APPROXIMATION NUMBERS OF HARDY-TYPE OPERATORS ON TREES

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Abstract. We present upper and lower estimates and an asymptotic result for the approximation numbers of the Hardy-type operator on a tree Γ. The results include a compactness criteria for the Hardy-type operator.

Introduction

It is well known that for a domain Ω in $\mathbb{R}^n$ with a smooth boundary, the inverse Dirichlet (or Neumann) Laplacian Δ^{-1} may be expressed as TT^* where T is the embedding map of $W_0^{1,2}(\Omega)$ ($W^{1,2}(\Omega)$ respectively) in $L^2(\Omega)$. If the boundary is not smooth, this may serve as the definition. In [3], [4] and [5] a connection is established, for a wide class of domains, between the embedding on the domain and a similar embedding on a line-segment or a tree, so that knowledge of the approximation numbers on the tree gives information about those on the domain and hence about the spectrum of the Laplacian. Since a tree is one-dimensional, its embedding map is in fact a Hardy-type operator. Then any information about approximation numbers of the Hardy-type operators on trees can give us information about the spectrum of Dirichlet (or Neumann) Laplacian.

What follows is a short account of our attempt, in collaboration with W. D. Evans, to estimate the approximation numbers of the operator from L^p to itself. This problem has been attacked also by Naimark and Solomyak but only for $p = 2$ (see [9]).

Let us start with some definitions.

In what follows we mean by a tree a graph whose edges are line segments and vertices the endpoints of the segments. In addition the vertices are all of finite degree and there are no loops nor cycles so that each pair of points

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may be joined by a unique polygonal path. There is an obvious metric on a tree Γ and Γ is locally compact. It is compact if it has only a finite number of edges. It has a natural one-dimensional Lebesgue measure and it is a σ-finite measure space. We may chose a point a of Γ as the root and define a partial ordering on Γ by $x \succ_a y$ if and only if y lies on the path joining a and x.

We define the Hardy-type operator T_a on a tree Γ by

$$T_a f(x) := v(x) \int_a^x f(t)u(t)\,dt$$

where $u \in L^{p'}_{\mathrm{loc}}(\Gamma)$, $v \in L^p(\Gamma)$, $1 \le p \le \infty$, and $\int_a^x$ denotes integration over the path (a, x). We may assume without loss of generality that $u, v \ge 0$.

For a measurable subset K of Γ we define the norm

$$\|f\|_{p,K} = \Big(\int_K |f|^p\Big)^{1/p}$$

on $L^p(K)$. The $L^p(\Gamma)$ norm will be denoted by $\|\cdot\|$ if there is no danger of confusion. We shall also denote the norm $\|T \mid L^p(\Gamma) \to L^p(\Gamma)\|$ by $\|T\|$.

In the case when Γ is an interval (a, b) it is well known that

$$\|T_a\| \sim \sup_{a \le x \le b} \|u\chi_{(a,x)}\|_{p'} \|v\chi_{(x,b)}\|_p.$$

But when Γ is a tree then the situation is more complicated. In [8] the following estimate for $\|T_a\|$ was obtained.

Theorem 1. *Let $1 < p < \infty$, $\alpha_k = \inf\{\|f\|_p : \int_a^t |f|\,|u| = 1$ for all $t \in \partial K_a\}$ and $d\mu = |v(t)|^p dt$. Then*

$$\|T_a\| \sim \sup\{\mu(\Gamma \setminus K_a)/\alpha_k\},$$

where the supremum is taken over all subtrees K_a of Γ which contain a and whose boundary points t are such that $x \notin K_a$ whenever $x \succ_a t$.

From this condition for boundedness we can see that the study of the Hardy operator on trees is more difficult than on intervals. This will lead us to skip many proofs and show only the main ideas in this presentation.

Let us recall the usual definition of approximation numbers. The n-th approximation number of a bounded operator $T : L^p(\Gamma) \to L^p(\Gamma)$ is defined by

$$a_n(T) = \inf \|T - P \mid L^p(\Gamma) \to L^p(\Gamma)\|, \quad n \in \mathbb{N},$$

where the infimum is taken over all bounded linear maps $P : L^p(\Gamma) \to L^p(\Gamma)$ of rank $P < n$.

Since L^p has the approximation property (see [12]) the measure of non-compactness of T, $\beta(T) = \inf\{\|T - F \mid L^p \to L^p\| : F$ is a compact linear map$\}$, satisfies $\beta(T) = \lim_{n\to\infty} a_n(T)$.

