

BOUNDEDNESS AND COMPACTNESS OF GENERAL KERNEL INTEGRAL OPERATORS FROM A WEIGHTED BANACH FUNCTION SPACE INTO L_∞

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ABSTRACT. We give necessary and sufficient conditions for boundedness and compactness of a general kernel integral operator $Lf(x) = \int_I \ell(x, t)f(t) dt$, where the kernel ℓ is assumed only to be measurable, from an arbitrary weighted Banach function space into L_∞ . We give lower and upper bounds for the distance of L from compact operators. The proofs are carried out by means a new method based on real-variable techniques.

1. INTRODUCTION

The problem of boundedness and compactness of kernel integral operators $Lf(x) = \int_I \ell(x, t)f(t) dt$, where $\ell(x, t)$ is a general measurable function on I^2 and I is an interval, and their distance from compact operators, have been studied by many authors (cf. e.g. [LZ], [EEH], [S], [O], or the monograph [EE]). Usually, for L_p - L_q , $q < \infty$ type estimates, the authors use rather restrictive assumptions on the kernels. Typically (see for example [O, (1.3)]), the kernel is supposed to be positive, monotone in each variable, locally uniformly continuous, and satisfying certain triangle inequality.

The situation turns out to be different when the target space is either L_∞ or BMO. For instance, in [LP], the boundedness and compactness of the Hardy operator $Hf(x) = \int_0^x f(t)dt$ as an operator from a weighted Banach function space (X, v) into either L_∞ or BMO was characterized by means of relatively simple conditions. The methods from [LP] can be immediately generalized to kernel operators $Tf(x) = \int_0^\infty k(x, t)f(t) dt$, but only when k is positive and monotone in the first variable.

In this paper we develop a different method based on real-variable methods and measure-theoretic considerations, which enables us to characterize completely

1991 *Mathematics Subject Classification.* 26D10, 47G10.

Key words and phrases. Integral operators, general kernels, boundedness, compactness, distance from compact operators, weighted Banach function spaces, L_∞ .

This research was supported by the research grants No. 201/94/1066 and No. 201/96/0431 of the Grant Academy of Czech Republic.

boundedness and compactness of the kernel operator, assuming only that the kernel is measurable. A remarkable fact is that ℓ is allowed to take negative values. We further give sharp lower and upper bounds for the distance of L from the set of compact linear operators. Notably, it turns out that every compact kernel operator can be approximated by operators with kernels of the form $k(x, t) = \sum_{i=1}^n \chi_{M_i}(x)\psi_i(t)$, where $M_i \subset I$ and $\frac{\psi_i}{v} \in (X', v)$ (here (X', v) denotes the associate space to (X, v)).

The paper is structured as follows: preliminary material and some basic facts on Banach function spaces are collected in Section 2 (the standard general reference is [L] or [BS]). In Section 3 we characterize the boundedness of L by means of the norm of ℓ in $L_\infty(X', v)$. This section also contains the key real-variable considerations. In Section 4 we characterize the compactness of L , and in Section 5 we present lower and upper bounds for the distance of the Hardy operator from compact operators, recovering thereby, in particular, a result from [LP].

2. PRELIMINARIES

Let $-\infty \leq a < b \leq \infty$ and let $I = (a, b)$. Let $\mathfrak{M}(I)$ and $\mathfrak{M}(I^2)$ denote the sets of all measurable functions on I and I^2 . Let v be a *weight* (that is, a measurable and a.e. positive and measurable function) on I .

2.1 Definition. We say that a normed linear subspace (X, v) of $\mathfrak{M}(I)$ is a *weighted Banach function space* if the following five axioms are satisfied:

- (2.1) the norm $\|f\|_{X,v}$ is defined for all $f \in \mathfrak{M}(I)$, and $f \in (X, v)$ if and only if $\|f\|_{X,v} < \infty$;
- (2.2) $\|f\|_{X,v} = \| |f| \|_{X,v}$ for all $f \in \mathfrak{M}(I)$;
- (2.3) $0 \leq f_n \nearrow f$ a.e. in I , then $\|f_n\|_{X,v} \nearrow \|f\|_{X,v}$;
- (2.4) if $v(E) = \int_E v(t)dt < \infty$, then $\chi_E \in (X, v)$, where χ_E denotes the characteristic function of E ;
- (2.5) for every E with $v(E) < \infty$ there exists a constant C_E such that $\int_E f(t)v(t)dt \leq C_E \|f\|_{X,v}$ for all $f \in (X, v)(I)$.

In what follows, (X, v) will be a fixed weighted Banach function space.

2.2 Definition. The set

$$(X', v) = \left\{ f; \int_I f(t)g(t)v(t)dt < \infty \text{ for all } g \in (X, v) \right\}$$

is called the *associate space* of (X, v) . The space (X', v) , equipped with the norm

$$\|f\|_{X',v} := \sup_{\|g\|_{X,v} \leq 1} \left| \int_I f(t)g(t)v(t)dt \right|,$$

is also a weighted Banach function space. The Hölder inequality

$$(2.6) \quad \int_I |fg|v \leq \|f\|_{X,v} \|g\|_{X',v}$$

holds, and it is saturated in the sense that for every $g \in \mathfrak{M}(I)$ and $\varepsilon > 0$ there exists a function, f , such that $\|f\|_{X,v} = 1$ and

$$(2.7) \quad (1 - \varepsilon)\|g\|_{X',v} \leq \int_I fg v.$$

Throughout the paper we shall work with a kernel operator L , defined for $f \in (X, v)$ by

$$Lf(x) = \int_I \ell(x, t)f(t)dt,$$

where ℓ is a *kernel*, that is, $\ell \in \mathfrak{M}(I^2)$.

Of course, $\int_I \ell(x, t)f(t)dt$ need not exist in the Lebesgue sense for some functions from (X, v) . We say that the kernel ℓ is *admissible*, $\ell \in \mathcal{A}$, if there is a set $J \subset I$, $|I \setminus J| = 0$, such that for every $f \in (X, v)$ the function $x \mapsto \int_I \ell(x, t)f(t)dt$ is defined everywhere in J in the Lebesgue sense. In the opposite case we say that ℓ is *inadmissible*.

Let us recall that $f \in L_\infty$ if

$$\|f\|_{L_\infty} = \inf_{|M|=0} \sup_{I \setminus M} |f(x)| < \infty,$$

where $|M|$ denotes the Lebesgue measure of M .

We define $L_\infty(X', v)$ as the set of all $\ell \in \mathfrak{M}(I^2)$ such that

$$\|\ell\|_{L_\infty(X', v)} = \operatorname{ess\,sup}_{x \in I} \left\| \frac{\ell(x, \cdot)}{v(\cdot)} \right\|_{X', v} < \infty.$$

2.3 Lemma. *The set $L_\infty(X', v)$, equipped with the norm $\|\cdot\|_{L_\infty(X', v)}$, is a Banach space. Moreover, it satisfies*

- (i) $\ell \in L_\infty(X', v)$ if and only if $\|\ell\|_{L_\infty(X', v)} < \infty$;
- (ii) $\|\ell\|_{L_\infty(X', v)} = \|\ell\|_{L_\infty(X', v)}$;
- (iii) if $0 \leq \ell_n \nearrow \ell$ a.e. in I^2 , then $\|\ell_n\|_{L_\infty(X', v)} \nearrow \|\ell\|_{L_\infty(X', v)}$.

