots,K-1\}} |\lambda_k|\|T\phi_k + \alpha_k v\ \|_{\infty,I_k}
\end{aligned}$$

where $\alpha_k = \lambda_k^{-1} \int_a^{c_k} \phi(t)u(t)dt$,

$$\begin{aligned} &\geq \sup_{k\in\{0,1,\dots,K-1\}} \inf_{\alpha\in\Re} |\lambda_k| \|T\phi_k - \alpha v\|_{\infty,I_k} \\ &\geq \sup_{k\in\{0,1,\dots,K-1\}} \lambda|\lambda_k|\varepsilon \\ &= \lambda\varepsilon\|\phi\|_\infty \end{aligned}$$

by (3.1). This implies that $a_K(T) \geq \lambda\varepsilon$, whence the result since $\lambda \in (0,1)$ is arbitrary. □

Corollary 3.3 *Suppose that T is compact (see Proposition 2.2). Then, for $\varepsilon \in (0, A(a,b))$,*

$$\begin{aligned} a_{M(\varepsilon)+1}(T) &\leq 2\varepsilon, \\ a_{[\frac{M(\varepsilon)}{2}]-1}(T) &> \varepsilon, \end{aligned}$$

where $M_\varepsilon \equiv M((a,b),\varepsilon)$ is defined in (2. 7) and $[.]$ denotes integer part.

Proof. This is an immediate consequence of Lemmas 3.1 and 3.2 □

4 Local asymptotic results.

We need some preliminary results and the functions v_s mentioned in §1, namely

$$v_s(x) := \lim_{\varepsilon\to 0_+} \|v\|_{\infty,(x-\varepsilon,x+\varepsilon)}$$

for $x \in (a,b)$.

Lemma 4.1 *For any $I \subseteq (a,b)$, $J(I;u,v) = J(I;u,v_s)$ and $A(I;u,v) = A(I;u,v_s)$, where $J(I;u,v)$ and $A(I;u,v)$ are the functions defined in (2. 3) and (2. 6) respectively.*

Proof. For any continuous function ϕ, it is readily shown that $\|v_s\phi\|_{\infty,I} = \|v\phi\|_{\infty,I}$, and this fact yields the lemma. □

Lemma 4.2 *Let $\bar{I} \subset (a,b)$, and let $\vartheta_n = \{I_i^n\}_{i=1}^{l(n)}$ be a partition of I by intervals I_i^n which are such that each $I_i^{(n+1)} \in \vartheta_{n+1}$ is a subinterval of some $I_j^{(n)} \in \vartheta_n$, and $|I_i^n| \to 0$ as $n \to \infty$. Define*

$$v_s^n(t) := \sum_{i=1}^{l(n)} \chi_{I_i^n}(t) c_i^n, \quad c_i^n = \|v_s\|_{\infty,I_i^n}.$$

Then for a.e. $t \in I$,

(i) $\|v_s\|_{\infty,I} \geq v_s^n(t) \geq v_s(t)$,

(ii) $v_s^n(t) \searrow v_s(t)$ *as* $n \to \infty$,

(iii) $\lim_{n\to\infty} \int_I u(t)[v_s^n(t) - v_s(t)]dt = 0$.

Proof. Since v_s is upper semi–continuous and bounded, it is known that it can be approximated from above by a decreasing sequence of step functions. However, we shall give a proof of the lemma for completeness and subsequent reference.

If $t \in \text{int}\, I_i^n$, the interior of I_i^n, then $v_s^n(t) = \|v_s\|_{\infty,I_i^n}$ satisfies

$$v_s(t) \leq v_s^n(t) \leq \|v_s\|_{\infty,I}.$$

This establishes (i), the exceptional set being $S = \cup_{n\in\mathbf{N}} S_n$, where S_n is the set of end points of the intervals $I_i^n \in \vartheta_n$. If $t \in \text{int}\, I_{i(n+1)}^{n+1} \subset \text{int}\, I_{i(n)}^n$ say, we have $c_{i(n+1)}^{n+1} \leq c_{i(n)}^n$ and so $v_s^{n+1}(t) \leq v_s^n(t)$ for $t \in I \ \setminus S$. Also, if $t \in \text{int}\, I_{i(n)}^n$,

$$v_s^n(t) = \|v_s\|_{\infty,I_{i(n)}^n} = \|v\|_{\infty,I_{i(n)}^n} \geq v(t)$$

as observed in the proof of Lemma 4.1. Moreover, given $\delta > 0$ there exists $\varepsilon_0 > 0$ such that

$$v_s(t) > \|v\|_{\infty,(t-\varepsilon_0,t+\varepsilon_0)} - \delta.$$

Now choose N such that for all $n \geq N$

$$t \in \text{int}\, I_{i(n)}^n \subset (t-\varepsilon_0, t+\varepsilon_0).$$

Then we have that for all $n \geq N$

$$0 < v_s^n(t) - v_s(t) < \delta$$

and hence $v_s^n(t) \to v_s(t)$ for all $t \in I \setminus S$.

Finally, (iii) follows by the dominated convergence theorem since $u \in L^1(I)$ and $\|v_s^n\|_{\infty,I} = \|v_s\|_{\infty,I} = \|v\|_{\infty,I} < \infty$. □

Lemma 4.3 *Let* u, v *be constant on* $I \subset\subset (a,b)$. *Then*

$$A(I) = \frac{1}{2}|u||v||I|. \tag{4. 1}$$

Proof. We have if $I = (c,d)$

$$\begin{aligned} A(I) &\geq |u||v| \inf_\alpha \|x - c - \alpha\|_{\infty,I} \\ &= |u||v|\|x - c - \frac{1}{2}(d-c)\|_{\infty,I} \\ &= \frac{1}{2}|u||v||I|. \end{aligned}$$