The function A,

$$A(K, u, v) = \sup_{\|f\|_p \le 1} \inf_{\alpha \in \mathcal{C}} \|T_{a,K} f - \alpha v\|_{p,K},$$

where

$$T_{a,K} f(x) := v(x)\chi_K(x) \int_a^x u(t) f(t) \chi_K(t)\, dt,$$

will be our main tool for the study of the Hardy operator. We shall show some of its properties.

A is monotone in the sense that if $X \subset Y$ then $A(X, u, v) \le A(Y, u, v)$.

Let K be a subtree of Γ, $v \in L^p(K)$ and $1 \le p \le \infty$, then there is a $w_K \in (L^p(K))^*$ such that

$$\inf_{c \in \mathcal{C}} \|\phi v - cv\|_{p,K} \le \|\phi v - w_K(\phi) v\|_{p,K} \le 2 \inf_{c \in \mathcal{C}} \|\phi v - cv\|_{p,K}$$

for all ϕ in $L^p(K)$. To see this for $1 < p < \infty$ take $w_K(\phi) = (\int_K \phi v)/\mu(K)$, where $\mu(K) = \int_K v^p$. For $p = 2$ we get an identity. (The cases $p = 1, \infty$ are trickier.)

Hence A almost represents the distance of the operator T_a from all one dimensional operators over K.

If $K = (a, x)$ and $1 < p < \infty$ then $A((a, x), u, v)$ is a continuous function of x. (This is not true when K is a subtree or $p = 1, \infty$.)

If K is a subtree of Γ then $\|T_{x,K}\|$ is a continuous function of x and $A(K) \le \inf_{x \in K} \|T_{x,K}\|$. If T_a is compact we can prove more.

Theorem 2. *Let T_a be compact. Then $A(K, u, v) = \min_{x \in \Gamma} \|T_{x,K}\|$.*

Sketch of the proof for $1 < p < \infty$.

(i) By the uniform convexity of $L^p(\mu)$ there is for every f in L^p a unique constant c_f for which $\|f - c_f\| = \inf_c \|f - c\|$.

(ii) The uniqueness of c_f implies its continuous dependence on f.

(iii) Let $T_{x,i}$ be the restriction of T_x to $\Gamma_{x,i}$ where $\Gamma_{x,i}$ are the subtrees which are the closures of the connected components of $\Gamma \setminus \{x\}$. Then $\|T_x\| = \max_i \|T_{x,i}\| = \|T_{x,1}\|$, say. We call the point x simple if there is only one i for which the norm is maximal. It is not difficult to show that there exists a point a which is not simple. Then $\|T_{a,1}\| = \|T_{a,2}\| = \|T_a\|$. If $Z < \|T_a\|$ there exist $f_1 \geq 0$, $f_2 \leq 0$, supported on $\Gamma_{x,1}$, $\Gamma_{x,2}$ and satisfying $\|T_x f_1\|, \|T_x f_2\| > Z$. Taking $g = \lambda f_1 + (1-\lambda) f_2$ $(0 \leq \lambda \leq 1)$ we have $\|T_x g\| > Z\|g\|$. Then, by continuity, λ can be chosen so that $c_g = 0$, since c_f obviously takes opposite signs for $f = f_1$, f_2. Then $A(K,u,v) > Z$. □

It follows from Theorem 2 that $A(K,u,v)$ is independent of a. Hence a change of the root of the tree Γ has no effect on $A(K,u,v)$. However, it can change the norm of T_a.

We define two classes of covering subtrees of Γ by means of A. For given $\varepsilon > 0$ we set

$$\begin{aligned} U_\varepsilon(\Gamma) = \{F : F &\text{ is a set of non-overlapping closed subtrees } X \text{ covering } \Gamma \\ &\text{such that } A(X,u,v) \leq \varepsilon\}, \\ L_\varepsilon(\Gamma) = \{G : G &\text{ is a set of non-overlapping closed subtrees } X \text{ covering } \Gamma \\ &\text{such that } \#\{X : X \in \Gamma,\ A(X,u,v) < \varepsilon\} \leq 1\} \end{aligned}$$

and we denote

$$\begin{aligned} N(\varepsilon,\Gamma,u,v) &= \min\{\#F : F \in U_\varepsilon(\Gamma)\}, \\ M(\varepsilon,\Gamma,u,v) &= \max\{\#G : G \in L_\varepsilon(\Gamma)\}. \end{aligned}$$

There is always a covering $F \in U_\varepsilon(\Gamma)$ such that $\#F = N(\varepsilon,\Gamma,u,v)$ and $G \in L_\varepsilon(\Gamma)$ such that $\#G = M(\varepsilon,\Gamma,u,v)$.

