The generalized Hardy operator with kernel and variable integral limits in Banach function spaces

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Let we have an integral operator

$$Kf(x) := v(x) \int_{a(x)}^{b(x)} k(x,y)u(y)f(y)dy \qquad \text{for } x > 0$$

where a and b are nondecreasing functions, u and v are non-negative and finite functions, and $k(x,y) \geq 0$ is non-decreasing in x, non-increasing in y and $k(x,z) \leq D[k(x,b(y)) + k(y,z)]$ for $y \leq x$ and $a(x) \leq z \leq b(y)$. We show that the integral operator $K : X \to Y$ where X and Y are Banach functions spaces with l-condition is bounded if and only if $A < \infty$. Where $A := A_0 + A_1$ and

$$\begin{aligned} A_0 &:= \sup_{x \leq y, a(y) \leq b(x)} \|\chi_{(x,y)}(.)v(.)k(., b(x))\|_Y \|\chi_{(a(y),b(x))}u\|_{X'} \\ A_1 &:= \sup_{x \leq y, a(y) \leq b(x)} \|\chi_{(x,y)}v\|_Y \|\chi_{(a(y),b(x))}(.)k(x,.)u(.)\|_{X'}. \end{aligned}$$

Keywords: Banach function spaces, Hardy operator, integral operator, l-condition

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1 Introduction

Let X and Y be two Banach function spaces on (c,d) and $\mathbb{R}$ respectively. We define the general Hardy operator

$$Kf(x) := v(x) \int_{a(x)}^{b(x)} k(x,y)u(y)f(y)dy \qquad \text{for } x \in \mathbb{R} \tag{1. 1}$$

where a, b are nondecreasing functions on $\mathbb{R}$, $-\infty \leq c \leq a(x) \leq b(x) \leq d \leq \infty$, and v and u are non-negative measurable and finite functions a.e. on $\mathbb{R}$ and

(c, d). The kernel $k(x, y) \geq 0$ is defined a.e. on $\{(x, y); x \in \mathbb{R}, a(x) \leq y \leq b(x)\}$ and satisfies the following conditions:

(i) it is non-decreasing in x and non-increasing in y;
(ii) $k(x, z) \leq D\,[k(x, b(y)) + k(y, z)]$ for every $y \leq x$ and $a(x) \leq z \leq b(y)$, where the constant $D > 1$ is independent of x, y, z. (1. 2)

In this paper we describe the necessary and sufficient condition for the boundedness of the operator (1. 1) in Banach function spaces.

This paper extends results of Lomakina and Stepanov [3] and Opic and Kufner [4]. In these papers the operator (1. 1) was characterized for $a(x) = 0$ and $b(x) = x$.

Section 2 and 3 contain the definitions, formulations of the main results and some comments. In Section 4 we treat the simpler case when the kernel $k(x, y)$ is equal to 1 and the spaces X, Y satisfy the l-condition. We use this result in section 5 to deal with the general kernel satisfying (1. 2).

2 Definitions

In this section we recall the definition and some basic properties of the Banach function spaces. In what follows $\mathcal{M}(\Omega)$ will be the set of all measurable functions on Ω, where Ω is any measurable subset of $\mathbb{R}$.

Definition 2.1 *A normed linear space $(X, \|.\|_X)$ on Ω is called a Banach function space (BFS) on Ω if the following conditions are satisfied:*

(2.1) *the norm $\|f\|_X$ is defined for all $f \in \mathcal{M}(\Omega)$ and $f \in X$ if and only if $\|f\|_X < \infty$;*

(2.2) $\|f\|_X = \|\,|f|\,\|_X$ *for every $f \in \mathcal{M}(\Omega)$;*

(2.3) *if $0 \leq f_n \nearrow f$ a.e. in Ω then $\|f_n\|_X \nearrow \|f\|_X$;*

(2.4) *if $|E| < \infty$, $E \subset \Omega$, then $\chi_E \in X$;*

(2.5) *for every set E, $|E| < \infty, E \subset \Omega$, there exists a positive constant C_E such that $\int_E |f(x)|dx \leq C_E \|f\|_X$.*

By l we denote a Banach sequence space (BSS), which means that the axioms (2.1)–(2.5) are fulfilled with respect to the counting measure and $\{e_k\}$ denotes the standard basis in l.

Recall that the condition (2.3) immediately yields the following property:

(2.6) if $0 \leq f \leq g$ then $\|f\|_X \leq \|g\|_X$.

Definition 2.2 *The set*

$$X' = \left\{ f; \left| \int_\Omega fgv \right| < \infty \text{ for every } g \in X \right\},$$

equipped with the norm

$$\|f\|_{X'} = \sup \left\{ \left| \int_\Omega fgv \right|; \|g\|_X \leq 1 \right\},$$

is called the associate space of X. It is known from Bennett and Sharpley [1] that $X'' = X$ and that X' is again a Banach function space.