Let $f \in L^\infty(I)$ and set $F(x) = \int_c^x f\,dt$. Then there exist $x_0, x_1 \in [c, d]$ such that

$$F(x_0) \leq F(x) \leq F(x_1), \qquad x \in [a, b]$$

and hence

$$\begin{aligned} \inf_\alpha \|F - \alpha\|_{\infty,I} &\leq \|F - \frac{1}{2}(F(x_0) + F(x_1))\|_{\infty,I} \\ &= \frac{1}{2}(F(x_1) - F(x_0)) \\ &= \frac{1}{2}\int_{x_0}^{x_1} f\,dt. \end{aligned}$$

This yields

$$A(I) \leq \sup_{\|f\|_{\infty,I}=1} \left\{ \frac{1}{2}\int_{x_0}^{x_1} f\,dt \right\} \leq \frac{1}{2}|I|$$

and the lemma is proved. □

Lemma 4.4 *Let $I \subset\subset (a, b)$ and $u_1, u_2 \in L^1(I)$. Then*

$$|A(I; u_1, u_2) - A(I; u_2, v)| \leq \|u_1 - u_2\|_{1,I}\|v\|_{\infty,I}. \qquad (4.\ 2)$$

Proof. We have

$$\begin{gathered} |A(I; u_1, v) - A(I; u_2, v)| \\ \leq \sup_{\|f\|_{\infty,I}=1} |\inf_\alpha \|v(x)(\textstyle\int_a^x u_1 f\,dt - \alpha)\|_{\infty,I} - \inf_\alpha \|v(x)(\int_a^x u_2 f\,dt - \alpha)\|_{\infty,I}|. \end{gathered}$$

Suppose f is such that

$$\inf_\alpha \|v(x)\left(\int_a^x u_1 f\,dt - \alpha\right)\|_{\infty,I} \geq \inf_\alpha \|v(x)\left(\int_a^x u_2 f\,dt - \alpha\right)\|_{\infty,I}. \qquad (4.\ 3)$$

Given $\varepsilon > 0$ there exists $\alpha_0 \in \Re$ such that

$$\inf_\alpha \|v(x)\left(\int_a^x u_2 f\,dt - \alpha\right)\|_{\infty,I} > \|v(x)\left(\int_a^x u_2 f\,dt - \alpha_0\right)\|_{\infty,I} - \varepsilon.$$

Hence

$$\begin{aligned} 0 &\leq \inf_\alpha \|v(x)\left(\int_a^x u_1 f\,dt - \alpha\right)\|_{\infty,I} - \inf_\alpha \|v(x)\left(\int_a^x u_2 f\,dt - \alpha\right)\|_{\infty,I} \\ &\leq \|v(x)\left(\int_a^x u_1 f\,dt - \alpha_0\right)\|_{\infty,I} - \|v(x)\left(\int_a^x u_2 f\,dt - \alpha_0\right)\|_{\infty,I} + \varepsilon \\ &\leq \|v(x)\int_a^x (u_1 - u_2) f\,dt\|_{\infty,I} + \varepsilon \\ &\leq \|v(x)\|_{\infty,I}\|u_1 - u_2\|_{1,I}\|f\|_{\infty,I} + \varepsilon. \end{aligned}$$

This remains valid if the inequality (4. 3) is reversed, and so

$$|A(I;u_1,v)-A(I;u_2,v)|\le\|v(x)\|_{\infty,I}\|u_1-u_2\|_{1,I}+\varepsilon$$

Since ε is arbitrary, the lemma is proved. □

In the next lemma g^* denotes the non–increasing rearrangement of a function g on an interval I: g^* is the generalised inverse of the non–increasing distribution function g_* of g, namely

$$g^*(x):=\inf\{t:g_*(t)\ge x\} \tag{4. 4}$$

where

$$g_*(t):=|\{x\in I:g(x)\ge t\}|. \tag{4. 5}$$

Note that since we have $\ge$ in the definitions above, g_* and g^* are left–continuous functions.

Lemma 4.5 *Let $I\subset\subset(a,b)$ and $\gamma,\delta\in\Re$ with $\delta\ge v_s(t)\ge 0$ on I. Then*

$$A(I;\gamma,\delta)\ge A(I;\gamma,v_s)\ge\frac{1}{2}|\gamma|\|(v_s\chi_I)^*(t)t\|_{\infty,(0,|I|)}. \tag{4. 6}$$

Proof. The first inequality in (4. 6) is obvious. The set

$$M_\beta:=\{y\in I:v_s(y)\ge\beta\}$$

is relatively closed in $\bar{I}$. For if $\{y_n\}\subset M_\beta$ and $y_n\to y\in\bar{I}$ as $n\to\infty$, then given $\varepsilon>0$ there exists N such that $(y-\varepsilon,y+\varepsilon)\supset(y_n-\frac{1}{2}\varepsilon,y_n+\frac{1}{2}\varepsilon)$ for $n>N$. Hence

$$\|v\|_{\infty,(y-\varepsilon,y+\varepsilon)}\ge\|v\|_{\infty,(y_n-\frac{1}{2}\varepsilon,y_n+\frac{1}{2}\varepsilon)}\ge v_s(y_n)\ge\beta$$

whence $v_s(y)\ge\beta$ and $y\in M_\beta$. From the observed left continuity of (4. 4) and (4. 5), we have

$$\|(v_s\chi_I)^*(t)t\|_{\infty,(0,|I|)}=\max_{(0,|I|]}|(v_s\chi_I)^*(t)t|=|(v_s\chi_I)^*(t_0)t_0|$$

for some $t_0\in(0,|I|]$, and there exist $\beta>0$ such that $|M_\beta|=t_0$. Choose the optimal c_0,d_0 such that $M_\beta\subseteq[c_0,d_0]\subseteq\bar{I}$. Then, with $I=(c,d)$,

$$\begin{aligned}A(I;\gamma,v_s)&\ge|\gamma|\inf_\alpha\|v_s(y)\left(\int_c^y dt-\alpha\right)\|_{\infty,I}\\&\ge|\gamma|\inf_\alpha\|\beta\chi_{M_\beta}(y)(y-c-\alpha)\|_{\infty,I}\\&=\beta|\gamma|\|y-c-\frac{1}{2}(c_0+d_0-2c)\|_{\infty,M_\beta}\\&=\frac{1}{2}\beta|\gamma|(d_0-c_0)\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{1}{2}\beta|\gamma||M_\beta| \\
&= \frac{1}{2}|\gamma|\left[(v_s\chi_I)^*(t_0)t_0\right] \\
&= \frac{1}{2}|\gamma|\|(v_s\chi_I)^*(t)t\|_{\infty,(0,|I|)}.
\end{aligned}
$$