The monotonicity of A yields $M(\varepsilon,\Gamma,u,v) \leq N(\varepsilon\Gamma,u,v)$.

For non-compact T_a we have $\beta(T_a) \sim \inf\{\varepsilon : N(\varepsilon,\Gamma,u,v) < \infty\}$.

If Γ is an interval and $1 < p < \infty$ then, using the continuity of A, it is possible to show that $M(\varepsilon,\Gamma,u,v) = N(\varepsilon,\Gamma,u,v)$. This is not true if Γ is a tree but we can get the following estimate.

Lemma 1. $M(\varepsilon,\Gamma,u,v) \geq \frac{1}{3} N(\varepsilon,\Gamma,u,v)$.

The proof by induction on $N(\varepsilon,\Gamma,u,v)$ is not difficult but somewhat time-consuming (see [7]).

Lemma 1 is fundamental for all results on trees and will play an important role in what follows.

Now we are ready to present estimates for approximation numbers of the Hardy operator T_a.

Lemma 2. *Let $\varepsilon > 0$, $1 \leq p \leq \infty$, $F \in U_\varepsilon(\Gamma)$ and set $N = \#F$. Then $a_{N+1}(T) \leq 2\varepsilon$. In particular, if $p = 2$ then $a_{N+1}(T_a) \leq \varepsilon$.*

Proof. Let $\{\Gamma_i\}_{i=1}^N \in U_\varepsilon(\Gamma)$ and set $Pf = \sum_{i=1}^N P_i f$ where

$$P_i f(x) = \chi_{\Gamma_i}(x) v(x) \Big[\int_a^{a_i} uf + w_{\Gamma_i} \Big(\int_{a_i}^x uf \chi_{\Gamma_i} \Big) \Big],$$

and where a_i is the point in Γ_i nearest to a. Then $\operatorname{rank} P \leq N$ and it is possible to show that $\|(T_a - P)f\|_p^p \leq 2\varepsilon$ for $1 < p < \infty$ and $\|(T_a - P)f\|_p^p = \varepsilon$ for $p = 2$, whence the lemma. □

Lemma 3. *Let $a \in \Gamma$. Let Γ_i $(i = 1, 2, \ldots, N)$ be closed non-empty subtrees covering Γ and such that $A(\Gamma_i, u, v) \geq \varepsilon$ for all i. Then*

$$a_N(T_a) \geq \varepsilon.$$

Proof. Consider ϕ_i such that $\|\phi_i\| = 1$ and $\|T\phi_i\|$ approximate ε from below. Then for any bounded linear operator P on $L^p\mu$ with rank $< N$ there exist $\lambda_1, \lambda_2, \ldots, \lambda_N$, not all zero and such that $P(\sum_i \lambda_i \phi_i) = 0$. The proof follows on considering $\phi = \sum_i \lambda_i \phi_i$ and showing that

$$a_N^p(T) \geq \frac{\|T_a\phi - P\phi\|_p^p}{\|\phi\|_p^p} \geq \frac{\min_i \|T\phi_i\|_p^p \|\phi\|_p^p}{\|\phi\|_p^p}.$$

□

From the preceding three lemmas we obtain the following theorem on upper and lower estimates for the approximation numbers.

Theorem 3. *Let $a \in \Gamma$ and $N = N(\varepsilon, \Gamma, u, v)$. Then*

$$a_{[N/3]-2}(T_a) \geq \varepsilon \qquad \text{and} \qquad a_{N+1}(T_a) \leq c_p \varepsilon \tag{1}$$

where

$$c_p = \begin{cases} 1 & \text{for } p = 2 \\ 2 & \text{for } p \neq 2. \end{cases}$$

We can write T instead of T_a in (1) because we know from Theorem 2 that A is independent of a and hence also a_n are independent of a.

Local asymptotic results

We begin with the investigation of asymptotic properties of N_ε as $\varepsilon \to 0$. First we consider the case of a compact subtree:

Lemma 4. *Let K be a compact subtree of Γ and $1 < p < \infty$. Then*

$$\alpha_p \int_K |u|\,|v| = \lim_{\varepsilon\to 0+} \varepsilon N(\varepsilon, K, u, v)$$

where $\alpha_p = A((0,1),1,1)$.