Let T be a linear operator from a Banach function space X into a Banach function space Y. Then T' is an associate operator to the operator T if $\int_\Omega (Tf)g = \int_\Omega f(T'g)$ for all $f \in X$ and $g \in Y$.

Lemma 2.3 *(Bennett and Sharpley [1]) Let T be a linear operator from a Banach function space X into a Banach function space Y. Then $\|Tf\|_Y \leq C\|f\|_X$ for all $f \in X$ with a finite positive constant C, if and only if $\|T'g\|_{X'} \leq C\|g\|_{Y'}$ for all $g \in Y'$.*

Moreover $\|T\|_{X \to Y} = \|T'\|_{Y' \to X'}$.

Definition 2.4 *(Lomakina and Stepanov [3]) Given a BFS X and a BSS l, X is l-concave, if for any sequence of disjoint intervals $\{J_k\}$ such that $\cup J_k = \Omega$, and for all $f \in X$*

$$\left\| \sum_k e_k \|\chi_{J_k} f\|_X \right\|_l \leq d_1 \|f\|_X,$$

where is d_1 a finite positive constant independent on $f \in X$ and $\{J_k\}$. Analogously, a BFS Y is said to be l-convex, if for any sequence of disjoint intervals $\{I_k\}$, $\cup I_k = \Omega$ and for all $g \in Y$

$$\|g\|_Y \leq d_2 \left\| \sum_k e_k \|\chi_{I_k} g\|_Y \right\|_l$$

with a finite positive constant d_2 independent on $g \in Y$ and $\{I_k\}$.

We say, that Banach function spaces X, Y satisfy the l-condition, if there exist a Banach sequence space l such that X is l-concave and Y is l-convex simultaneously.

Lemma 2.5 *(Lomakina and Stepanov [3]) Let Y be a l-convex BFS. Then Y' is an l'-concave BFS and*

$$\left\| \sum_k e_k \|\chi_{I_k} f\|_{Y'} \right\|_{l'} \leq d_2 \|f\|_{Y'}$$

for all $f \in Y'$ and $\{I_k\}$, $\cup I_k = \Omega$.

3 Main results

Assume X and Y are two Banach function spaces on (c,d) and $\mathbb{R}$, respectively. Then we denote

$$\begin{aligned} A_0 &:= \sup_{x\leq y, a(y)\leq b(x)} \|\chi_{(x,y)}(.)v(.)k(.,b(x))\|_Y \|\chi_{(a(y),b(x))}u\|_{X'} \\ A_1 &:= \sup_{x\leq y, a(y)\leq b(x)} \|\chi_{(x,y)}v\|_Y \|\chi_{(a(y),b(x))}(.)k(x,.)u(.)\|_{X'} \end{aligned}$$

and $A := A_0 + A_1$.

Note that $A_0 = A_1$ if $k(x,y) = 1$.

Theorem 3.1 *Let X and Y be two BFS on (c,d) and $\mathbb{R}$, respectively, satisfying the l-condition. Let K be the integral operator of the form (1. 1) with kernel $k(x,y)\geq 0$ satisfying (1. 2). Then $K : X \to Y$ is bounded, if and only if, A is finite. Moreover*

$$\|K\|_{X\to Y} \asymp A.$$

To prove Theorem 3.1 we need a corresponding result for for the general Hardy operator with kernel $k(x,y) = 1$.

Let

$$Hf(x) := v(x)\int_{a(x)}^{b(x)} u(y)f(y)dy \tag{3. 1}$$

where $-\infty \leq c \leq a(x) \leq b(x) \leq d \leq \infty$ are nondecreasing functions on $\mathbb{R}$, v and u are real measurable and finite functions a.e. on $\mathbb{R}$ and (α,β), respectively.

Theorem 3.2 *Let X and Y be two BFS on (c,d) and $\mathbb{R}$, respectively, satisfying the l-condition, and let H be the operator defined by (3. 1). Then $H : X \to Y$ is bounded, if and only if,*

$$A_H := \sup_{x\leq y, a(y)\leq b(x)} \|\chi_{(x,y)}v\|_Y \|\chi_{(a(y),b(x))}u\|_{X'} < \infty.$$

Moreover $A_H \asymp \|H\|_{X\to Y}$.

4 Boundedness of the operator H

In this section we prove Theorem 3.2. At first we prove a lemma.