The lemma is therefore proved. □

Lemma 4.6 *Let $I \subset\subset (a,b)$ and $\gamma, \delta \in \Re$ with $\delta \geq v_s(t) \geq 0$ on I. Then, for any $\alpha > 1$*

$$
A(I;\gamma,\delta) - A(I;\gamma,v_s) \leq \frac{\alpha}{2}\int_I |\gamma|\left(\delta - v_s(t)\right)dt + \frac{|\gamma|\delta|I|}{2\alpha}. \qquad (4.\ 7)
$$

Proof. We first observe that

$$
(v_s\chi_I)^*(t) \geq v_0(t) := \left(\delta - \frac{V\alpha}{|\gamma||I|}\right)\chi_{(0,|I|-\frac{|I|}{\alpha})} \qquad (4.\ 8)
$$

where $V = |\gamma|\int_I(\delta - v_s(t))dt$. For, with $S := \left\{x : v_s(x) < \delta - \frac{V\alpha}{|\gamma||I|}\right\}$

$$
\frac{V}{|\gamma|} > \int_S \left(\delta - \delta + \frac{V\alpha}{|\gamma||I|}\right)dt = \frac{V\alpha}{|\gamma||I|}|S|
$$

which implies that

$$
\left|\left\{x : v_s(x) > \delta - \frac{V\alpha}{|\gamma||I|}\right\}\right| > |I| - \frac{|I|}{\alpha}
$$

and hence (4. 8). Note that (4. 8) is trivially true if $\delta - \frac{V\alpha}{|\gamma||I|} < 0$. On using (4. 1) and (4. 6)

$$
\begin{aligned}
0 &\leq A(I;\gamma,\delta) - A(I;\gamma,v_s) \leq \frac{1}{2}|\gamma|\delta|I| - \frac{1}{2}|\gamma|\|(v_s\chi_I)^*(t)t\|_{\infty,(0,|I|)} \\
&\leq \frac{1}{2}|\gamma|\delta|I| - \frac{1}{2}\max_{(0,|I|]}(tv_0(t)) \\
&= \frac{1}{2}|\gamma|\delta|I| - \frac{1}{2}|\gamma|\left(\delta - \frac{V\alpha}{|\gamma||I|}\right)\left(|I| - \frac{|I|}{\alpha}\right) \\
&= \frac{\alpha V}{2} + \frac{|\gamma|\delta|I|}{2\alpha} - \frac{V}{2} \\
&\leq \frac{\alpha}{2}\int_I |\gamma|\left(\delta - v_s(t)\right)dt + \frac{|I|}{2\alpha}|\gamma|\delta
\end{aligned}
$$

which is (4. 7). □

Theorem 4.7 *For any $I \subset\subset (a,b)$*

$$\frac{1}{2}\int_I |u(t)|v_s(t)dt \leq \liminf_{\varepsilon\to 0_+} \varepsilon M(I,\varepsilon) \leq \limsup_{\varepsilon\to 0_+} \varepsilon M(I,\varepsilon) \leq \int_I |u(t)|v_s(t)dt. \tag{4. 9}$$

Proof. On using Lemma 4.2, we infer that for each $\eta > 0$ there exist step functions u_η, v_η on I such that

$$\begin{aligned} \|u-u_\eta\|_{1,I} &< \eta \\ \int_I |u(t)|\,(v_\eta(t)-v_s(t))\,dt &< \eta \end{aligned}$$

and

$$\|v_s\|_{\infty,I} \geq v_\eta(t) \geq v_s(t)$$

on I. We may assume that

$$u_\eta = \sum_{j=1}^m \xi_j \chi_{W(j)}, \qquad v_\eta = \sum_{j=1}^m \eta_j \chi_{W(j)},$$

where the $W(j)$ are disjoint subintervals of I, and $\eta_j \geq 0$.

Let $\varepsilon > 0, M \equiv M(I,\varepsilon)$, and let $c_k \equiv c_k(\varepsilon), k = 1,2,\ldots,M+1$, be the end-points of the intervals in (2. 7): with $I = [c,d]$ and $I_k \equiv I_k(\varepsilon) = [c_k, c_{k+1}]$, we have $c = c_1 < c_2 < \ldots < c_{M+1} = d$ and

$$\begin{aligned} A(I_k) \equiv A(I_k;u,v) \leq \varepsilon \quad &, k = 1,2,\ldots,M, \\ A(I_k \cup I_{k+1}) > \varepsilon \quad &, k = 1,2,\ldots,M-1. \end{aligned}$$

Then

$$\begin{aligned} \left|\int_I |u(t)|\,v_s(t)dt \right. &- \left. \int_I |u_\eta(t)|v_\eta(t)dt\right| \\ &\leq \int_I |u(t)|\,(v_\eta(t)-v_s(t))\,dt + \int_I |u(t)-u_\eta(t)|v_\eta(t)dt \\ &< \eta(1+\|v_\eta\|_{\infty,I}) \\ &\leq \eta(1+\|v_s\|_{\infty,I}). \end{aligned} \tag{4. 10}$$