Proof. Clearly, $L_\infty(X', v)$ is a linear space, and $\|\cdot\|_{L_\infty(X', v)}$ defines a norm. Further, $\|\ell\|_{L_\infty(X', v)}$ is defined for every $\ell \in \mathfrak{M}(I^2)$. The properties (i) and (ii) are obvious. Let us prove (iii). Let $0 \leq \ell_n(x, t) \nearrow \ell(x, t)$ a.e. in I^2 . Set $F_n(x) = \left\| \frac{\ell_n(x, \cdot)}{v(\cdot)} \right\|_{X', v}$, $F(x) = \left\| \frac{\ell(x, \cdot)}{v(\cdot)} \right\|_{X', v}$. Since (X', v) and L_∞ are Banach function spaces we have $F_n(x) \nearrow F(x)$ for a.e. $x \in I$ and, consequently, $\|F_n\|_{L_\infty} \nearrow \|F\|_{L_\infty}$, which proves (iii). The completeness of $L_\infty(X', v)$ follows from the proof of [BS, Chapter 1, Theorem 1.6]. (This proof does not use (2.4) nor (2.5) – the essential ingredient is the Fatou's property (2.3).) \square

We note that $L_\infty(X', v)$ need not be a Banach function space as it does not necessarily satisfy (2.4) or (2.5).

By $\|L\|$ we shall denote the norm of an operator L , given by a kernel $\ell \in \mathcal{A}$, as an operator from (X, v) into L_∞ , i.e.,

$$\|L\| = \sup_{\|f\|_{X, v} \leq 1} \operatorname{ess\,sup}_{x \in I} \left| \int_I \ell(x, t)f(t)dt \right|.$$

3. Boundedness of a general kernel operator

The main aim of this section is to establish important relations between a kernel ℓ and the corresponding operator L .

3.1 Lemma. *Let $\ell \in L_\infty(X', v)$. Then $\ell \in \mathcal{A}$ and $\|L\| \leq \|\ell\|_{L_\infty(X', v)}$.*

Proof. We denote $I_1 = \{x \in I; \ell(x, t) \text{ is defined for a.e. } t \in I\}$. Since $\ell \in \mathfrak{M}(I^2)$ we have $|I \setminus I_1| = 0$. The definition of $\|\ell\|_{L_\infty(X', v)}$ guarantees that there is a set I_2 , $|I \setminus I_2| = 0$, such that

$$\|\ell\|_{L_\infty(X', v)} = \sup_{x \in I_2} \left\| \frac{\ell(x, \cdot)}{v(\cdot)} \right\|_{X', v}.$$

We now fix an f , $\|f\|_{X, v} < \infty$. According to Lemma 2.3 and (2.6) we have for $x \in I_1 \cap I_2$

$$\left| \int_I \ell(x, t) f(t) dt \right| \leq \left\| \frac{\ell(x, \cdot)}{v(\cdot)} \right\|_{X', v} \|f\|_{X, v}.$$

Thus $\ell \in \mathcal{A}$.

Moreover, for $\|f\|_{X, v} = 1$ we obtain

$$\begin{aligned} \|Lf\|_\infty &= \inf_{|M|=0} \sup_{x \in I \setminus M} \left| \int_I \ell(x, t) f(t) dt \right| \\ &\leq \sup_{x \in I_1 \cap I_2} \left| \int_I \ell(x, t) f(t) dt \right| \leq \sup_{x \in I_1 \cap I_2} \left\| \frac{\ell(x, \cdot)}{v(\cdot)} \right\|_{X', v} \\ &\leq \sup_{x \in I_2} \left\| \frac{\ell(x, \cdot)}{v(\cdot)} \right\|_{X', v} = \|\ell\|_{L_\infty(X', v)}, \end{aligned}$$

which yields $\|L\| \leq \|\ell\|_{L_\infty(X', v)}$. The proof is complete. \square

The rest of this section is devoted to the proof of the converse inequality, which is much more complicated. It will follow from a series of lemmas. We start with two measure theory lemmas. As usual, $A \div B$ denotes the symmetric difference of A and B .

For $A \subset I^2$, we denote by A_x the intersection of A with $\{x\} \times I$, i.e., $A_x = \{y, (x, y) \in A\}$.

3.2 Lemma. *Let $\Omega \subset I^2$ be an open set. Let $M \subset I$ be a measurable set such that $|(x - \delta, x + \delta) \cap M| > 0$ for all $x \in M$ and $\delta > 0$. Then for every $\varepsilon > 0$ there exist a $Z \subset I$ and an $N \subset M$ such that $|N| > 0$ and $|\Omega_x \div Z| < \varepsilon$ for every $x \in N$.*

Proof. Assume the contrary. Let $\varepsilon > 0$ be such that for every $Z \subset I$ and $N \subset M$, $|N| > 0$, there is an $x \in N$ such that $|\Omega_x \div Z| > \varepsilon$. Let $x_0 \in M$. Then Ω_{x_0} is an open subset of I , whence either $\Omega_{x_0} = \emptyset$ or $\Omega_{x_0} = \bigcup_{i=1}^\infty (a_i, b_i)$ for some $0 \leq a_i < b_i \leq 1$. By the regularity of measure, there is a $K_0 = \bigcup_{i=1}^{n_0} [c_i, d_i]$ such that $K_0 \subset \Omega_{x_0}$ and

$$(3.1) \quad |\Omega_{x_0} \setminus K_0| < \frac{\varepsilon}{4}.$$

Now, K_0 is compact. Therefore, the distance of $\{x_0\} \times K_0$ from $I^2 \setminus \Omega$ positive. Thus, for a $\delta_0 > 0$ small enough we have

$$(3.2) \quad (x_0 - \delta_0, x_0 + \delta_0) \times K_0 \subset \Omega.$$

Set $Z = \Omega_{x_0}$ and $N = (x_0 - \delta_0, x_0 + \delta_0) \cap M$. By our assumption, there is an $x_1 \in (x_0 - \delta_0, x_0 + \delta_0) \cap M$ such that

$$(3.3) \quad |\Omega_{x_0} \div \Omega_{x_1}| > \varepsilon.$$

Now, by (3.2), $K_0 \subset \Omega_{x_1}$, (3.1), and (3.3),

$$(3.4) \quad |\Omega_{x_1} \setminus K_0| > \frac{3\varepsilon}{4}.$$

Since $\Omega_{x_1} \setminus K_0$ is open, there exists a set $R_1 = \bigcup_{i=n_0+1}^{n_1} [c_i, d_i] \subset (\Omega_{x_1} \setminus K_0)$ such that

$$(3.5) \quad |\Omega_{x_1} \setminus (R_1 \cup K_0)| < \frac{\varepsilon}{4},$$

and as a consequence of (3.4) and (3.5) we have

$$(3.6) \quad |R_1| > \frac{\varepsilon}{2}.$$

Denote $K_1 = R_1 \cup K_0 = \bigcup_{i=1}^{n_1} [c_i, d_i]$. Now, as above, K_1 is compact, whence, for $\delta_1 > 0$ small enough, we have

$$(3.7) \quad (x_1 - \delta_1, x_1 + \delta_1) \times K_1 \subset \Omega.$$

Let $Z = \Omega_{x_1}$ and $N = (x_1 - \delta_1, x_1 + \delta_1) \cap M$. By our assumption, there is an $x_2 \in (x_1 - \delta_1, x_1 + \delta_1) \cap M$ such that $|\Omega_{x_1} \div \Omega_{x_2}| > \varepsilon$. As above, $K_1 \subset \Omega$ and $|\Omega_{x_2} \setminus K_1| > \frac{3\varepsilon}{4}$. Since $\Omega_{x_2} \setminus K_1$ is an open set, there is a set $R_2 = \bigcup_{i=n_1+1}^{n_2} [c_i, d_i] \subset (\Omega_{x_2} \setminus K_1)$ such that $|\Omega_{x_2} \setminus (R_2 \cup K_1)| < \frac{\varepsilon}{4}$, and, consequently, $|R_2| > \frac{\varepsilon}{2}$. Let $K_2 = K_1 \cup R_2 = \bigcup_{i=1}^{n_2} [c_i, d_i]$. Then $|K_2| > \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Since K_2 is a compact set, we have for δ_2 small enough $(x_2 - \delta_2, x_2 + \delta_2) \times K_2 \subset \Omega$. Let $Z = \Omega_{x_2}$ and $N = (x_2 - \delta_2, x_2 + \delta_2) \cap M$. Continuing this process we obtain after m steps for large $m \in \mathbb{N}$ $|K_m| > |I|$, which is a contradiction. \square