Definition 4.1 *Let v be a non-negative measurable function on an interval (α,β) where $-\infty \leq \alpha < \beta \leq \infty$. Let $c \in \mathbb{R}$, let $-\infty \leq a(x) \leq c \leq b(x) \leq \infty$*

be nondecreasing functions, and let u *be a non-negative measurable function on* (e,d) *where* $e := \liminf_{x\to\alpha} a(x)$ *and* $d := \limsup_{x\to\beta} b(x)$. *Then we define*

$$H_b f(x) := v(x) \int_c^{b(x)} u(t) f(t) dt$$

for every measurable function f *on* (c,d), *and*

$$H_a f(x) := v(x) \int_{a(x)}^{c} u(t) f(t) dt$$

for every measurable function f *on* (e,c).

Lemma 4.2 *Let* X *and* Y *be two BFS on* (e,d) *and* (α,β), *respectively, satisfying the* l*-condition. Then* $H_b : X \to Y$ *is bounded, if and only if,*

$$A_b := \sup_{\alpha\le x\le\beta} \|v\chi_{(x,\beta)}\|_Y \|u\chi_{(c,b(x))}\|_{X'} < \infty.$$

Moreover

$$\|H_b\|_{X\to Y} \asymp A_b.$$

Also $H_a : X \to Y$ *is bounded, if and only if,*

$$A_a := \sup_{\alpha\le x\le\beta} \|v\chi_{(x,\beta)}\|_Y \|u\chi_{(a(x),c)}\|_{X'} < \infty.$$

Moreover

$$\|H_a\|_{X\to Y} \asymp A_a.$$

Proof. We will give the proof only for H_b. The proof for H_a is similar. Necessity. Given $x \in (\alpha,\beta)$ and $f \in X$ such that $fu \ge 0$, we have

$$\begin{aligned}
\left\| v(.) \int_c^{b(.)} u(t) f(t) dt \right\|_Y &\ge \left\| v(.)\chi_{(x,\beta)}(.) \int_c^{b(.)} u(t) f(t) dt \right\|_Y \\
&\ge \left\| v(.)\chi_{(x,\beta)}(.) \int_c^{b(x)} u(t) f(t) dt \right\|_Y \\
&= \|v\chi_{(x,\beta)}\|_Y \int_c^{b(x)} |u(t) f(t)| dt.
\end{aligned}$$

Taking the supremum over all such f and $x \in (\alpha,\beta)$ we obtain

$$\|H_b\|_{X\to Y} \ge \|v\chi_{(x,\beta)}\|_Y \|u\chi_{(c,b(x))}\|_{X'}$$

and so,

$$\|H_b\|_{X\to Y} \ge A_b.$$

Sufficiency. If $A_b = \infty$ then $\|H\|_{X\to Y} \le A_b$. If $A_b = 0$ then $\|H_b\|_{X\to Y} = 0$.

Let $0 < A_b < \infty$. Choose $f \in X$ such that $\|f\|_X = 1$. Define $C_i = \{t; t \in (\alpha,\beta), \int_c^{b(t)} |fu| \ge 2^i\}$, $D_i = C_i \setminus C_{i+1}$ and $E_i = \{x; x \in (c,d), b(x) \in C_i\}$, $B_i = E_i \setminus E_{i+1}$. Then $|(c,d) \setminus \cup_{i\in\mathbf{Z}} B_i| = 0$ and $|(\alpha,\beta) \setminus \cup_{i\in\mathbf{Z}} D_i| = 0$ and we have

$$\begin{aligned}
\int_\alpha^\beta gH_bf &\le \sum_{i\in\mathbf{Z}} \int_{D_i} 2^{i+1} gv \le \sum_{i\in\mathbf{Z},|D_i|>0} 2^{i+1} \|v\chi_{D_i}\|_Y \|g\chi_{D_i}\|_{Y'} \\
&\le \sum_{i\in\mathbf{Z},|D_i|>0} 2^{i+1} \|v\chi_{C_i}\|_Y \|g\chi_{D_i}\|_{Y'} \\
&\le \sum_{i\in\mathbf{Z},|D_i|>0} 2^{i+1} \frac{A_b}{\|u\chi_{(c,\sup E_{i-1})}\|_{X'}} \|g\chi_{D_i}\|_{Y'} \\
&\quad \text{(using } 2^{i-1} \le \textstyle\int_c^d |f\chi_{B_{i-1}}||u| \\
&\qquad\qquad \le \|f\chi_{B_{i-1}}\|_X \|u\chi_{(c,\sup B_{i-1})}\|_{X'}\text{)} \\
&\le \sum_{i\in\mathbf{Z},|D_i|>0} A_b 2^{i+1} \frac{1}{2^{i-1}} \|f\chi_{B_{i-1}}\|_X \|g\chi_{D_i}\|_{Y'} \\
&\quad \text{(using Hölders inequality and } l\text{-condition)} \\
&\le 4A_b \left\| \sum_{i\in\mathbf{Z},|D_i|>0} e_i \|f\chi_{(B_{i-1})}\|_X \right\|_l \left\| \sum_{i\in\mathbf{Z},|D_i|>0} e_i \|g\chi_{D_i}\|_{Y'} \right\|_{l'} \\
&\le 4d_1d_2A_b \|f\|_X \|g\|_{Y'}.
\end{aligned}$$

Then we have

$$\|H_bf\|_Y = \sup_{\|g\|_{Y'}\le 1} \int_a^b gH_bf \le 4d_1d_2A_b\|f\|_X.$$

□

Now we prove Theorem 3.2.