Next, let $\mathbf{K} := \{k : \text{there exist } j \text{ such that} I_{2k} \cup I_{2k+1} \subset W(j)\}$. Then $\#\mathbf{K} \geq [\frac{M}{2}] - 2m \geq \frac{M}{2} - 1 - 2m$, and, by Lemmas 4.4 and 4.6,

$$\left(\frac{M}{2} - 1 - 2m\right)\varepsilon \quad \leq \quad \sum_{k\in\mathbf{K}} A(I_{2k} \cup I_{2k+1}; u, v)$$

$$
\begin{aligned}
&\le \sum_{k\in\mathbf{K}} \{A(I_{2k}\cup I_{2k+1}; u_\eta, v_\eta) \\
&\qquad + (A(I_{2k}\cup I_{2k+1}; u, v_s) - A(I_{2k}\cup I_{2k+1}; u_\eta, v_s)) \\
&\qquad + (A(I_{2k}\cup I_{2k+1}; u_\eta, v_s) - A(I_{2k}\cup I_{2k+1}; u_\eta, v_\eta))\} \\
&\le \frac{1}{2}\sum_j |\xi_j|\eta_j|W(j)| \\
&\qquad + \sum_j \{\|u-u_\eta\|_{1,W(j)}\|v_s\|_{\infty,W(j)} \\
&\qquad\quad + \frac{\alpha}{2}\int_{W(j)} |\xi_j|(v_\eta - v_s)dt + \frac{|\xi_j|\ \eta_j}{2\alpha}|W(j)|\} \\
&\le \frac{1}{2}\int_I |u_\eta|v_\eta dt + \|u-u_\eta\|_{1,I}\|v_s\|_{\infty,I} \\
&\qquad + \frac{\alpha}{2}\int_I |u_\eta|(v_\eta - v_s)dt + \frac{1}{2\alpha}\int_I |u_\eta|v_\eta dt \\
&\le \frac{1}{2}\int_I |u_\eta|v_\eta dt + K\ (\alpha\eta + \frac{1}{\alpha}) \\
&\le \frac{1}{2}\int_I |u(t)|v_s(t)dt + K\left(\alpha\eta + \frac{1}{\alpha}\right)
\end{aligned}
$$

by (4. 10), for some constant K independent of ε. We therefore conclude that

$$
\limsup_{\varepsilon\to 0_+} \varepsilon M(I,\varepsilon) \le \int_I |u(t)|v_s(t)dt + K\left(\alpha\eta + \frac{1}{\alpha}\right)
$$

and the right–hand inequality in (4. 9) follows since $\eta > 0$ and $\alpha > 1$ are arbitrary.

For the left–hand inequality in (4. 9), we add the end–points of the intervals $W(j), j = 1, 2\ldots, m$ to the $c_k, k = 1, 2, \ldots, M-1$, to form the partition $c = e_1 < \ldots < e_n = d$, say, where $n \le M+1+m$. Note that each interval $J_i := [e_i, e_{i+1}]$ is a subinterval of some $W(j)$ and hence u_η, v_η have constant values on each J_i. We again use Lemmas 4.3, 4.4 and 4.6 to get

$$
\begin{aligned}
\frac{1}{2}\int_I |u_\eta|v_\eta dt &= \sum_{j=1}^{m}\sum_{J_i\subseteq W(j)} A(J_i; u_\eta, v_\eta) \\
&\le \sum_{i=1}^{n}\Bigg\{ A(J_i; u, v_s) + \|u-u_\eta\|_{1,J_i}\|v_s\|_{\infty,J_i} \\
&\qquad \frac{\alpha}{2}\int_{J_i} |u_\eta|(v_\eta - v_s)dt + \frac{1}{2\alpha}\int_{J_i} |u_\eta|v_\eta dt \Bigg\} \\
&\le (M+1+m)\varepsilon + K\left(\alpha\eta + \frac{1}{\alpha}\right).
\end{aligned}
$$

Hence, from (4. 10),

$$\frac{1}{2}\int_I |u(t)|v_s(t)dt \le (M+1+m)\varepsilon + K\left(\alpha\eta + \frac{1}{\alpha}\right)$$

and the left–hand inequality in (4. 9) follows. □

5 The main result.

With $U(x) := \int_a^x |u(t)|dt$, we define $\xi_k \in \Re^+$ by

$$U(\xi_k) = 2^k; \tag{5. 1}$$

if $u \notin L^1(a,b)$, k may by any integer, but if $u \in L^1(a,b), 2^k \le \|u\|_1$. For each admissible k we set

$$\sigma_k := \|uv\|_{\infty, Z_k}, \qquad Z_k = (\xi_k, \xi_{k+1}), \tag{5. 2}$$

so that

$$2^k \|v\|_{\infty, Z_k} \le \sigma_k \le 2^{k+1}\|v\|_{\infty, Z_k}. \tag{5. 3}$$

For non–admissible k we set $\sigma_k = 0$. The sequence $\{\sigma_k\}$ is the analogue of that defined in [2,§3], which in turn was motivated by a similar sequence introduced in [5].

The following technical lemma has a central role in this section.

Lemma 5.1 *Let $k_0, k_1, k_2 \in \mathbf{Z}$ with $k_0 < k_1 < k_2$, and let $I_j = (a_j, b_j)$ $(j = 0, 1, \ldots, l)$ be non–overlapping intervals in (a,b) which are such that $I_j \subset Z_{k_2}$ $(j = 1, \ldots, l)$, $a_0 \in Z_{k_0}$, $b_0 \in Z_{k_2}$. Let $x_j \in I_j$ $(j = 0, 1, \ldots, l)$ and $x_0 \in Z_{k_1}$. Then, if $\alpha \ge 1$,*

$$\begin{aligned} S &:= \sum_{j=0}^{l} \left(\int_{a_j}^{x_j} |u(t)|dt\right)^\alpha \|v\|^\alpha_{\infty,(x_j,b_j)} \\ &\le (2^\alpha + 1) \max_{k_0 \le n \le k_2} \sigma_n^\alpha. \end{aligned} \tag{5. 4}$$