3.3 Lemma. *Let $A \subset I^2$ be a measurable set and let $M \subset I$, $|M| > 0$. Then for every $\varepsilon > 0$ there is a set $N \subset M$, $|N| > 0$, such that*

$$|A_x \div A_y| < \varepsilon \quad \text{for all } x, y \in N.$$

Proof. Let $\varepsilon > 0$ be fixed. In the case $|A| = 0$ it suffices to put $N = M$. Let $|A| > 0$. Define $P = \{x; |A_x| > 0\}$. Clearly, $|P| > 0$. Since $M = (M \setminus P) \cup (M \cap P)$ then either $|M \setminus P| > 0$ or $|M \cap P| > 0$. In the case $|M \setminus P| > 0$, it suffices

to put $N = M \setminus P$. Assume that $|M \setminus P| = 0$. Denote $M_1 = M \cap P$ and $B = \cup_{x \in M_1} \{x\} \times A_x$. Clearly, $|B| > 0$.

The regularity of the Lebesgue measure gives the existence of an open set Ω , $B \subset \Omega$, such that $|\Omega \setminus B| < \frac{\varepsilon}{4}|M_1|$. Set

$$(3.8) \quad Q = \{x \in M_1; |\Omega_x \setminus B_x| \geq \frac{\varepsilon}{4}\}.$$

If $|Q| = |M_1|$, then the Fubini theorem implies $|\Omega \setminus B| \geq \frac{\varepsilon}{4}|Q| = \frac{\varepsilon}{4}|M_1|$, which is a contradiction. Therefore, $|Q| < |M_1|$.

Let M_2 be a set of all density points of $M_1 \setminus Q$. By the Lebesgue density theorem we have $|M_2| = |M_1 \setminus Q| > 0$, and by (3.8) we obtain

$$(3.9) \quad |\Omega_x \setminus B_x| = |\Omega_x \setminus A_x| < \frac{\varepsilon}{4} \quad \text{for all } x \in M_2.$$

According to Lemma 3.2, there are sets $Z \subset I$ and $N \subset M_2$, $|N| > 0$, such that

$$(3.10) \quad |\Omega \div Z| < \frac{\varepsilon}{8} \quad \text{for all } x \in N.$$

Now, we fix $x, y \in N$. We shall estimate $|A_x \div A_y|$. Since

$$A_x \setminus A_y \subset \Omega_x \setminus A_y \subset (\Omega_x \setminus \Omega_y) \cup (\Omega_y \setminus A_y),$$

(3.10) implies

$$(3.11) \quad |A_x \setminus A_y| < |\Omega_x \setminus \Omega_y| + \frac{\varepsilon}{4}.$$

Moreover, $\Omega_x \setminus \Omega_y \subset (\Omega_x \setminus Z) \cup (Z \setminus \Omega_y)$ and $\Omega_y \setminus \Omega_x \subset (\Omega_y \setminus Z) \cup (Z \setminus \Omega_x)$, which together with (3.10) yields

$$|\Omega_x \div \Omega_y| \leq |\Omega_x \div Z| + |\Omega_y \div Z| < \frac{\varepsilon}{4}.$$

Using (3.11), we obtain $|A_x \setminus A_y| < \frac{\varepsilon}{2}$, and, consequently, $|A_x \div A_y| < \varepsilon$. The proof is complete. \square

In the sequel we denote by $f|_K$ the restriction of a function f to a set K .

3.4 Lemma. *Let $I = [0, 1]$ and let $K \subset I^2$ be a compact set. Let $\ell \geq 0$ be a kernel such that $\ell|_K$ is continuous and $\ell = 0$ on $I^2 \setminus K$. Let $C > 0$ and $\mathcal{F} = \{f, \|f\|_{X,v} \leq 1 \text{ and } 0 \leq f \leq C\}$. We define*

$$F(x) = \sup_{f \in \mathcal{F}} \int_I \ell(x, t) f(t) dt.$$

Then F is a measurable function on I . Moreover, setting $P = \text{ess sup}_{x \in I} F(x)$, we have for $\ell \in \mathcal{A}$ an inequality $\|L\| \geq P$.

Proof. By our assumptions on K and ℓ , there exists a constant $D > 0$ such that $0 \leq \ell \leq D$ on I^2 . Let $\varepsilon > 0$. We divide the proof into three steps. In the first two steps we prove that F is measurable and in the third step we show $\|L\| \geq P$.

Step 1. We claim that for a set $M \subset I$, $|M| > 0$, there is a set $N \subset M$, $|N| > 0$, and $f \in \mathcal{F}$, such that

$$F(x) - \varepsilon(1 + 2C + 4CD) \leq \int_I \ell(x, t)f(t)dt \leq F(x) \quad \text{for all } x \in N.$$

Let $|M| > 0$. By Lemma 3.3, there is $M_0 \subset M$, such that $|M_0| > 0$ and

$$(3.12) \quad |K_x \div K_y| < \varepsilon \quad \text{for all } x, y \in M_0.$$

Now, choose $x_0 \in M_0$ such that for every $\delta > 0$ we have $|(x_0 - \delta, x_0 + \delta) \cap M_0| > 0$. By the uniform continuity of $\ell|_K$, there is a $\delta_0 > 0$ small enough and such that, for $x \in M_1 = (x_0 - \delta_0, x_0 + \delta_0) \cap M_0$ and $t \in K_x \cap K_{x_0}$,

$$(3.13) \quad |\ell(x, t) - \ell(x_0, t)| < \varepsilon.$$

From the definition of $F(x)$ we obtain the existence of $f_0 \in \mathcal{F}$ satisfying

$$(3.14) \quad F(x_0) - \varepsilon \leq \int_I \ell(x_0, t)f_0(t)dt \leq F(x_0).$$

Set

$$G(x) = \int_I \ell(x, t)f_0(t)dt \quad \text{for } x \in I.$$

Clearly,

$$(3.15) \quad G(x) \leq F(x) \quad \text{for a.e. } x \in I.$$

Let $x \in M_1$ and $f \in \mathcal{F}$ be fixed. We denote

$$R(x, f) = \int_I (\ell(x, t) - \ell(x_0, t))f(t)dt.$$

Using (3.12), (3.13), and $0 \leq \ell \leq D$, we get

$$\begin{aligned} |R(x, f)| &\leq \int_I |\ell(x, t) - \ell(x_0, t)|f(t)dt \\ &= \int_{K_x \cap K_{x_0}} |\ell(x, t) - \ell(x_0, t)|f(t)dt \\ &\quad + \int_{K_x \setminus K_{x_0}} \ell(x, t)f(t)dt + \int_{K_{x_0} \setminus K_x} \ell(x_0, t)f(t)dt \\ &\leq \varepsilon C + 2\varepsilon CD = \varepsilon(C + 2CD). \end{aligned}$$

Now, setting $f \equiv f_0$ we have

$$(3.16) \quad |G(x) - G(x_0)| \leq \varepsilon(C + 2CD) \quad \text{for } x \in M_1.$$