Proof. Since K is a compact tree it has a bounded number of vertices, i.e. K is a finite union of intervals. The proof can be finished using the argument of [2, Theorem 5], with A replacing the function L there. □

In the case when Γ is a non-compact tree we know that there exist compact subtrees Γ_n of Γ, $n = 1, 2, \dots$, such that $\Gamma_n \subset \Gamma$ and $\Gamma_n \to \Gamma$ as $n \to \infty$. By Lemma 4, for $1 < p < \infty$ we have

$$\alpha_p \int_{\Gamma_n} |u|\,|v| = \lim_{\varepsilon\to 0+} \varepsilon N(\varepsilon, \Gamma_n, u, v) \le \liminf_{\varepsilon\to 0+} \varepsilon N(\varepsilon, \Gamma, u, v).$$

This yields

$$\alpha_p \int_{\Gamma} |u|\,|v| \le \liminf_{\varepsilon\to 0+} \varepsilon N(\varepsilon, \Gamma, u, v). \tag{2}$$

To prove the converse estimate we shall assume that $u \in L^{p'}(\Gamma)$ and $v \in L^p(\Gamma)$.

Lemma 5. *Let $1 < p < \infty$, $u \in L^{p'}(\Gamma)$ and $v \in L^p(\Gamma)$. Then*

$$\lim_{\varepsilon\to 0+} \varepsilon N(\varepsilon, \Gamma, u, v) = \alpha_p \int_\Gamma |u|\,|v|.$$

Proof. Let us consider a sequence of compact subtrees K_n such that $a \in K_n$ and $K_n \nearrow \Gamma$. Let $\{\Gamma_j\}_{j=1}^{N(\varepsilon,\Gamma,u,v)}$ be a covering set of subtrees of Γ from the definition of $N(\varepsilon, \Gamma, u, v)$ and set

$$S_n(\varepsilon) := \big\{\Gamma_i : i \in \{1, 2, \dots, N(\varepsilon, \Gamma, u, v)\}, \Gamma_i \subset \Gamma \setminus K_n\big\},$$
$$N_n(\varepsilon) := \# S_n(\varepsilon).$$

Then

$$\begin{aligned}\varepsilon\big(N(\varepsilon,\Gamma,u,v)-N(\varepsilon,K_n,u,v)\big) &\le \varepsilon\big(\#\partial K_n+N_n(\varepsilon)\big)\\ &\le \varepsilon(\#\partial K_n)+\varepsilon N_n(\varepsilon)\\ &\le \varepsilon(\#\partial K_n)+\sum_{\Gamma_i\in S_n(\varepsilon)}\|u\chi_{\Gamma_i}\|_{p'}\|v\chi_{\Gamma_i}\|_p\\ &\le \varepsilon(\#\partial K_n)+\|u\chi_{\Gamma\setminus K_n}\|_{p'}\|v\chi_{\Gamma\setminus K_n}\|_p;\end{aligned}$$

the inequalities follows from properties of A and of norms on trees. It follows that

$$\limsup_{\varepsilon\to 0+}\varepsilon N(\varepsilon,\Gamma,u,v)\le \alpha_p\int_{K_n}|u|\,|v|+\|u\chi_{\Gamma\setminus K_n}\|_{p'}\|v\chi_{\Gamma\setminus K_n}\|_p,$$

which yields

$$\limsup_{\varepsilon\to 0+}\varepsilon N(\varepsilon,\Gamma,u,v)\le \alpha_p\int_{\Gamma}|u|\,|v|.$$

The converse inequality was proved in (2). □

From Lemma 5 and Theorem 3 we get the following asymptotic estimates for the approximation numbers of the Hardy type operator in the case when $u\in L^{p'}(\Gamma)$ and $v\in L^p(\Gamma)$.

Theorem 4. *Let $1<p<\infty$, $u\in L^{p'}(\Gamma)$ and $v\in L^p(\Gamma)$. Then*

$$\frac{\alpha_p}{3}\int_\Gamma|u|\,|v|\le\liminf_{n\to\infty}na_n(T)\le\limsup_{n\to\infty}na_n(T)\le 2\alpha_p\int_\Gamma|u|\,|v|. \tag{3}$$

In particular, for $p=2$ we have

$$\lim_{n\to\infty}na_n(T)=\frac{1}{\pi}\int_\Gamma|u|\,|v|.$$

Proof. The inequalities (3) follow from Lemma 5 and Theorem 3.