Proof. On using Jensen's inequality, we have

$$\begin{aligned} S &\le \left(\int_{\xi_{k_0}}^{\xi_{k_1+1}} |u(t)|dt\right)^\alpha \|v\|^\alpha_{\infty,(\xi_{k_1},\xi_{k_2+1})} + \sum_{j=1}^{l}\left(\int_{I_j} |u(t)|dt\right)^\alpha \|v\|^\alpha_{\infty, I_j} \\ &\le \left\{(2^{k_1+1} - 2^{k_0}) \max_{k_1 \le n \le k_2} \frac{\sigma_n}{2^n}\right\}^\alpha + \left(\int_{Z_{k_2}} |u(t)|dt\right)^\alpha \|v\|^\alpha_{\infty, Z_{k_2}} \end{aligned}$$

by (5. 3),

$$\le \left\{2 \max_{k_1 \le n \le k_2} \sigma_n\right\}^\alpha + \left\{2^{k_2}\frac{\sigma_{k_2}}{2^{k_2}}\right\}^\alpha$$

whence (5. 4). □

Lemma 5.2 *The quantity $J(a,b)$ defined in (2. 3) satisfies*

$$\frac{1}{3}J(a,b) \le \sup_k \sigma_k \le 2J(a,b). \qquad (5.\ 5)$$

Proof. From (2. 4) and Lemma 5.1

$$J(a,b) \le 3 \sup_k \sigma_k.$$

Also

$$\begin{aligned} \sigma_k &\le 2^{k+1}\|v\|_{\infty,Z_k} \\ &\le 2\int_a^{\xi_k} |u(t)|dt\|v\|_{\infty,(\xi_k,b)} \\ &\le 2J(a,b). \end{aligned}$$

□

Corollary 5.3 *The operator $T : L^\infty(a,b) \to L^\infty(a,b)$ is bounded if and only if the sequence $\{\sigma_k\}$ is bounded, in which case their norms are equivalent:*

$$\|T\| \asymp \|\{\sigma_k\}\|_\infty. \qquad (5.\ 6)$$

Also T is compact if and only if $\lim_{k\to\pm\infty} \sigma_k = 0$.

Proof. The first part is an immediate consequence of Proposition 2.2 and Lemma 5.2. We also have from Lemma 5.2, as in the proof of Lemma 5.2,

$$\frac{1}{3}J(a,\xi_{k_2}) \le \max_{n\le k_2} \sigma_n \le 2J(a,\xi_{k_2+1})$$

and

$$\frac{1}{3}J(\xi_{k_0},b) \le \max_{n\ge k_0} \sigma_n \le 2J(\xi_{k_0-1},b).$$

Since $\xi_{k_2} \to a$ if and only if $k_2 \to -\infty$, and $\xi_{k_0} \to b$ if and only if $k_0 \to \infty$ in the case $u \notin L^1(a,b)$ and otherwise to the largest admissible value of k in the definition of σ_k, the corollary follows. □

The main result is

Theorem 5.4 *Suppose that (2. 1) and (2. 2) are satisfied, T is compact, and that $\sum_{n\in\mathbf{Z}} \sigma_n$ is convergent. Then*

$$\frac{1}{4}\int_a^b |u(t)|v_s(t)dt \le \liminf_{n\to\infty} na_n(T) \le \limsup_{n\to\infty} na_n(T) \le 2\int_a^b |u(t)|v_s(t)dt. \qquad (5.\ 7)$$

Proof. Let $I=[c,d]\subset\subset(a,b)$ and suppose that $c\in[\xi_{k_0},\xi_{k_0+1}]$, $d\in[\xi_{k_1},\xi_{k_1+1})$. With $I_j(\varepsilon), j=1,2,\dots,M(\varepsilon)$, the covering of (a,b) in (2. 7), where $M(\varepsilon)\equiv M((a,b),\varepsilon)$, let

$$m_0(\varepsilon)=\#\{j:I_j(\varepsilon)\subset[a,c]\}$$
$$m_1(\varepsilon)=\#\{j:I_j(\varepsilon)\subset[a,d]\}.$$

Then

$$m_1(\varepsilon)-m_0(\varepsilon)\le M(I,\varepsilon)+1$$

and

$$\begin{aligned}\frac{\varepsilon}{2}\left(M(\varepsilon)-M(I,\varepsilon)-9\right) &\le \varepsilon\left(\left[\frac{m_0(\varepsilon)}{2}\right]+\left[\frac{M(\varepsilon)}{2}\right]-\left[\frac{m_1(\varepsilon)}{2}\right]-2\right)\\ &\le \sum_{j=1}^{[m_0(\varepsilon)/2]}A(I_{2j-1}\cup I_{2j};u,v)+\sum_{j=[m_1(\varepsilon)/2]+2}^{[M(\varepsilon)/2]}A(I_{2j-1}\cup I_{2j};u,v)\\ &\le \sum_{j=1}^{[m_0(\varepsilon)/2]}J(I_{2j-1}\cup I_{2j};u,v)+\sum_{j=[m_1(\varepsilon)/2]+2}^{[M(\varepsilon)/2]}J(I_{2j-1}\cup I_{2j};u,v)\\ &\le 3\sum_{n\le k_0}\sigma_n+3\sum_{n\ge k_1}\sigma_n\end{aligned}$$

on using (2. 9) and (5. 5).

It follows from Theorem 4.7 that

$$\limsup_{\varepsilon\to 0_+}\varepsilon M(\varepsilon)\le\int_{\xi_{k_0}}^{\xi_{k_1+1}}|u(t)|v_s(t)dt+3\left(\sum_{n\le k_0}\sigma_n+\sum_{n\ge k_1}\sigma_n\right)$$

which yields,

$$\limsup_{\varepsilon\to 0_+}\varepsilon M(\varepsilon)\le\int_a^b|u(t)|v_s(t)dt.$$

On setting $n=M(\varepsilon)+1$ in Corollary 3.3, we get $\varepsilon\ge\frac{1}{2}a_n(T)$ and hence

$$\limsup_{n\to\infty}na_n(T)\le 2\int_a^b|u(t)|v_s(t)dt.$$

Similarly, from Theorem 4.7,

$$\liminf_{\varepsilon\to 0_+}\varepsilon M(\varepsilon)\ge\frac{1}{2}\int_a^b|u(t)|v_s(T)dt$$

and from Corollary 3.3

$$\liminf_{n\to\infty}na_n(T)\ge\frac{1}{4}\int_a^b|u(t)|v_s(T)dt.$$

□

6 l^q and weak–l^q estimates.