Let $R(x) = \sup_{f \in \mathcal{F}} |R(x, f)|$. We immediately obtain

$$(3.17) \quad 0 \leq R(x) \leq \varepsilon(C + 2CD) \quad \text{for all } x \in M_1.$$

Since

$$\int_I \ell(x, t) f(t) dt = \int_I \ell(x_0, t) f(t) dt + R(x, f),$$

we have for $x \in M_1$

$$F(x) = \sup_{f \in \mathcal{F}} \left(\int_I \ell(x_0, t) f(t) dt + R(x, f) \right),$$

and, consequently,

$$F(x) \leq \sup_{f \in \mathcal{F}} \int_I \ell(x_0, t) f(t) dt + R(x).$$

We can rewrite the last inequality as $F(x) \leq F(x_0) + R(x)$, $x \in M_1$. Thus, by the above inequalities, (3.16), (3.14), and (3.17), we obtain for $x \in M_1$

$$\begin{aligned} G(x) &\geq G(x_0) - \varepsilon(C + 2CD) \geq F(x_0) - \varepsilon(1 + C + 2CD) \\ &\geq F(x) - R(x) - \varepsilon(1 + C + 2CD) \geq F(x) - \varepsilon(1 + 2C + 4CD). \end{aligned}$$

Now, it suffices to use (3.15) and the last inequality, and to set $f = f_0$ and $N = M_1$ to prove our claim.

Step 2. Let $M = I$. By Step 1, there exist $N_1 \subset I$, $|N_1| > 0$, and $f_1 \in \mathcal{F}$ such that

$$F(x) - \varepsilon(1 + 2C + 4CD) \leq \int_I \ell(x, t) f_1(t) dt \leq F(x) \quad \text{for } x \in N_1.$$

Assume that we have constructed sets N_β and functions f_β for all ordinal numbers $\beta < \alpha$, where α is a fixed countable ordinal number. Set $M = I \setminus \bigcup_{\beta < \alpha} N_\beta$. If $|M| = 0$, we stop the construction. If $|M| > 0$, then by Step 1 we have $N_\alpha \subset I \setminus \bigcup_{\beta < \alpha} N_\beta$, $|N_\alpha| > 0$, and there is an $f_\alpha \in \mathcal{F}$ such that

$$F(x) - \varepsilon(1 + 2C + 4CD) \leq \int_I \ell(x, t) f_\alpha(t) dt \leq F(x) \quad \text{for } x \in N_\alpha.$$

This process will stop after countably many steps. Hence, there exists a countable ordinal number γ such that for $x \in N_\beta$, $\beta < \gamma$,

$$(3.18) \quad F(x) - \varepsilon(1 + 2C + 4CD) \leq \int_I \ell(x, t) f_\beta(t) dt \leq F(x),$$

and moreover

$$(3.19) \quad N_\alpha \cap N_\beta = \emptyset \quad \text{for } \alpha \neq \beta, \quad |N_\beta| > 0 \quad \text{for } \beta < \gamma;$$

$$(3.20) \quad |I \setminus \bigcup_{\beta < \gamma} N_\beta| = 0.$$

Define a function H_ε by $H_\varepsilon(x) = \sum_{\beta < \gamma} \chi_{N_\beta}(x) \int_I \ell(x, t) f_\beta(t) dt$. Since for every $\beta < \gamma$ the functions $x \mapsto \int_I \ell(x, t) f_\beta(t) dt$ and $\chi_{N_\beta}(x)$ are measurable, H_ε is measurable as well. Moreover, according to (3.18), (3.19) and (3.20) we have for a.e. $x \in I$

$$F(x) - \varepsilon(1 + 2C + 4CD) \leq H_\varepsilon(x) \leq F(x),$$

which implies

$$F(x) = \lim_{n \rightarrow \infty} H_{1/n}(x) \quad \text{for a.e. } x \in I,$$

and, consequently, F is measurable.

Step 3. We claim $\|L\| \geq P$. Observe that

$$\|L\| = \sup_{\|f\|_{X,v} \leq 1} \operatorname{ess\,sup}_{x \in I} \left| \int_I \ell(x, t) f(t) dt \right| = \sup_{\|f\|_{X,v} \leq 1} \operatorname{ess\,sup}_{x \in I} \int_I \ell(x, t) f(t) dt.$$

By the definition of P , there is a set M , $|M| > 0$, such that

$$P - \varepsilon \leq F(x) \leq P \quad \text{for all } x \in M.$$

Now, by Step 1, there is a set $N \subset M$, $|N| > 0$, and a function $f_0 \in \mathcal{F}$ such that

$$\int_I \ell(x, t) f_0(t) dt \geq F(x) - \varepsilon(1 + 2C + 4CD) \quad \text{for all } x \in N.$$

Thus,

$$\begin{aligned} \|L\| &= \sup_{\|f\|_{X,v} \leq 1} \operatorname{ess\,sup}_{x \in I} \int_I \ell(x, t) f(t) dt \\ &\geq \sup_{f \in \mathcal{F}} \operatorname{ess\,sup}_{x \in N} \int_I \ell(x, t) f(t) dt \\ &\geq \operatorname{ess\,sup}_{x \in N} \int_I \ell(x, t) f_0(t) dt \geq F(x) - \varepsilon(1 + 2C + 4CD) \\ &\geq P - \varepsilon(2 + 2C + 4CD). \end{aligned}$$

Letting ε tend to zero we complete the proof. \square

3.5 Lemma. *Let $I = [0, 1]$ and let ℓ be a measurable kernel on I^2 , $0 \leq \ell \leq D$, for some $D > 0$. Let $C > 0$. Define $\mathcal{F} = \{f, \|f\|_{X,v} \leq 1 \text{ and } 0 \leq f \leq C\}$. Then the function F , from Lemma 3.4 is measurable. If $\ell \in \mathcal{A}$, then $\|L\| \geq P$.*

Proof. Let K_n be a sequence of compact sets, $K_n \nearrow I^2$, such that $\ell|_{K_n}$ are continuous functions. Set $\ell_n(x, t) = \ell(x, t)\chi_{K_n}(x, t)$ and

$$\begin{aligned} L_n f(x) &= \int_I \ell_n(x, t) f(t) dt \quad \text{for } f \in (X, v), \\ F_n(x) &= \sup_{f \in \mathcal{F}} L_n f(x) \text{ and,} \\ P_n &= \text{ess sup}_{x \in I} F_n(x). \end{aligned}$$

Clearly, $0 \leq \ell_n \nearrow \ell$ a.e. in I^2 . Then there is a set $I_1 \subset I$, $|I \setminus I_1| = 0$, such that for every $x \in I_1$ we have $0 \leq \ell_n(x, \cdot) \nearrow \ell(x, \cdot)$ a.e. in I . Thus, $0 \leq \ell_n(x, t)f(t) \nearrow \ell(x, t)f(t)$ for $x \in I_1$ and $f \geq 0$, whence

$$(3.21) \quad 0 \leq \int_I \ell_n(x, t) f(t) dt \nearrow \int_I \ell(x, t) f(t) dt \quad \text{for } x \in I_1 \text{ and } f \geq 0.$$

Now, it is not difficult to verify that

$$0 \leq F_n(x) \nearrow F(x) \text{ for } x \in I_1.$$

By Lemma 3.4, F_n are measurable. Thus F is measurable as a pointwise limit of F_n . Moreover, it is readily seen that

$$(3.22) \quad 0 \leq P_n \nearrow P.$$

Now, since ℓ_n and ℓ are non-negative, for every $f \in \mathfrak{M}(I)$ we have

$$(3.23) \quad \begin{cases} (\ell_n(x, t)f(t))^+ = \ell_n(x, t)f^+(t), \\ (\ell(x, t)f(t))^+ = \ell(x, t)f^+(t), \end{cases}$$

and, similarly,

$$(3.24) \quad \begin{cases} (\ell_n(x, t)f(t))^- = \ell_n(x, t)f^-(t), \\ (\ell(x, t)f(t))^- = \ell(x, t)f^-(t). \end{cases}$$