Proof of Theorem 3.2. Necessity. Let $f \in X$ be such that $fu \ge 0$ and $\|f\|_X = 1$, and let x, y be such that $\alpha \le x \le y \le \beta$ and $b(x) \ge a(y)$. Then

$$\|Hf\|_Y \ge \|v(.)\chi_{(x,y)}(.) \int_{a(.)}^{b(.)} u(t)f(t)dt\|_Y = \|v\chi_{(x,y)}\|_Y \int_{a(y)}^{b(x)} u(t)f(t)dt.$$

Taking the supremum over all such x, y and f we we obtain

$$\|H\|_{X\to Y} \ge \|v\chi_{(x,y)}\|_Y \|u\chi_{(a(y),b(x))}\|_{X'}.$$

Sufficiency. Define $M := \{(x,t); x \in \mathbb{R}, a(x) < t < b(x)\}$. M is a measurable set. If $|M| = 0$ then it is easy to see that $\|H\|_{X\to Y} = 0$.

Suppose that $|M| > 0$. We set $M_y := \{x; (x,t) \in M, t = y\}$, $y \in \mathbb{R}$, and $P := \{y; (x,t) \in M, |M_y| > 0\}$. Then $P = \cup_{i=1}^{m} P_i$, where P_i are intervals, $|P_i| > 0$, and $m \le \infty$.

Let $y_0 \in \operatorname{int} P_i$; then we have a set M_{y_0} and $c_0 = a(\inf M_{y_0}), d_0 = b(\sup M_{y_0})$.

Suppose we have defined y_i, c_i, d_i and M_{y_i}.

If $i \ge 0$ and $d_i \in \operatorname{int} P_i$ then we define $y_{i+1} = d_i$, $c_{i+1} = a(\inf M_{y_{i+1}})$, $d_{i+1} = b(\sup M_{y_{i+1}})$. If $i \le 0$ and $c_i \in \operatorname{int} P_i$ then we define $y_{i-1} = c_i$, $c_{i-1} = a(\inf M_{y_{i-1}})$, $d_{i-1} = b(\sup M_{y_{i-1}})$.

By using this method we can construct, for every P_i sequences $\{y_j^i\}_{j=m_i}^{n_i}$, $\{c_j^i\}_{j=m_i}^{n_i}$, $\{d_j^i\}_{j=m_i}^{n_i}$, $\{M_{y_j^i}\}_{j=m_i}^{n_i}$, where $-\infty \le m_i \le n_i \le \infty$.

We can rewrite these all sequences in the following way: $\{y_i\}_{i=1}^{k}$, $\{c_i\}_{i=1}^{k}$, $\{d_i\}_{i=1}^{k}$ and $\{M_{y_i}\}_{i=1}^{k}$ where $k = \sum_{i=1}^{m}(n_i - m_i + 1)$.

Then we have

$$Hf(x) = \sum_{i=1}^{k} \chi_{M_{y_i}}(x) \left(v(x) \int_{y_i}^{b(x)} u(t)f(t)dt + v(x) \int_{a(x)}^{y_i} u(t)f(t)dt \right) \quad \text{a.e.}$$

and

$$\begin{aligned}
\int_{\mathbb{R}} gHf &= \sum_{i=1}^{k} \int_{M_{y_i}} gHf \\
&= \sum_{i=1}^{k} \int_{M_{y_i}} \left(\left[v(x) \int_{y_i}^{b(x)} f(t)u(t)dt + v(x) \int_{a(x)}^{y_i} f(t)u(t)dt \right] g(x) \right) dx \\
&= \sum_{i=1}^{k} \Bigg[\int_{M_{y_i}} \left(v(x) \int_{y_i}^{b(x)} f(t)\chi_{(y_i,d_i)}u(t)dt \right) g(x)dx \\
&\qquad + \int_{M_{y_i}} \left(v(x) \int_{a(x)}^{y_i} f(t)\chi_{(c_i,y_i)}u(t)dt \right) g(x)dx \Bigg] \\
&\qquad \text{(Using Lemma 4.2 and } A_a + A_b \le A_H) \\
&\le 4d_1d_2 \sum_{i=1}^{k} A_H \|f\chi_{(c_i,y_i)}\|_X \|g\chi_{M_i}\|_{Y'} + 4d_1d_2 \sum_{i=1}^{k} A_H \|f\chi_{(y_i,d_i)}\|_X \|g\chi_{M_i}\|_{Y'} \\
&\le 8d_1d_2A_H \sum_{i=1}^{k} \|f\chi_{(c_i,d_i)}\|_X \|g\chi_{M_i}\|_{Y'} \\
&\qquad \text{(use Hölder's inequality and } l\text{-condition)} \\
&\le 8A_H d_1^2 d_2^2 \|f\|_X \|g\|_{Y'}.
\end{aligned}$$

□

5 Boundedness of the operator K

In this section we prove Theorem 3.1.