The case $p=2$ is quite technical. At first we can see that from Lemma 5 and Theorem 3 we have

$$\frac{\alpha_p}{3}\int_\Gamma|u|\,|v|\le\liminf_{n\to\infty}na_n(T)\le\limsup_{n\to\infty}na_n(T)\le \alpha_p\int_\Gamma|u|\,|v|.$$

Then we can improve the first estimate in (1) for a compact tree Γ; it is possible to show that

$$a_{N-3E(\Gamma)}(T_a)\ge\varepsilon, \tag{4}$$

where $E(\Gamma)$ is the number of edges in Γ.

Consider a sequence of compact subtrees K_l as in the proof of Lemma 5. By Lemma 4 and (4),

$$\lim_{n\to\infty} n a_n(T_{a,K_l}) = \alpha_2 \int_{K_l} |u|\,|v|.$$

Since $K_l \nearrow \Gamma$, we have $a_n(T_{a,K_l}) \le a_n(T_{a,K_{l+1}})$ and so

$$\lim_{n\to\infty} n a_n(T_a) \ge \alpha_2 \int_{\Gamma} |u|\,|v|.$$

From [2] we know that $\alpha_2 = 1/\pi$. □

The main result for $1 < p < \infty$

In this section we shall study the relationship between the approximation numbers σ_k and $\sigma_{k,i}$. The numbers σ_k were introduced in [10] for certain class of integral operators and they were found to be very useful to study of the approximation numbers. They were used in [2] and [6]. We shall show that in some sense σ_k and $\sigma_{k,i}$ are similar to a_k.

We set

$$Z_k := \{x : x \in \Gamma,\ 2^{kp'/p} \le U(x) < 2^{(k+1)p'/p}\}, \tag{5}$$

where $U(x) := \int_a^x |u(t)|^{p'}\, dt$ $(x \in \Gamma)$ and k is any integer if $u \in L^{p'}_{\text{loc}}(\Gamma)\setminus L^{p'}(\Gamma)$ and $2^k \le \|u\|^p_{p',\Gamma}$ if $u \in L^{p'}(\Gamma)$; we refer to these values of k as to *admissible* values.

We have $Z_k = \bigcup_{i=1}^{n_k} Z_{k,i}$ where $Z_{k,i}$ are connected components of Z_k. For every admissible k we set

$$\sigma^p_{k,i} := 2^k \mu(Z_{k,i}),\ i \in \{1,\dots,n_k\}, \quad \text{and} \quad \sigma^p_k := 2^k \mu(Z_k).$$

For all other k we set $\sigma_k = 0$.

Let $\Gamma'_b \subset \Gamma$ be a subtree and let b be the nearest point of Γ'_b to a. Suppose that the set $R := \{k : |Z_k \cap \Gamma'_b| > 0\}$ has more than one element. Take K_b from Theorem 1 for Γ'_b then we have $1/\alpha_{K_b} \le \min_{z\in\partial K_b\setminus\{b\}} \|u\|_{p',(b,z)} \le 2^{(m+1)/p}$ where $m := \inf\{k \in Z : |Z_k \cap (\Gamma'_b \setminus K_b)| > 0\}$, and

$$\begin{aligned}
\int_{\Gamma'_b\setminus K_b} v(x)^p dx &= \mu(\Gamma'_b \setminus K_b) \le \mu\Big(\bigcup_{(k,l)\in S} (Z_{k,l}\cap\Gamma'_b)\Big)\\
&= \sum_{(k,l)\in S} \mu(Z_{k,l}\cap\Gamma'_b) \le \sum_{(k,l)\in S} 2^{-k}\sigma^p_{k,l}\\
&\le \sum_{k\in R} 2^{-k} \sum_{l:(k,l)\in S} \sigma^p_{k,l} \le 2^{1-m} \sup_{k\in R} \sum_{l:(k,l)\in S} \sigma^p_{k,l},
\end{aligned}$$

where $S := \{(k,l) : |Z_{k,l} \cap (\Gamma'_b \setminus K_b)| > 0\}$. It follows from Theorem 1 that

$$\|T_{\Gamma'_b}\| \le 4 \sup_{K_b} \frac{\|v\chi_{(\Gamma'_b \setminus K_b)}\|_{p,\Gamma'_b}}{\alpha_{K_b}} \le 4^{1+1/p} \sup_{k\in R} \Big(\sum_{(l,k)\in S_0} \sigma^p_{k,l} \Big)^{1/p},$$

where $S_0 := \{(k,l) : |Z_{k,l} \cap \Gamma'_b| > 0\}$.

This observation leads us to the following theorem.