Since $\ell \in \mathcal{A}$, there is a set J , $|I \setminus J| = 0$, such that $\int_I \ell(x, t)f(t) dt$ exists for any $x \in J$ and $f \in (X, v)$. Let us fix $x \in J \cap I_1$ and $f \in (X, v)$. Since $0 \leq \ell_n(x, t) \leq \ell(x, t)$, we have $0 \leq \ell_n(x, t)f^+(t) \leq \ell(x, t)f^+(t)$ and $0 \leq \ell_n(x, t)f^-(t) \leq \ell(x, t)f^-(t)$. Together with (3.23) and (3.24), this implies

$$(3.25) \quad \int_I (\ell_n(x, t)f(t))^+ dt \leq \int_I (\ell(x, t)f(t))^+ dt = A^+,$$

and

$$(3.26) \quad \int_I (\ell_n(x, t)f(t))^- dt \leq \int_I (\ell(x, t)f(t))^- dt = A^-.$$

Since $\int_I \ell(x, t)f(t)dt$ is well-defined, either $A^+ < \infty$ or $A^- < \infty$ and, consequently, by (3.25) and (3.26), $\int_I \ell_n(x, t)f(t)dt$ is well-defined. It immediately follows that $\ell_n \in \mathcal{A}$. Moreover, it is clear that

$$\|L\| = \sup_{\|f\|_{x,v} \leq 1, f \geq 0} \text{ess sup}_{x \in I} \int_I \ell(x, t)f(t)dt,$$

and

$$\|L_n\| = \sup_{\|f\|_{x,v} \leq 1} \text{ess sup}_{x \in I} \int_I \ell_n(x, t)f(t)dt.$$

For $f \geq 0$ we have $\int_I \ell_n(x, t)f(t)dt \leq \int_I \ell(x, t)f(t)dt$. We thus obtain

$$(3.27) \quad \|L\| \geq \|L_n\| \quad \text{for any } n \in \mathbb{N}.$$

Observe that the kernels ℓ_n satisfy the assumptions of Lemma 3.4 and therefore $\|L_n\| \geq P_n$, which via (3.22) and (3.27) implies $\|L\| \geq P$. The proof is complete. \square

For $M \subset \mathbb{R}$ measurable we denote by $\mathfrak{D}(M)$ the set of all Lebesgue density points of M . Recall that $|M \setminus \mathfrak{D}(M)| = 0$.

3.6 Lemma. *Let $I = [0, 1]$, $A \in \mathfrak{M}(I^2)$ and $M \in \mathfrak{M}(I)$, $|M| > 0$. Then there exists $N \subset M$, $|M \setminus N| = 0$, $N = \mathfrak{D}(N)$ with the following property: for every $x \in N$ and $\varepsilon > 0$ there is a set $N_{\varepsilon, x} \subset N$, such that*

$$(3.28) \quad N_{\varepsilon, x} = \mathfrak{D}(N_{\varepsilon, x}),$$

$$(3.29) \quad x \in N_{\varepsilon, x},$$

$$(3.30) \quad |A_y \div A_z| < \varepsilon \quad \text{for } y, z \in N_{\varepsilon, x}.$$

Proof. Fix $\varepsilon > 0$. By Lemma 3.3, there is a set $\widetilde{M}_{\varepsilon, 1} \subset M$, $|\widetilde{M}_{\varepsilon, 1}| > 0$, such that $|A_y \div A_z| < \varepsilon$ for $y, z \in \widetilde{M}_{\varepsilon, 1}$. Set $M_{\varepsilon, 1} = \mathfrak{D}(\widetilde{M}_{\varepsilon, 1})$. Clearly, $|\widetilde{M}_{\varepsilon, 1} \setminus M_{\varepsilon, 1}| = 0$ and, consequently, $|M_{\varepsilon, 1}| > 0$. Assume that we have constructed for an ordinal number α the sets $M_{\varepsilon, \beta}$, $\beta < \alpha$, such that for any β we have

$$|A_y \div A_z| < \varepsilon \quad \text{for } y, z \in M_{\varepsilon, \beta} \quad \text{and} \quad M_{\varepsilon, \beta} = \mathfrak{D}(M_{\varepsilon, \beta}).$$

If $|M \setminus \bigcup_{\beta < \alpha} M_{\varepsilon, \beta}| = 0$, we set $M_\varepsilon = \bigcup_{\beta < \alpha} M_{\varepsilon, \beta}$ and we stop the construction. If $|M \setminus \bigcup_{\beta < \alpha} M_{\varepsilon, \beta}| > 0$, then, by Lemma 3.3, there is an $\widetilde{M}_{\varepsilon, \alpha} \subset M \setminus \bigcup_{\beta < \alpha} M_{\varepsilon, \beta}$, $|\widetilde{M}_{\varepsilon, \alpha}| > 0$ and $|A_y \div A_z| < \varepsilon$ for $y, z \in \widetilde{M}_{\varepsilon, \alpha}$. Set $M_{\varepsilon, \alpha} = \mathfrak{D}(\widetilde{M}_{\varepsilon, \alpha})$.

This process will stop after a countable number of steps. Hence there is a countable ordinal γ_ε such that

$$|M \setminus \bigcup_{\beta < \gamma_\varepsilon} M_{\varepsilon, \beta}| = 0,$$

and for $\beta < \gamma_\varepsilon$ we have

$$M_{\varepsilon, \beta} = \mathfrak{D}(M_{\varepsilon, \beta}), \quad \text{and} \quad |A_y \div A_z| < \varepsilon \quad \text{for} \quad y, z \in M_{\varepsilon, \beta}.$$

Let us define

$$M_\varepsilon = \bigcup_{\beta < \gamma_\varepsilon} M_{\varepsilon, \beta}.$$

Set

$$\tilde{N} = \bigcap_{n=1}^{\infty} M_{\frac{1}{n}} \quad \text{and} \quad N = D(\tilde{N}).$$

Evidently, $N \subset M$. Moreover, $|M \setminus M_{\frac{1}{n}}| = 0$ for $n \in \mathbb{N}$, hence $|M \setminus \bigcap_{n=1}^{\infty} M_{\frac{1}{n}}| = |M \setminus \tilde{N}| = 0$, and, as $|\tilde{N} \setminus N| = 0$, we have $|M \setminus N| = 0$. Clearly, $N = \mathfrak{D}(N)$.