In this section we show that the sequences $\{a_n(T)\}_{n\in\mathbf{N}}, \{\sigma_n\}_{n\in\mathbf{Z}}$ belong to l^q and weak–l^q sequence spaces with the same exponent q, and have equivalent norms. We first need some preparatory results.

Lemma 6.1 *Let $I = [c,d] \subset (a,b)$ and, for $\varepsilon > 0$, suppose that*

$$\sigma(\varepsilon) := \{k \in \mathbf{Z} : Z_k \subset I, \sigma_k > \varepsilon\}$$

has at least 4 distinct elements. Then $A(I) > \varepsilon/8$.

Proof. Let $Z_{k_i}, i = 1,2,3,4$, $k_1 < k_2 < k_3 < k_4$, be 4 distinct members of $\sigma(\varepsilon)$, and set $I_1 = (\xi_{k_1}, \xi_{k_2})$, $I_2 = (\xi_{k_2+1}, \xi_{k_4})$. Then, with $f_0 = \chi_{I_1} + \chi_{I_2}$,

$$\begin{aligned}
A(I) &\geq \inf_\alpha \|v(x)\left(\int_c^x |u(t)|f_0(t)dt - \alpha\right)\|_{\infty,I} \\
&\geq \inf_\alpha \max\left\{\|v\|_{\infty,Z_{k_2}}\left|\int_{I_1}|u(t)|dt - \alpha\right|; \|v\|_{\infty,Z_{k_4}}\left|\int_{I_1\cup I_2}|u(t)|dt - \alpha\right|\right\} \\
&= \inf_\alpha \max\left\{\|v\|_{\infty,Z_{k_2}}|2^{k_2} - 2^{k_1} - \alpha|; \|v\|_{\infty,Z_{k_4}}|2^{k_2} - 2^{k_1} + 2^{k_4} - 2^{k_2+1} - \alpha|\right\} \\
&\geq \inf_\alpha \max\left\{\frac{\varepsilon}{2^{k_2+1}}|2^{k_2} - 2^{k_1} - \alpha|; \frac{\varepsilon}{2^{k_4+1}}|2^{k_2} - 2^{k_1} + 2^{k_4} - 2^{k_2+1} - \alpha|\right\} \\
&\geq \frac{\varepsilon}{2^{k_4}+1}\frac{1}{2}\left(2^{k_4} - 2^{k_2+1}\right) \geq \frac{\varepsilon}{8}.
\end{aligned}$$

□

Lemma 6.2 *Let $\varepsilon > 0$ and $M(\varepsilon) = M((a,b),\varepsilon)$. Then*

$$\#\{k \in \mathbf{Z} : \sigma_k > 8\varepsilon\} \leq 5M(\varepsilon) + 3. \tag{6. 1}$$

Proof. Clearly, with $I_i = (c_i, c_{i+1})$ the intervals in (2. 7) when $I = (a,b)$,

$$\#\left\{k \in \mathbf{Z} : c_i \in \bar{Z}_k \text{ for some } i \in \{1,2,\ldots,M(\varepsilon)\}\right\} \leq 2M(\varepsilon).$$

Also, for every $k \in \mathbf{Z}$ not included in the above set, $\bar{Z}_k \subset I_i$ for some $i \in \{1,2,\ldots,M(\varepsilon)\}$. Hence, by Lemma 6.1

$$\begin{aligned}
\#\{k \in \mathbf{Z} : \sigma_k > 8\varepsilon\} &\leq 2M(\varepsilon) + 3(M(\varepsilon)+1) \\
&= 5M(\varepsilon) + 3.
\end{aligned}$$

□

Lemma 6.3 *For all $t > 0$*

$$\#\{k \in \mathbf{Z} : \sigma_k > t\} \leq 10\#\{k \in \mathbf{N} : a_k(T) > t/8\} + 23. \tag{6. 2}$$

Proof. By Corollary 3.3,

$$\#\{k \in \mathbf{N} : a_k(T) > \varepsilon\} \geq \frac{M(\varepsilon)}{2} - 2$$

. Hence, by Lemma 6.2,

$$\begin{aligned} \#\{k \in \mathbf{Z} : \sigma_k > t\} &\leq 5M(t/8) + 3 \\ &\leq \#\{k \in \mathbf{N} : a_k(T) > t/8\} + 23. \end{aligned}$$

□

Lemma 6.4 *For all $q > 0$*

$$\|\{\sigma_k\}\|^q_{l^q(\mathbf{Z})} \leq 10\ (8^q)\|\{a_k(T)\}\|^q_{l^q(\mathbf{N})} + 23\|\{\sigma_k\}\|^q_{l^\infty(\mathbf{Z})}. \tag{6. 3}$$

Proof. Let $\lambda = \|\{\sigma_k\}\|_{l^\infty(\mathbf{Z})}$. Then, by Lemma 6.3,

$$\begin{aligned} \|\{\sigma_k\}\|^q_{l^q(\mathbf{Z})} &= q\int_0^\lambda t^{q-1}\#\{k \in \mathbf{Z} : \sigma_k > t\}\,dt \\ &\leq 10\ q\int_0^\lambda t^{q-1}\#\{k \in \mathbf{N} : a_k(T) > t/8\}\,dt + 23\lambda^q \\ &\leq 10\ (8^q)\|\{a_k(T)\}\|^q_{l^q(\mathbf{N})} + 23\lambda^q. \end{aligned}$$

□

Corollary 6.5 *For any $q > 0$ there exists a constant $C > 0$ such that*

$$\|\{\sigma_k\}\|_{l^q(\mathbf{Z})} \leq C\|\{a_k(T)\}\|_{L^q(\mathbf{N})}. \tag{6. 4}$$

Proof. By (5. 6),

$$\begin{aligned} \|\{\sigma_k\}\|_{l^\infty(\mathbf{Z})} &\leq C\|T\| = Ca_1(T) \\ &\leq C\|\{a_k(T)\}\|_{l^q(\mathbf{N})}. \end{aligned}$$