Theorem 5. *Let $u \in L^{p'}_{\mathrm{loc}}(\Gamma)$, $v \in L^p(\Gamma)$ and $1 < p < \infty$. Then*

$$2^{-1/p} \sup_k \max_{1\le j \le n_k} \sigma_{k,j} \le \|T \mid L^p(\Gamma) \to L^p(\Gamma)\| \le 4^{1+1/p} \sup_k \sigma_k \tag{6}$$

Proof. The right inequality follows from the previous observation with Γ'_b replaced by Γ. For the left inequality we first note that, by [8, Proposition 5.1], $\sup_{x\in\Gamma} \|u\chi_{(a,x)}\|_{p'} \|v\chi_{(a,x)^c}\|_p \le \|T\|$ where $(a,x)^c := \{y \in \Gamma : y \succ_a x\}$. Then, by the definition of Z_k and $Z_{k,i}$,

$$\begin{aligned}
\|T\| &\ge \sup_{x\in\Gamma} U(x)^{1/p'} \left(\int_{y\succ_a x} |v(y)|^p dy \right)^{1/p} \\
&\ge \sup_k \sup_{x\in Z_k} U(x)^{1/p'} \left(\int_{y\succ_a x} |v(y)|^p dy \right)^{1/p} \\
&\ge \sup_k \sup_{x\in Z_k} 2^{k/p} \left(\int_{y\succ_a x} |v(y)|^p dy \right)^{1/p} \\
&\ge \sup_k \big\{ 2^{k/p} \max_{1\le j\le n_{k+1}} \mu(Z_{k+1,j})^{1/p} \big\} \\
&= \sup_k \big\{ 2^{k/p} \max_{1\le j\le n_{k+1}} \sigma_{k+1,j} 2^{(-k-1)/p} \big\}.
\end{aligned}$$

□

Writing σ_k in place of $\sigma_{k,i}$ would improve the left inequality in (6). However, this is possible only for some special trees.

Theorem 6. *Let $1 \le p \le \infty$, $u \in L^{p'}_{\mathrm{loc}}(\Gamma)$, $v \in L^p(\Gamma)$, and suppose that T is compact and $\sum_k n^{1/p'}_{k+1} \sigma_k < \infty$. Then there exist positive constants c_1, c_2 such that*

$$c_1 \int_\Gamma |u|\,|v| \le \liminf_{n\to\infty} n a_n(T) \le \limsup_{n\to\infty} n a_n(T) \le c_2 \int_\Gamma |u|\,|v|. \tag{7}$$

In particular, if $p = 2$ then

$$\lim_{n\to\infty} n a_n(T) = \frac{1}{\pi} \int_\Gamma |u|\,|v|.$$

Proof. Let $m \in \mathbb{N}$ and set $K_m = \bigcup_{k=-m}^{m} \overline{Z}_k$, where Z_k are the sets defined in (5). Then K_m is a compact subset of Γ.

Let $\{\Gamma_j\}_{j=1}^{M(\varepsilon,\Gamma,u,v)}$ be a minimizing set of subtrees of Γ in the definition of $M(\varepsilon,\Gamma,u,v)$ and set

$$S_1(\varepsilon) = \Big\{\Gamma_i : i \in \{1,2,\dots,M(\varepsilon,\Gamma,u,v)\},\ \Gamma_i \subset \bigcup_{k=-\infty}^{-m-1} \overline{Z}_k\Big\},$$

$$S_2(\varepsilon) = \Big\{\Gamma_i : i \in \{1,2,\dots,M(\varepsilon,\Gamma,u,v)\},\ \Gamma_i \subset \bigcup_{k=m+1}^{\infty} \overline{Z}_k\Big\},$$

$$M_i(\varepsilon) = \#S_i(\varepsilon), \quad i = 1,2.$$

Then

$$\begin{aligned}
\varepsilon\big(M(\varepsilon,\Gamma,u,v) - M(\varepsilon,K_m,u,v)\big) &\le \varepsilon\big(n_{-m} + n_m + M_1(\varepsilon) + M_2(\varepsilon)\big)\\
&\le \varepsilon(n_m + n_{-m}) + \varepsilon M_1(\varepsilon) + \varepsilon M_2(\varepsilon)\\
&\le \varepsilon(n_m + n_{-m}) + \sum_{\Gamma_i \in S_1(\varepsilon)} \|T_{\Gamma_i}\| + \sum_{\Gamma_i \in S_2(\varepsilon)} \|T_{\Gamma_i}\|\\
&\le \varepsilon(n_m + n_{-m}) + c'\left(\varepsilon + \sum_{k\le -m} \sigma_k n_{k+1}^{1/p'} + \sum_{k \ge m} \sigma_k n_{k+1}^{1/p'}\right);
\end{aligned}$$