Let $\varepsilon > 0$ and $x \in N$. Fix n such that $\frac{1}{n} < \varepsilon$. Then $x \in M_{\frac{1}{n}} = \bigcup_{\beta < \gamma_{\frac{1}{n}}} M_{\frac{1}{n}, \beta}$. Let $\alpha < \gamma_{\frac{1}{n}}$ be an ordinal number such that $x \in M_{\frac{1}{n}, \alpha}$. By the construction we have $M_{\frac{1}{n}, \alpha} = \mathfrak{D}(M_{\frac{1}{n}, \alpha})$. Moreover,

$$|A_y \div A_z| < \frac{1}{n} < \varepsilon \quad \text{for} \quad y, z \in M_{\frac{1}{n}, \alpha}.$$

Now, to prove (3.28), (3.29) and (3.30), it suffices to take $N_{\varepsilon, x} = M_{\frac{1}{n}, \alpha} \cap N$. \square

3.7 Lemma. *Let $I = [0, 1]$, $\ell \in \mathfrak{M}(I^2)$ and let $D > 0$ be such that $|\ell| \leq D$ a.e. in I^2 . Let $C > 0$. Set $\mathcal{F} = \{f; \|f\|_{X, v} \leq 1 \text{ and } |f| \leq C\}$. Then the function F from Lemma 3.4 is measurable. Moreover, if $\ell \in \mathcal{A}$, then, $\|L\| \geq P$.*

Proof. Since $\ell \in \mathcal{A}$, there is a $J \subset I$, $|I \setminus J| = 0$, and the function $x \mapsto \int_I \ell(x, t) f(x) dt$ is well-defined for all $x \in J$ and $f \in (X, v)$. Fix $x \in J$ and $f \in (X, v)$. Then

$$(3.31) \quad F(x) = \sup_{f \in \mathcal{F}} \int_I |\ell(x, t)| f(t) dt$$

By Lemma 3.5, the last expression is a measurable function. Thus, F is measurable.

Let $\varepsilon > 0$. Then there is a set $M \subset J$, $|M| > 0$, such that

$$(3.32) \quad P - \varepsilon \leq F(x) \leq P \quad \text{for all } x \in M.$$

Set

$$K^+ = \{(x, y) \in I^2, \ell(x, t) > 0\}, \quad K^- = \{(x, y) \in I^2, \ell(x, t) < 0\}, \quad K = K^+ \cup K^-,$$

and

$$PK^+ = \{x \in J, |K_x^+| > 0\}, \quad PK^- = \{x \in J, |K_x^-| > 0\}.$$

Let further

$$\begin{aligned} M_1 &= PK^+ \cap PK^- \cap M, & M_2 &= (PK^+ \setminus PK^-) \cap M, \\ M_3 &= (PK^- \setminus PK^+) \cap M, & M_4 &= (I \setminus (PK^+ \cup PK^-)) \cap M. \end{aligned}$$

Clearly, $M = \bigcup_{i=1}^4 M_i$, and at least one of these sets has a positive measure.

Fix $i \in \{2, 3, 4\}$ and assume $|M_i| > 0$. Set

$$\begin{aligned} \ell_i(x, t) &= \ell(x, t)\chi_{M_i}(t), \\ (L_i f)(x) &= \int_I \ell_i(x, t)f(t)dt. \end{aligned}$$

It is easy to see that, $0 \leq \ell_2 \leq D$, $-D \leq \ell_3 \leq 0$ and $\ell_4 = 0$, $\ell_i \in \mathcal{A}$ and

$$(3.33) \quad \|L\| \geq \|L_i\|.$$

Define

$$P_i = \operatorname{ess\,sup}_{x \in I} \sup_{\|f\|_{x,v} \leq 1, 0 \leq f \leq C} \int_I |\ell_i(x, t)|f(t)dt.$$

Then, by the definition of ℓ_i and (3.31), we have

$$\begin{aligned} P_i &\geq \operatorname{ess\,sup}_{x \in M_i} \sup_{\|f\|_{x,v} \leq 1, 0 \leq f \leq C} \int_I |\ell_i(x, t)|f(t)dt \\ &= \operatorname{ess\,sup}_{x \in M_i} \sup_{\|f\|_{x,v} \leq 1, 0 \leq f \leq C} \int_I |\ell(x, t)|f(t)dt = \operatorname{ess\,sup}_{x \in M_i} F(x). \end{aligned}$$

By Lemma 3.5 we have $\|L_i\| \geq P_i$. Therefore, using also (3.32) and (3.33), we have

$$(3.34) \quad \|L\| \geq P - \varepsilon.$$

Now let us assume that $|M_2| = |M_3| = |M_4| = 0$. Then $|M_1| > 0$. Let K_n^+, K_n^- be sequences of compact sets such that $K_n^+ \nearrow K^+$, $K_n^- \nearrow K^-$. Then $\ell|_{K_n^+}, \ell|_{K_n^-}$ are continuous functions. Set $K_n = K_n^+ \cup K_n^-$. Fix $n \in \mathbb{N}$. Now, Lemma 3.6 guarantees the existence of sets $N_n^+ \subset M_1$, $|M_1 \setminus N_n^+| = 0$ and $N_n^+ = \mathfrak{D}(N_n^+)$ such that for any $x \in N_n^+$ there is a set $N_{x,n}^+ \subset N_n^+$ which satisfies

$$(3.35) \quad N_{x,n}^+ = \mathfrak{D}(N_{x,n}^+),$$

$$(3.36) \quad x \in N_{x,n}^+,$$

$$(3.37) \quad |K_{n,y}^+ \div K_{n,z}^+| < \varepsilon \quad \text{for } y, z \in N_{x,n}^+.$$

Analogously, there is a set N_n^- with $N_n^- \subset M_1$, $|M_1 \setminus N_n^-| = 0$, $N_n^- = \mathfrak{D}(N_n^-)$ and for any $x \in N_n^-$ we have a set $N_{x,n}^- \subset N_n^-$ such that

$$(3.38) \quad N_{x,n}^- = \mathfrak{D}(N_{x,n}^-),$$

$$(3.39) \quad x \in N_{x,n}^-,$$

$$(3.40) \quad |K_{n,y}^- \div K_{n,z}^-| < \varepsilon \quad \text{for } y, z \in N_{x,n}^-.$$

Set $\tilde{N}_1 = \bigcap_{n=1}^{\infty} (N_n^+ \cap N_n^-)$ and $N_1 = \mathfrak{D}(\tilde{N}_1)$. Obviously, $|M_1 \setminus N_1| = 0$ and, consequently, $|N_1| > 0$. Denote for $n \in \mathbb{N}$

$$PK_n^+ = \{x \in J, |K_{n,x}^+| > 0\}, \quad PK_n^- = \{x \in J, |K_{n,x}^-| < 0\}.$$

By the Fubini theorem, $|M_1| > 0$, and $K_n \nearrow K$, we can choose an $n_0 \in \mathbb{N}$ large enough in order that

$$|PK_n^+ \cap PK_n^- \cap M_1| > 0 \quad \text{for all } n \geq n_0.$$

Set

$$A_n = \{x \in J; |K_x^+ \setminus K_{n,x}^+| < \varepsilon \text{ and } |K_x^- \setminus K_{n,x}^-| < \varepsilon\}.$$

Since for any $n \in \mathbb{N}$ the functions $x \mapsto |K_x^+ \setminus K_{n,x}^+|$ and $x \mapsto |K_x^- \setminus K_{n,x}^-|$ are measurable, A_n are measurable as well. Moreover, A_n is a non-decreasing sequence of sets.

Since $K_n^+ \nearrow K^+$ and $K_n^- \nearrow K^-$, we obtain by the Fubini theorem $K_{n,x}^+ \nearrow K_x^+$ and $K_{n,x}^- \nearrow K_x^-$ for a.e. $x \in J$. So, there is a set $J_1 \subset J$, $|J \setminus J_1| = 0$ such that

$$(3.41) \quad K_{n,x}^+ \nearrow K_x^+, \quad K_{n,x}^- \nearrow K_x^- \quad \text{for all } x \in J_1.$$

Using the Fubini theorem again, we can find an $n_1 \in \mathbb{N}$ such that $|A_n \cap N_1| > 0$ for any $n \geq n_1$. Let $n_2 = \max\{n_0, n_1\}$. Then we see that

$$(3.42) \quad |K_x^+ \setminus K_{n,x}^+| < \varepsilon \quad \text{and} \quad |K_x^- \setminus K_{n,x}^-| < \varepsilon \quad \text{for all } x \in A_n \cap N_1, \quad n \geq n_2.$$

Define $\ell_n^+ = \ell \chi_{K_n^+}$ and $\ell_n^- = \ell \chi_{K_n^-}$. Set $\ell_n = \ell_n^+ - \ell_n^-$. Let $N_2 = A_{n_1} \cap N_1 \cap J_1$. Clearly, $|N_2| > 0$.