The result then follows from Lemma 6.4. □

Theorem 6.6 *For $q \in (1,\infty)$, $\{a_k(T)\} \in l^q(\mathbf{N})$ if and only if $\{\sigma_k\} \in l^q(\mathbf{Z})$, and*

$$\|\{\sigma_k\}\|_{l^q(\mathbf{Z})} \asymp \|\{a_k(T)\}\|_{l^q(\mathbf{N})}.$$

Proof. Let $I_i, i = 1, 2, \ldots, N(\varepsilon)$, be the intervals in (2. 8) with $I = (a, b)$ and $N(\varepsilon) \equiv N((a, b), \varepsilon)$: note that in view of Lemma 2.1, we have $J(I_i) = \varepsilon$. We group the intervals I_i into families $\mathbf{F}_j, j = 1, 2, \ldots$ such that each $\mathbf{F}_j$ consists of the maximal number of those intervals satisfying the hypothesis of Lemma 5.1

: they lie within (ξ_{k_0}, ξ_{k_2+1}) for some k_0, k_2, and the next interval I_k intersects Z_{k_2+1}. Hence, by Lemma 5.1, there is a positive constant c such that

$$\varepsilon \# \mathbf{F}_j \leq c \max_{k_0 \leq n \leq k_2} \sigma_n = c\sigma_{k_j}$$

say. It follows that, with $n_j = [c\sigma_{k_j}/\varepsilon]$,

$$\begin{aligned} N(\varepsilon) &= \sum_j \#\mathcal{F}_j \\ &\leq \sum_j \sum_{n=1}^{n_j} 1 = \sum_{n=1}^{\infty} \sum_{j:n_j \geq n} 1 \\ &= \sum_{n=1}^{\infty} \# \left\{ j : \frac{c\sigma_{k_j}}{\varepsilon} \geq n \right\} \\ &\leq \sum_{n=1}^{\infty} \# \left\{ k : \sigma_k \geq \frac{n\varepsilon}{c} \right\}. \end{aligned} \tag{6. 5}$$

Thus, if $\{\sigma_k\} \in l^q(\mathbf{Z})$ for some $q \in (1, \infty)$,

$$\begin{aligned} q \int_0^\infty t^{q-1} N(t) dt &\leq q \int_0^\infty \sum_{n=1}^{\infty} t^{q-1} \# \left\{ k : \sigma_k > \frac{nt}{c} \right\} dt \\ &= qc^q \int_0^\infty \sum_{n=1}^{\infty} n^{-q} s^{q-1} \# \{ k : \sigma_k > s \} ds \\ &\preceq \| \{\sigma_k\} \|_{l^q(\mathbf{Z})}^q \end{aligned} \tag{6. 6}$$

where $\preceq$ stands for less than or equal to a constant multiple of what follows. From Corollary 3.3, $a_{M(\varepsilon)+1}(T) \leq 2\varepsilon$ and so

$$\begin{aligned} \# \{ k \in \mathbf{N} : a_k(T) > t \} &\leq M(t/2) + 1 \\ &\leq N(t/2) + 1. \end{aligned}$$

This yields

$$\begin{aligned} \| \{a_k(T)\} \|_{l^q(\mathbf{N})}^q &= q \int_0^\infty t^{q-1} \# \{ k \in \mathbf{N} : a_k(T) > t \} dt \\ &\leq q \int_0^{\|T\|} t^{q-1} \left[N(\frac{t}{2}) + 1 \right] dt \\ &\preceq \| \{\sigma_k\} \|_{l^q(\mathbf{Z})}^q + \|T\|^q \\ &\preceq \| \{\sigma_k\} \|_{l^q(\mathbf{Z})}^q \end{aligned}$$

by (6. 6) and since $\|T\| \preceq \| \{\sigma_k(T)\} \|_{l^\infty(\mathbf{Z})} \leq \| \{\sigma_k\} \|_{l^q(\mathbf{Z})}$, by (5. 6). The theorem follows from (6. 4). □

The final result in this section concerns the weak l^q spaces, which we denote by l^q_ω ($l^{q,\infty}$ in the Lorentz scale). Recall that $l^q_\omega(\mathbf{Z})$ is the space of sequences $x = \{x_k\}$ such that

$$\|x\|_{l^q_\omega(\mathbf{Z})} := \sup_{t>0} \left\{ t \left[\# \{k \in \mathbf{Z} : |x_k| > t\} \right]^{1/q} \right\} < \infty.$$

The space $l^q_\omega(\mathbf{N})$ is defined analogously.

Theorem 6.7 *For $q \in (1, \infty)$, $\{a_k(T)\} \in l^q_\omega(\mathbf{N})$ if and only if $\{\sigma_k\} \in l^q_\omega(\mathbf{Z})$, and*

$$\| \{\sigma_k\} \|_{l^q_\omega(\mathbf{Z})} \asymp \| \{a_k(T)\} \|_{l^q_\omega(\mathbf{N})}.$$

Proof. Suppose $\{\sigma_k\} \in l^q_\omega(\mathbf{Z})$. From Corollary 3.3 and (6. 5)

$$\begin{aligned} \| \{a_k(T)\} \|^q_{l^q_\omega(\mathbf{N})} &\preceq \sup_{t>0} \{t^q M(t)\} \\ &\leq \sup_{t>0} \{t^q N(t)\} \\ &\leq \sum_{n=1}^{\infty} t^q \# \{k : \sigma_k \geq nt/c\} \\ &\leq \sum_{n=1}^{\infty} \| \{\sigma_k\} \|^q_{l^q_\omega(\mathbf{Z})} \left(\frac{c}{n}\right)^q \preceq \| \{\sigma_k\} \|^q_{l^q_\omega(\mathbf{Z})}. \end{aligned}$$

Now suppose that $\{a_k(T)\} \in l^q_\omega(\mathbf{N})$. From Lemma 6.3

$$\sup_{t>0} \left(t^q \# \{k \in \mathbf{Z} : \sigma_k > t\}\right) \preceq \sup_{t>0} \left(t^q \left(\# \left\{ k \in \mathbf{N} : a_k(T) > \frac{t}{8} \right\} + 1 \right) \right).$$