the last inequality follows from properties of a norm on a tree. Then

$$\limsup_{\varepsilon\to 0+} \varepsilon M(\varepsilon,\Gamma,u,v) \le \alpha_p \int_{K_m} |u|\,|v| + c'\left(\sum_{k\le -m} \sigma_k n_k^{1/p'} + \sum_{k\ge m} \sigma_k n_{k+1}^{1/p'}\right)$$

which yields

$$\limsup_{\varepsilon\to 0+} \varepsilon M(\varepsilon,\Gamma,u,v) \le \alpha_p \int_{\Gamma} |u|\,|v|;$$

whence the last inequality in (7) follows by the use of Theorem 3. The first inequality in (7) follows from (1) and (2).

The case $p = 2$ can be proved similarly as in Theorem 4. □

Next we study the relationship between σ_k and $a_k(T)$ for a general tree.

Theorem 7. *Let $q \in (1,\infty)$. Then*

$$\|\{a_k(T)\}\|_{l^q(\mathbb{N})} \le c\, \|\{n_{k+1}^{1/p'}\sigma_k\}\|_{l^q(\mathbb{Z})}$$

and

$$\|\{a_k(T)\}\|_{l^q_w(\mathbb{N})} \le c\, \|\{n_{k+1}^{1/p'}\sigma_k\}\|_{l^q_w(\mathbb{Z})}.$$

Proof. Let Γ_i, $i = 1, \ldots, M(\varepsilon, \Gamma, u, v)$, be a minimizing set of subtrees of Γ from the definition of $M(\varepsilon, \Gamma, u, v)$, so that $A(\Gamma_i) \geq \varepsilon$. We group the subtrees into families $\mathcal{F}_j$, $j \in \mathbb{Z}$, consisting of the maximal number of subtrees Γ_i such that $\Gamma_i \subset Z_j$ or $|\Gamma_i \cap Z_j| > 0$ and $\|T_{\Gamma_i}\| \leq 8(2^k \|v\|^p_{p,\Gamma_i \cap Z_k})^{1/p}$. Then

$$\varepsilon \# \mathcal{F}_j \leq c\, n^{1/p'}_{k_j+1} \sigma_{k_j}.$$

Since $A(\Gamma_i) \leq \|T_{\Gamma_i}\|$ and $\bigcup_{j \in \mathbb{Z}} \mathcal{F}_j = \{\Gamma_i\}_{i=1}^{M(\varepsilon,\Gamma,u,v)} = \Gamma$ we have

$$M(\varepsilon, \Gamma, u, v) = \sum_j \# \mathcal{F}_j \leq \sum_j \sum_{n=1}^{m_{k_j}} 1 = \sum_{n=1}^{\infty} \# \{ j : \varepsilon^{-1} 4^{1+1/p} n^{1/p'}_{j+1} \sigma_j \geq n \}$$
$$\leq \sum_{n=1}^{\infty} \# \Big\{ j : n^{1/p'}_{j+1} \sigma_j \geq \frac{n\varepsilon}{4^{1+1/p}} \Big\},$$

where $m_{k_j} := \varepsilon^{-1} 4^{1+1/p} n^{1/p'}_{k_j+1} \sigma_{k_j}$. Thus, if $\{n^{1/p'}_{k+1} \sigma_k\} \in l^q(\mathbb{Z})$ for some $q \in (1, \infty)$, then

$$q \int_0^\infty t^{q-1} M(t, \Gamma, u, v)\, dt \leq q \int_0^\infty \sum_{n=1}^{\infty} t^{q-1} \# \Big\{ j : n^{1/p'}_{j+1} \sigma_j \geq \frac{nt}{4^{1+1/p}} \Big\}\, dt$$
$$= 4^{(1+1/p)q} q \int_0^\infty \sum_{n=1}^{\infty} n^{-q} s^{q-1} \# \{ k : n^{1/p'}_{k+1} \sigma_k \geq s \} ds$$
$$\leq c \| \{ n^{1/p'}_{k+1} \sigma_k \} \|_{l^q(\mathbb{Z})}.$$