Let $x_0 \in N_2$, be a Lebesgue density point of N_2 , i.e., $x_0 \in \mathfrak{D}(N_2)$. Since $x_0 \in N_2 \subset N_1 \subset M_1 \subset M$, we have from (3.32) and the definition of $F(x_0)$ a function $f_0(t)$ such that

$$(3.43) \quad P - 2\varepsilon \leq \left| \int_I \ell(x_0, t) f_0(t) dt \right| \leq P.$$

Clearly, (3.41) and the fact that $x_0 \in J_1$ imply $\ell_n^+(x_0, t) \nearrow \ell^+(x_0, t)$, $\ell_n^-(x_0, t) \nearrow \ell^-(x_0, t)$. Since the constant function CD can serve as an integrable majorant, we can write

$$\int_I \ell_n^+(x_0, t) f_0(t) dt \rightarrow \int_I \ell^+(x_0, t) f_0(t) dt,$$

and

$$\int_I \ell_n^-(x_0, t) f_0(t) dt \rightarrow \int_I \ell^-(x_0, t) f_0(t) dt.$$

Hence, there exists an $n_3 \geq n_2$ such that

$$\int_I \ell^+(x_0, t) f_0(t) dt - \varepsilon \leq \int_I \ell_{n_3}^+(x_0, t) f_0(t) dt \leq \int_I \ell^+(x_0, t) f_0(t) dt + \varepsilon,$$

and

$$\int_I \ell^-(x_0, t) f_0(t) dt - \varepsilon \leq \int_I \ell_{n_3}^-(x_0, t) f_0(t) dt \leq \int_I \ell^-(x_0, t) f_0(t) dt + \varepsilon.$$

Using these inequalities, $\ell_n(x_0, t) = \ell_n^+(x_0, t) - \ell_n^-(x_0, t)$, and $\ell(x_0, t) = \ell^+(x_0, t) - \ell^-(x_0, t)$, we obtain

$$\left| \int_I \ell(x_0, t) f_0(t) dt \right| - 2\varepsilon \leq \left| \int_I \ell_{n_3}(x_0, t) f_0(t) dt \right| \leq \left| \int_I \ell(x_0, t) f_0(t) dt \right| + 2\varepsilon.$$

Together with (3.43), this yields

$$(3.44) \quad P - 4\varepsilon \leq \left| \int_I \ell_{n_3}(x_0, t) f_0(t) dt \right| \leq P + 2\varepsilon.$$

Since $\ell|_{K_{n_3}}$ is continuous, there is an $\alpha_0 > 0$ such that, for $x \in (x_0 - \alpha_0, x_0 + \alpha_0)$ and $t \in K_{n_3, x} \cap K_{n_3, x_0}$,

$$(3.45) \quad |\ell_{n_3}(x, t) - \ell_{n_3}(x_0, t)| < \varepsilon$$

Set $N_3 = (x_0 - \alpha_0, x_0 + \alpha_0) \cap N_2 \cap N_{x_0, n_3}^+ \cap N_{x_0, n_3}^-$. We know that $x_0 \in \mathfrak{D}(N_2)$. By (3.35), (3.36), (3.38) and (3.39) we have $x_0 \in N_3$, $x_0 \in \mathfrak{D}(N_3)$ and, consequently, $|N_3| > 0$.

Recall that N_3 satisfies the following inclusions:

$$(3.46) \quad N_3 \subset N_2 \subset A_{n_1},$$

$$(3.47) \quad N_3 \subset (x_0 - \alpha_0, x_0 + \alpha_0),$$

$$(3.48) \quad N_3 \subset N_{x_0, n_3}^+ \cap N_{x_0, n_3}^-.$$

We shall estimate

$$G(x) = \int_I \ell(x, t) f_0(t) dt, \quad x \in N_3$$

Clearly, for a fixed $x \in N_3$, we have

$$\begin{aligned}
G(x) &= \int_I (\ell(x, t) - \ell_{n_3}(x, t)) f_0(t) dt \\
&\quad + \int_{K_{n_3, x_0} \cap K_{n_3, x}} (\ell_{n_3}(x, t) - \ell_{n_3}(x_0, t)) f_0(t) dt \\
&\quad + \int_{K_{n_3, x_0} \setminus K_{n_3, x}} \ell_{n_3}(x, t) f_0(t) dt \\
&\quad + \int_{K_{n_3, x} \setminus K_{n_3, x_0}} \ell_{n_3}(x_0, t) f_0(t) dt \\
&\quad + \int_I \ell_{n_3}(x_0, t) f_0(t) dt = I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}$$

Now, evidently,

$$\begin{aligned}
|I_1| &\leq \int_I |\ell(x, t) - \ell_{n_3}(x, t)| |f_0(t)| dt \\
&\leq \int_{K_x^+ \setminus K_{n_3, x}^+} \ell^+(x, t) |f_0(t)| dt + \int_{K_x^- \setminus K_{n_3, x}^-} |\ell^-(x, t)| |f_0(t)| dt.
\end{aligned}$$

By (3.46), $x \in N_3 \subset A_{n_1}$. Since $n_3 \geq n_2 \geq n_1$, we have by (3.42) $|I_1| \leq 2\varepsilon CD$. By (3.45) and (3.47), $|I_2| \leq \varepsilon C$. Using (3.37), (3.40) and (3.48), we get $|K_{n, x}^+ \div K_{n, x_0}^+| < \varepsilon$ and $|K_{n, x}^- \div K_{n, x_0}^-| < \varepsilon$, and therefore

$$|I_3| \leq \varepsilon CD, \quad |I_4| < \varepsilon CD.$$

Now, (3.44) and the estimates of I_1, I_2, I_3, I_4 give

$$\begin{aligned}
\|L\| &\geq \sup_{f \in \mathcal{F}} \operatorname{ess\,sup}_{x \in I} \left| \int_I \ell(x, t) f(t) dt \right| \geq \operatorname{ess\,sup}_{x \in N_3} \left| \int_I \ell(x, t) f_0(t) dt \right| \\
&= \operatorname{ess\,sup}_{x \in N_2} |G(x)| \geq |I_5| - |I_1| - |I_2| - |I_3| - |I_4| \geq P - \varepsilon(4 + C + 4CD).
\end{aligned}$$

We have proved that if $|M_1| > 0$, then $\|L\| \geq P - \varepsilon(4 + C + CD)$. Together with (3.34) this yields

$$\|L\| \geq P - \varepsilon \max\{1, 4 + C + 4CD\} = P - \varepsilon(4 + C + 4CD).$$

Letting ε tend to zero, we obtain $\|L\| \geq P$, and the proof is complete. \square

3.8 Lemma. *Let $I = [0, 1]$ and $\ell \in \mathcal{A}$. Let $D > 0$, $|\ell| \leq D$ in I^2 . Then F is measurable. Moreover, $\|L\| \geq P$.*

Proof. Let $\mathcal{F}_C = \{f, \|f\|_{X, v} \leq 1, |f| \leq C\}$ for any $C > 0$. Let $a > 0$ and $h \in \mathfrak{M}(I)$. We denote

$$h_a(t) = \begin{cases} a & \text{if } h(t) > a \\ h(t) & \text{if } |h(t)| \leq a \\ -a & \text{if } h(t) < -a. \end{cases}$$

We define $F_C(x) = \sup_{f \in \mathcal{F}_C} \left| \int_I \ell(x, t) f(t) dt \right|$.