Lemma 5.1 *Let $b(x)$ be a nondecreasing right continuous function on (α,β) and let $b(\alpha)=c$, $b(\beta)=d$. Let $k_0(x,y)\geq 0$ be a kernel satisfying (1. 2), and $k_0(x,y)>0$ on set of positive measure. Suppose that $k_0(x,y)$ is right continuous with respect to x for all $x\in[\alpha,\beta]$ and for a.e. $y\in(c,b(x))$.*

Let u, f be measurable functions on (c,d), $fu\geq 0$, and

$$G_0(x)=\int_c^{b(x)}k_0(x,y)u(y)f(y)dy.$$

For a fixed number $\delta>D$ (where D is a constant from (1. 2)), we define $\Delta_k:=\{x\in(\alpha,\beta);G_0(x)\geq(\delta+1)^k\}$, $k\in\mathbf{Z}$, and $N=\sup\{k;\Delta_k\neq\emptyset\}$. Then there exist sequences $\{x_k\}$, $\{\gamma_k\}$ such that $\alpha<\ldots<x_{k-1}<x_k<\ldots<\beta$ and the inequality

$$\begin{aligned}(\delta+1)^{\gamma_k-1} \quad\leq\quad & \int_{b(x_{k-1})}^{b(x_k)}k_0(x_k,y)u(y)f(y)dy\\ &+D\int_{b(x_{k-2})}^{b(x_{k-1})}k_0(x_{k-1},y)u(y)f(y)dy\\ &+Dk_0(x_k,b(x_{k-1}))\int_{b(x_{k-1})}^{b(x_k)}u(y)f(y)dy\\ &+Dk_0(x_k,b(x_{k-2}))\int_c^{b(x_{k-2})}u(y)f(y)dy.\end{aligned}$$

holds for all $k\leq N$, and $G_0(x)\leq(1+\delta)^{\gamma_{k-1}+1}$ when $x\in[x_{k-1},x_k)$.

Proof. By the Lebesgue Dominated Convergence Theorem $G_0(x)$ is a nondecreasing right continuous function for all $\alpha\leq x\leq\beta$ and $\lim_{x\to\alpha}G_0(x)=0$.

Set $a_k=\inf\Delta_k$, for $k\leq N$.

Fix $i\in\mathbf{Z}$ such that $|\Delta_i|>0$. We set $x_0=a_i$, $\gamma_0=\max\{i;a_i=x_0\}$, $x_k=a_{\gamma_k}$ where $\gamma_k=\max\{i;a_i=a_{\gamma_{k-1}+1}\}$ for $k>0$ and $\gamma_k=\max\{i;a_i=a_{\gamma_{k+1}}\}$ for $k<0$.

It is obvious that $\{\gamma_k\}$ is an increasing sequence of integers, therefore $\gamma_{k_1}\leq\gamma_k-1$, $G(x_k)=G(a_{\gamma_k})\geq(1+\delta)^{\gamma_k}$.

If $x\in[x_k,x_{k+1})$, then we have $a_{\gamma_k+1}=x_{k+1}$, and therefore $x<a_{\gamma_k+1}$ $G(x)<(1+\delta)^{\gamma_k+1}$. Next on using (1. 2) we find that

$$\begin{aligned}(1+\delta)^{\gamma_k} &= \int_c^{b(x_k)}k_0(x_k,y)u(y)f(y)dy\\ &= \int_{b(x_{k-1})}^{b(x_k)}k_0(x_k,y)u(y)f(y)dy+\int_{b(x_{k-2})}^{b(x_{k-1})}k_0(x_k,y)u(y)f(y)dy\end{aligned}$$

$$
\begin{aligned}
&+\int_c^{b(x_{k-2})} k_0(x_k,y)u(y)f(y)dy \\
\leq\ & \int_{b(x_{k-1})}^{b(x_k)} k_0(x_k,y)u(y)f(y)dy \\
&+D\int_{b(x_{k-2})}^{b(x_{k-1})} k_0(x_{k-1},y)u(y)f(y)dy \\
&+Dk_0(x_k,b(x_{k-1}))\int_{b(x_{k-2})}^{b(x_{k-1})} u(y)f(y)dy \\
&+Dk_0(x_k,b(x_{k-2}))\int_c^{b(x_{k-2})} u(y)f(y)dy + DG_0(x_{k-2}).
\end{aligned}
$$