Since $a_{3M(\varepsilon,\Gamma,u,v)+1}(T) \leq 2\varepsilon$ it follows that

$$\#\{ k \in \mathbb{N} :\ a_k(T) > t \} \leq 3M(t/2, \Gamma, u, v) + 1.$$

This yields:

$$\| \{ a_k(T) \} \|_{l^q(\mathbb{N})} = q \int_0^\infty t^{q-1} \# \{ k \in \mathbb{N} : a_k(T) > t \}\, dt$$
$$\leq cq \int_0^{\|T\|} t^{q-1} \left[3M(t/2, \Gamma, u, v) + 1 \right] dt$$
$$\leq c\, \| \{ n^{1/p'}_{k+1} \sigma_k \} \|^q_{l^q(\mathbb{Z})} + \|T\|^q$$
$$\leq c\, \| \{ n^{1/p'}_{k+1} \sigma_k \} \|^q_{l^q(\mathbb{Z})}.$$

It is not hard to modify this proof for weak-l^q spaces and to obtain

$$\| \{ a_k(T) \} \|_{l^q_w(\mathbb{N})} \leq c\, \| \{ n^{1/p'}_{k+1} \sigma_k \} \|_{l^q_w(\mathbb{Z})}$$

for $q \in (1, \infty)$. □

Theorem 8. *Let* $q \in (p, \infty)$. *Then*

$$\|\{a_k(T)\}\|_{l^q(\mathbb{N})} \le c\,\|\{\sigma_k\}\|_{l^q(\mathbb{Z})} \tag{8}$$

and

$$\|\{a_k(T)\}\|_{l^q_w(\mathbb{N})} \le c\,\|\{\sigma_k\}\|_{l^q_w(\mathbb{Z})}. \tag{9}$$

Proof. The proof is similar to that of Theorem 7. We group the subtrees into families $\mathcal{F}_j$, $j \in \mathbb{Z}$, consisting of the maximal number of subtrees Γ_i such that $\Gamma_i \subset Z_j$ or $|\Gamma_i \cap Z_j| > 0$ and $\|T_{\Gamma_i}\|^p \le 8(2^k \|v\|^p_{p,\Gamma_i \cap Z_k})$. Then

$$\begin{aligned}\varepsilon^p \# \mathcal{F}_j &= \varepsilon^p \#\{i : \Gamma_i \subset Z_j\} \\ &\quad + \varepsilon^p \#\{i : |\Gamma_i \cap Z_j| > 0,\ \|T_{\Gamma_i}\|^p \le 8 \cdot 2^k \|v\|^p_{p,\Gamma_i \cap Z_k}\} \\ &\le \varepsilon^p \sigma_j^p + \varepsilon^p \sigma_j^p.\end{aligned}$$

From this estimates we can derive (8) and (9). □

Let us now suppose that the tree Γ satisfies the condition

$$\#\partial Z_{k,i} < B < \infty \text{ for every admissible } k \text{ and } i. \tag{10}$$

For such trees we can obtain lower estimates as counterparts to Theorems 7 and 9.

Theorem 9. *Let* Γ *satisfy* (10). *Then for any* $q > 0$ *there exists a constant* $c > 0$ *such that*

$$\|\{\sigma_{k,i}\}\|_{l^q(\mathbb{Z}\times\mathbb{N})} \le c\,\|\{a_k(T)\}\|_{l^q(\mathbb{N})}.$$

Proof. We shall present only a sketch of the rather long and technical proof. At first we show that for T compact and $\varepsilon > 0$,

$$\#\{(k,i) \in \mathbb{Z}\times\mathbb{N},\ \sigma_{k,i} > \varepsilon\} \le (B+1)2N(\varepsilon/c(p), \Gamma) + B + 1$$

where $c(p)$ is a constant dependent only on p. From this we get

$$\#\{(k,i) : \sigma_{k,i} > t\} \le 6(B+1)B\ \#\{k \in \mathbb{N} : a_k(T) > \varepsilon/c(p)\} + 4$$

for all $t > 0$, and, as a consequence,

$$\|\{\sigma_{k,i}\}\|^q_{l^q(\mathbb{Z}\times\mathbb{N})} \le c_1 \|\{a_k(T)\}\|^q_{l^q(\mathbb{N})} + c_2 \|\{\sigma_{k,i}\}\|^q_{l^\infty(\mathbb{Z}\times\mathbb{N})}$$

for all $q > 0$. It remains to use the estimates

$$\|\{\sigma_{k,i}\}\|_{l^\infty(\mathbb{Z}\times\mathbb{N})} \le c\,\|T\| = ca_1(T) \le c\|\{a_k(T)\}\|_{l^q(\mathbb{N})}.$$

□

Remark. For $p = 1$ and $p = \infty$ we can get similar results using the corresponding local asymptotic estimates (see [7]).

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