Since

$$\# \left\{ k \in \mathbf{N} : a_k(T) > \frac{t}{8} \right\} \geq \frac{M(t/8)}{2} - 2 \geq 1$$

for sufficiently small t, we conclude that

$$\sup_{t>0} \left(t^q \# \{k \in \mathbf{Z} : \sigma_k > t\}\right) \preceq \sup_{t>0} \left(t^q \# \left\{ k \in \mathbf{N} : a_k(T) > \frac{t}{8} \right\} \right).$$

This implies that $\{\sigma_k\} \in l^q_\omega(\mathbf{Z})$ and and $\| \{\sigma_k\} \|_{l^q_\omega(\mathbf{Z})} \preceq \| \{a_k(T)\} \|_{l^q_\omega(\mathbf{N})}$. The theorem is therefore proved. □

7 The operator T on L^1.

In this case the assumptions (2. 1) and (2. 2) on u and v are replaced by

$$u \in L^\infty(a, x), \qquad (7.\ 1)$$

$$v \in L^1(x, b), \qquad (7.\ 2)$$

for all $x \in (a,b)$. On setting $a = -B$, $b = -A$, $\hat{f}(x) = f(-x)$, and similarly for u, v in (1. 4), we see that

$$T\hat{f}(x) = \hat{v}\ (x) \int_x^B \hat{u}\ (t)\hat{f}(t)dt, \quad A \leq x \leq B.$$

But this is the adjoint of the map $S : L^\infty(A,B) \to L^\infty(A,B)$ defined by

$$Sg(x) = \hat{u}\ (x) \int_A^x \hat{v}\ (t)g(t)dt, \quad A \leq x \leq B.$$

Hence, T and S have the same norms and their approximation numbers are equal if one, and hence both, are compact (see [3; Proposition II.2.5]). The results for $T : L^1(a,b) \to L^1(a,b)$ therefore follow from those proved for the $L^\infty(a,b)$ case on interchanging u and v. Before stating the results, we need some new terminology.

Let $\eta_k \in \Re^+$ be defined by

$$V(x) := \int_x^b |v(t)|dt, \qquad V(\eta_k) = 2^k, \tag{7. 3}$$

where $k \in \mathbf{Z}$ if $v \in L^1(a,b)$, but otherwise $2^k \leq \|v\|_1$. Set

$$\zeta_k := \|uv\|_{\infty, W_k}, \qquad W_k = (\eta_k, \eta_{k+1})$$

with $\zeta_k = 0$ if $v \in L^1(a,b)$ and $2^k > \|v\|_1$.

Theorem 7.1 *Suppose that (7. 1) and (7. 2) are satisfied. Then*

(i) *T in (1. 4), as a map from $L^1(a,b)$ into $L^1(a,b)$ is bounded if and only if $\{\zeta_k\} \in l_\infty(\mathbf{Z})$, in which case*

$$\|T\| \asymp \| \{\zeta_k\} \|_{l_\infty(\mathbf{Z})};$$

(ii) *T is compact if and only if $\lim_{k\to\pm\infty} \zeta_k = 0$;*

(iii) *if $\{\zeta_k\} \in l_1(\mathbf{Z})$*

$$\frac{1}{4}\int_a^b u_s(t)|v(t)|dt \leq \liminf_{n\to\infty} na_n(T) \leq \limsup_{n\to\infty} na_n(T) \leq 2\int_a^b u_s(t)|v(t)|dt;$$

(iv) *for $q \in (1,\infty)$, $\{a_k(T)\} \in l^q(\mathbf{N})$ if and only if $\{\zeta_k\} \in l^q(\mathbf{Z})$ and*

$$\| \{\zeta_k\} \|_{l^q(\mathbf{Z})} \asymp \| \{a_k(T)\} \|_{l^q(\mathbf{N})};$$

(v) *for $q \in (1,\infty)$, $\{a_k(T)\} \in l^q_\omega(\mathbf{N})$ if and only if $\{\zeta_k\} \in l^q_\omega(\mathbf{Z})$ and*

$$\| \{\zeta_k\} \|_{l^q_\omega(\mathbf{Z})} \asymp \| \{a_k(T)\} \|_{l^q_\omega(\mathbf{N})}.$$

Remark 7.2 *Let M be a dense subset of* $(0,1)$ *with measure* $|M| = \alpha < 1$ *and let* $u = 1, v = \chi_M$. *Then* $v_s = 1,\ (v - v_s)_s = 1$ *on* $(0,1)$ *and so*

$$\|v\|_{\infty,(x,1)} = \|v_s\|_{\infty,(x,1)} = \|v - v_s\|_{\infty,(x,1)}$$

for any $x \in (0,1)$. *Since*

$$\|T_{u,v}|L^\infty(0,1) \to L^\infty(0,1)\| = \sup_{0<x<1} \left\{ \int_0^x dt \|v\|_{\infty,(x,1)} \right\}$$

(see[6]), where $T_{u,v}$ *denotes the operator in (1. 4), it follows that*

$$\|T_{u,v}\| = \|T_{u,v_s}\| = \|T_{u,v} - T_{u,v-v_s}\|,$$

for the operator norms from $L^\infty(0,1)$ *to* $L^\infty(0,1)$. *Also*

$$\int_0^1 |u(t)v(t)|dt = |M| < 1 = \int_0^1 |u(t)|v_s(t)dt.$$

The choice $u = \chi_M,\ v = 1$ *gives an analogous example in the* $L^1(0,1)$ *case.*

Acknowledgement. J. Lang wishes to record his gratitude to the Royal Society and Nato for support to visit the School of Mathematics at Cardiff during 1997/8, under their Postdoctoral Fellowship programme. He also thanks the Grant Agency of the Czech Republic for partial support under grant No. 201/96/0431.

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School of Mathematics,
University of Wales, Cardiff,
Senghennydd Road,
Cardiff CF2 4YH, U.K.