As $DG_0(x_{k-2}) \leq D(1+\delta)^{\gamma_{k-2}+1} \leq D(1+\delta)^{\gamma_k-1} \leq \delta(1+\delta)^{\gamma_k-1}$ and $(1+\delta)^{\gamma_k} - \delta(1+\delta)^{\gamma_k-1} = (1+\delta)^{\gamma_k-1}$ the lemma follows. □

Theorem 5.2 *Let X and Y be two BFS on (c,d) and (α,β), respectively, (where $b(\beta)=d$ and $b(\alpha)=c$) satisfying the l-condition and*

$$K_b f(x) := v(x)\int_c^{b(x)} k(x,y)f(y)u(y)dy,$$

where $k(x,y)$ satisfies (1. 2). Then

$$\|K_b\|_{X\to Y} \asymp A_b^1 + A_b^0$$

where

$$
\begin{aligned}
A_b^1 &:= \sup_{\alpha<z<\beta} \|\chi_{(z,\beta)}v\|_Y \|\chi_{(c,b(z))}(.)k(z,.)u(.)\|_{X'}\ , \\
A_b^0 &:= \sup_{\alpha<z<\beta} \|\chi_{(z,\beta)}(.)v(.)k(.,b(z))\|_Y \|\chi_{(c,b(z))}u\|_{X'}.
\end{aligned}
$$

Proof. Necessity. Let $x>\alpha$. Then $b(x)\geq c$. Since $k(x,y)$ is nondecreasing in x and nonincreasing in y, for every $\alpha<x<z<\beta$ we have

$$Kf(x) \geq v(z)\int_c^{b(x)} k(x,t)f(t)u(t)dt$$

and

$$Kf(z) \geq v(z)k(z,b(x))\int_c^{b(x)} u(t)f(t)dt.$$

Hence,

$$
\begin{aligned}
\|Kf\|_Y &\geq \|\chi_{(x,\beta)}(.)v(.)\int_c^{b(x)} k(x,t)f(t)u(t)dt\|_Y \\
&\geq \|\chi_{(x,\beta)}v\|_Y \|\chi_{(c,b(x))}(.)k(x,.)u(.)\|_{X'} \|f\chi_{(c,b(x))}\|_X
\end{aligned}
$$

for all $f \in X$ and $\alpha < x < \beta$, and

$$\begin{aligned} \|Kf\|_Y &\geq \|\chi_{(x,\beta)}(.)v(.)k(.,b(x))\int_c^{b(x)} u(t)f(t)dt\|_Y \\ &\geq \|\chi_{(x,\beta)}(.)v(.)k(.,b(x))\|_Y\|\chi_{(c,b(x))}u\|_{X'}\|\chi_{(c,b(x))}f\|_X \end{aligned}$$

for all $f \in X$ and $\alpha < x < \beta$.

Sufficiency. Let D be the constant from condition (1. 2) and let $\delta > D$ be fixed. Without loss of generality we may assume that $k(x,y)$ and $b(x)$ satisfy the assumptions of Lemma 5.1. Otherwise we replace $k(x,y)$ by $k(x_+,y)$ and $b(x)$ by $b(x_+)$.

By the principle of duality it is sufficient to show that

$$J = \left|\int_\alpha^\beta v(t)G(t)g(t)dt\right| \leq A\ \|f\|_X\|g\|_{Y'}, \quad \text{for all } f \in X \text{ and } g \in Y',$$

where $G(t) = \int_c^{b(t)} |k(t,y)f(y)u(y)|dy$. By Lemma 5.1 we obtain

$$\begin{aligned} J &\leq \sum_{k\leq N}\int_{x_k}^{x_{k+1}} |v(t)G(t)g(t)|dt \\ &\leq \sum_{k\leq N}(1+\delta)^{\gamma_k+1}\int_{x_k}^{x_{k+1}} |v(t)g(t)|dt \\ &\leq (1+\delta)^2[J_{11}+J_{12}+J_{21}+J_{22}], \end{aligned}$$

where

$$\begin{aligned} J_{11} &:= \sum_{k\leq N}\int_{b(x_{k-1})}^{b(x_k)} |k(x_k,t)u(t)f(t)|dt\int_{x_k}^{x_{k+1}} |g(t)v(t)|dt, \\ J_{12} &:= D\sum_{k\leq N}\int_{b(x_{k-2})}^{b(x_{k-1})} |k(x_{k-1},t)u(t)f(t)|dt\int_{x_k}^{x_{k+1}} |g(t)v(t)|dt, \\ J_{21} &:= D\sum_{k\leq N} k(x_k,b(x_{k-1}))\int_{b(x_{k-2})}^{b(x_{k-1})} |u(t)f(t)|dt\int_{x_k}^{x_{k+1}} |g(t)v(t)|dt, \\ J_{22} &:= D\sum_{k\leq N} k(x_k,b(x_{k-2}))\int_c^{b(x_{k-2})} |u(t)f(t)|dt\int_{x_k}^{x_{k+1}} |g(t)v(t)|dt. \end{aligned}$$

Applying the Hölder inequality and the l-condition we find

$$\begin{aligned} J_{11} \leq \sum_{k\leq N} \big(&\|\chi_{(b(x_{k-1}),b(x_k))}k(x_k,.)u\|_{X'}\|\chi_{(b(x_{k-1}),b(x_k))}f\|_X \times \\ &\times \|\chi_{(x_k,x_{k+1})}g\|_{Y'}\|\chi_{(x_k,x_{k+1})}v\|_Y\big) \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{k\leq N} \|\chi_{(c,b(x_k))}k(x_k,.)u\|_{X'}\|\chi_{(x_k,\beta)}v\|_Y\|\chi_{(b(x_{k-1}),b(x_k))}f\|_X\|\chi_{(x_k,x_{k+1})}g\|_{Y'}\\
&\leq A_b^1\sum_{k\leq N}\|\chi_{(b(x_{k-1}),b(x_k))}f\|_X\|\chi_{(x_k,x_{k+1})}g\|_{Y'}\\
&\leq A_b^1\left\|\sum_{k\leq N} e_k\|\chi_{(b(x_{k-1}),b(x_k))}f\|_X\right\|_l\left\|\sum_{k\leq N} e_k\|\chi_{(x_k,x_{k+1})}g\|_{Y'}\right\|_{l'}\\
&\leq d_1d_2A_b^1\|f\|_X\|g\|_{Y'}.
\end{aligned}$$

Analogously, we obtain

$$J_{12}\leq d_1d_2A_b^1\|f\|_X\|g\|_{Y'}.$$

The estimate for J_{21} is similar to that for J_{11} on applying the knowledge that $k(x,y)$ is nondecreasing in x and proceeding like for J_{11}: we find that

$$J_{21}\leq d_1d_2A_b^0\|f\|_X\|g\|_{Y'}.$$

For J_{22} we write

$$\begin{aligned}
J_{22} &= \int_\alpha^\beta\int_c^{b_1(x)}|u(t)f(t)|dt\sum_{k\leq N}k(x_k,b(x_{k-2}))\chi_{(x_k,x_{k+1})}(x)|g(x)v(x)|dx\\
&\leq \|K_1f\|_Y\|g\|_{Y'}\ ,
\end{aligned}$$

where $K_1f(x):=\phi(x)v(x)\int_c^{b_1(x)}|u(t)f(t)|dt$ and $b_1(x):=\sum_{k\leq N}b(x_{k-2})\chi_{(x_k,x_{k+1})}(x)$ and $\phi(x):=\sum_{k\leq N}k(x_k,b(x_{k-2}))\chi_{(x_k,x_{k+1})}(x)$. By Theorem 3.2 we have that

$$\|K_1\|_{X\to Y}\leq C\sup_{\alpha<z<\beta}\|\chi_{(z,\beta)}v\ \phi\|_Y\|\chi_{(c,b_1(z))}u\|_{X'}$$

if $x_{k_0}<z<x_{k_0+1}$ when $b_1(z)=b(x_{k_0-2})$ and

$$\phi(t)\chi_{(z,\beta)}(t)\leq\sum_{k=k_0}^\infty k(x_k,b(x_{k-2}))\chi_{(x_k,x_{k+1})}(t)\leq k(t,b(x_{k_0-2})).$$

Therefore we have

$$\|\chi_{(z,\beta)}\phi v\|_Y\|\chi_{(c,b_1(z))}u\|_{X'}\leq\|\chi_{(x_{k_0-2},\beta)}k(.,b(x_{k_0-2}))v\|_Y\|\chi_{(c,b(x_{k_0-2}))}u\|_{X'}\leq A_b^0.$$

Thus $J_{22}\leq CA_b^0\|f\|_X\|g\|_{Y'}$. Then we have that $\|K_b\|_{X\to Y}\leq C(A_b^1+A_b^0)$. □

Proof of Theorem 3.1. Necessity. Let $fu\geq 0$ a.e., $x<y$ and $b(x)\geq a(y)$. Since $k(x,y)$ is nondecreasing in x and nonincreasing in y, for every $x<z<y$ we have

$$Kf(z)\geq v(z)\int_{a(y)}^{b(x)}k(x,t)f(t)u(t)dt$$

and

$$Kf(z) \geq v(z)k(z,b(x)) \int_{a(y)}^{b(x)} f(t)u(t)dt.$$

Therefore we get

$$\int_{a(y)}^{b(x)} k(x,t)u(t)f(t)dt \|\chi_{(x,y)}\|_Y \leq \|Kf\|_Y \leq \|K\|_{X\to Y}\|f\|_X$$

and

$$\int_{a(y)}^{b(x)} u(t)f(t)dt \|v\chi_{(x,y)}k(.,b(x))\|_Y \leq \|Kf\|_Y \leq \|K\|_{X\to Y}\|f\|_X$$

for all $f \in X$ such that $fu \geq 0$.

Then by duality we have

$$A_0 + A_1 \leq 2\|K\|_{X\to Y}.$$

Sufficiency. We use the same technique as in the proof of Theorem 3.2. For $a(x), b(x)$ we define $\{c_i\}_{i=1}^k, \{d_i\}_{i=1}^k, \{y_i\}_{i=1}^k$ and $\{M_{y_i}\}_{i=1}^k$ as in that proof. Then we have

$$\begin{aligned} Kf(x) &= \sum_{i=1}^k \chi_{M_{y_i}}(x)(v(x)\int_{y_i}^{b(x)} k(x,t)u(t)f(t)dt + v(x)\int_{a(x)}^{y_i} k(x,t)u(t)f(t)dt) \\ &= \sum_{i=1}^k \chi_{M_{y_i}}(x)K_i^1 f(x) + \sum_{i=1}^k \chi_{M_{y_i}}(x)K_i^2 f(x), \end{aligned}$$

where $K_i^1 = v(x)\int_{y_i}^{b(x)} k(x,t)u(t)f(t)dt\chi_{M_{y_i}}(x)$ and $K_i^2 = v(x)\int_{a(x)}^{y_i} k(x,t)u(t)f(t)dt\chi_{M_{y_i}}(x)$.

By the l-condition we obtain

$$\begin{aligned} \|Kf\|_Y &\leq \left\|\sum_{i=1}^k e_i\|K_i^1(f)\chi_{M_{y_i}}\|_Y\right\|_l + \left\|\sum_{i=1}^k e_i\|K_i^2(f)\chi_{M_{y_i}}\|_Y\right\|_l \\ &= I_1 + I_2. \end{aligned}$$

By Theorem 5.2 we have

$$\|K_i^1(f)\chi_{M_{y_i}}\|_Y \leq CA\ \|\chi_{(c_i,d_i)}f\|_X$$

and therefore we obtain

$$I_1 \leq CA\ \|\sum_{i=1}^k e_i\|\chi_{(c_i,d_i)}f\|_X\|_l \leq CA\ \|f\|_X.$$

To estimate I_2 we use the condition (1. 2) for $x_i = \inf(M_{y_i})$ and $a(x) \leq t \leq y_i = b(x_i)$, $x_i < x$. Then $k(x,t) \leq D[k(x,y_i) + k(x_i,t)]$ and we have

$$\begin{aligned}\chi_{M_{y_i}}(x)K_i^2 f(x) \qquad &= \chi_{M_{y_i}}(x)v(x)\textstyle\int_{a(x)}^{y_i} k(x,t)u(t)f(t)dt \\ &\leq D\chi_{M_{y_i}}(x)v(x)k(x,y_i)\textstyle\int_{a(x)}^{y_i} u(t)f(t)dt \\ &\quad + D\chi_{M_{y_i}}(x)v(x)\textstyle\int_{a(x)}^{y_i} k(x_i,t)u(t)f(t)dt.\end{aligned}$$

Theorem 3.2 yields

$$\left\|\chi_{M_{y_i}} v(x)k(x,y_i)\int_{a(x)}^{y_i} u(t)f(t)dt\right\|_Y \leq A\ \|f\chi_{(c_i,d_i)}\|_X$$

and

$$\left\|\chi_{M_{y_i}} v(x)\int_{a(x)}^{y_i} k(x_i,t)u(t)f(t)dt\right\|_Y \leq A\ \|f\chi_{(c_i,d_i)}\|_X.$$

Therefore

$$\|\chi_{M_{y_i}} K_i^2 f\|_Y \leq 2CA\|f\chi_{(c_i,d_i)}\|_X$$

and by the l-condition we obtain that

$$I_2 \leq 2cA\ \left\|\sum_{i=1}^k e_i\|f\chi_{(c_i,d_i)}\|_X\ \right\|_l \leq 2CA\ \|f\|.$$

Combining the estimates of I_1 and I_2 we arrive at

$$\|Kf\|_Y \leq AC\|f\|_X.$$

Theorem 3.1 is proved.

Remark. When this paper was finished we learned [by oral communications] that this problem for Hardy operators in Lebesgue spaces was considered by Heinig and Sinnamon [2].

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