Let $J \subset I$, $|I \setminus J| = 0$ and assume that $\int_I \ell(x, t) f(t) dt$ exists for any $x \in J$ and $f \in (X, v)$. Fix $x \in J$. Clearly, $\int_I h_C \rightarrow \int_I h$ for $C \rightarrow \infty$ if $\int_I h$ exists in the Lebesgue sense. Then

$$\left| \int_I \ell(x, t) f_C(t) dt \right| \rightarrow \left| \int_I \ell(x, t) f(t) dt \right| \quad \text{for } C \rightarrow \infty.$$

It is not difficult to prove from the above convergence that

$$F_C(x) \nearrow F(x).$$

By Lemma 3.7, the functions $F_C(x)$ are measurable and therefore $F(x)$ is measurable as a monotone pointwise limit of $F_C(x)$.

Moreover, as $F_C(x) \nearrow F(x)$ a.e. in I , we have $P_C \nearrow P$. Now, Lemma 3.7 implies $\|L\| \geq P_C$ for any $C > 0$, and thus $\|L\| \geq P$. The proof is complete. \square

3.9 Lemma. *Let $I = [0, 1]$. Let $\delta > 0$ and denote*

$$\mathcal{M} = \{A, A \in \mathfrak{M}(I), |A| < \delta\}.$$

Then there exists a countable system $\mathcal{C} = \{M_i\}_{i \in \mathbb{N}}$ of sets, $|M_i| < \delta$, such that for every $A \in \mathcal{M}$ and $\varepsilon > 0$ there is a $k \in \mathbb{N}$ such that $|A \div M_k| < \varepsilon$.

Proof. Set

$$\mathcal{C} = \left\{ M = \bigcup_{i=1}^n (a_i, b_i), n \in \mathbb{N}, a_i, b_i \text{ rational}, (a_i, b_i) \cap (a_j, b_j) = \emptyset \text{ for } i \neq j \right.$$

$$\left. \text{and } \sum_{i=1}^n (b_i - a_i) < \delta \right\}.$$

Clearly, \mathcal{C} is a countable system, i.e. $\mathcal{C} = \{M_i\}_{i \in \mathbb{N}}$.

Now, let $A \in \mathcal{M}$ and $\varepsilon > 0$. Fix a γ such that $0 < \gamma < \min\{\frac{\varepsilon}{3}, \delta - |A|\}$. By the regularity of the Lebesgue measure, there is an open set $G = \bigcup_{i=1}^{\infty} (c_i, d_i)$ such that $A \subset G$ and

$$(3.49) \quad |G \setminus A| < \gamma.$$

Since $G \subset I$, we have $\sum_{i=1}^{\infty} (c_i - d_i) \leq 1$, and there exists an $n \in \mathbb{N}$ such that the set G_n , defined by $G_n = \bigcup_{i=1}^n (c_i, d_i)$, satisfies

$$(3.50) \quad |G \setminus G_n| < \gamma.$$

Let $a_i, b_i \in \mathbb{Q}$, $i = 1, 2, \dots, n$, are such that

$$(a_i, b_i) \subset (c_i, d_i) \quad \text{and} \quad |(c_i, d_i) \setminus (a_i, b_i)| < \frac{\gamma}{n}.$$

Set $M = \bigcup_{i=1}^n (a_i, b_i)$. Clearly, $M \subset G_n$ and

$$(3.51) \quad |G_n \setminus M| \leq \sum_{i=1}^n |(c_i, d_i) \setminus (a_i, b_i)| < \gamma.$$

Evidently, we have from (3.49)

$$|M| \leq |M \setminus A| + |A| \leq |G \setminus A| + |A| \leq \gamma + |A| < \delta,$$

which implies $M \subset \mathcal{C}$. Moreover, due to (3.49), (3.50), and (3.51) we can write

$$|A \div M| = |A \setminus M| + |M \setminus A| \leq |G \setminus G_n| + |G_n \setminus M| + |G \setminus A| \leq 3\gamma < \varepsilon,$$

which completes the proof. \square

3.10 Lemma. *Let $\ell \in \mathcal{A}$. Then F is measurable and $\|L\| \geq P$.*

Proof. Since $\ell \in \mathcal{A}$, there is a set $J \subset I$, $|I \setminus J| = 0$, and such that $\int_I \ell(x, t)f(t)dt$ is well-defined for any $x \in J$ and $f \in (X, v)$. Set $B_n = \{(x, y) \in I^2; |\ell(x, t)| \leq n\}$. Define

$$\ell_n(x, t) = \ell(x, t)\chi_{B_n}(x, t), \quad (L_n f)(x) = \int_I \ell_n(x, t)f(t)dt.$$

Step 1. We claim that $\ell_n \in \mathcal{A}$. Fix $x \in J$ and $f \in (X, v)$. If for every $t \in I$ $\ell(x, t)f(t) > 0$, then $\ell_n(x, t)f(t) \geq 0$ for every $t \in I$. Analogously, if for every $t \in I$ $\ell(x, t)f(t) < 0$, then $\ell_n(x, t)f(t) \leq 0$ for every $t \in I$. Together with the definition of ℓ_n this yields

$$(3.52) \quad \begin{cases} 0 \leq (\ell_n(x, t)f(t))^+ \leq (\ell(x, t)f(t))^+ \\ 0 \leq (\ell_n(x, t)f(t))^- \leq (\ell(x, t)f(t))^- \end{cases}$$

Since $\int_I \ell(x, t)f(t)dt = \int_I (\ell(x, t)f(t))^+ dt - \int_I (\ell(x, t)f(t))^- dt$ exists, either $\int_I (\ell(x, t)f(t))^+ dt < \infty$ or $\int_I (\ell(x, t)f(t))^- dt < \infty$. Therefore, by (3.52), either $\int_I (\ell_n(x, t)f(t))^+ dt < \infty$ or $\int_I (\ell_n(x, t)f(t))^- dt < \infty$, which yields the existence of $\int_I \ell(x, t)f(t)dt$. Hence $\ell_n \in \mathcal{A}$.

Step 2. Now we claim that F is measurable. Note that $F(x)$ is defined for every $x \in J$. Fix $x \in J$. Clearly, $|\ell_n(x, t)| \nearrow |\ell(x, t)|$ in I^2 , and, consequently,

$$(3.53) \quad |\ell_n(x, t)|f(t) \nearrow |\ell(x, t)|f(t) \quad \text{for any } f \geq 0.$$

It is easy to see that

$$F(x) = \sup_{\|f\|_{X, v} \leq 1, f \geq 0} \int_I |\ell(x, t)|f(t)dt.$$

According to (3.53), we have for $f \geq 0$

$$\int_I |\ell_n(x, t)|f(t)dt \nearrow \int_I |\ell(x, t)|f(t)dt,$$

and, consequently,

$$F_n(x) := \sup_{\|f\|_{x,v} \leq 1, f \geq 0} \int_I |\ell_n(x,t)| f(t) dt \nearrow F(x).$$

Similarly as in (3.54), the function F_n can be expressed by

$$(3.55) \quad F_n(x) = \sup_{\|f\|_{x,v} \leq 1} \left| \int_I \ell_n(x,t) f(t) dt \right|.$$

Since $\ell_n \in \mathcal{A}$, we have from Lemma 3.8 that F_n are measurable. Then the fact that $F_n(x) \nearrow F(x)$ for any $x \in J$ shows that F is measurable.

Step 3. We will prove the inequality

$$(3.56) \quad \|L\| \geq \|L_n\| \text{ for any } n \in \mathbb{N}.$$

Fix $n \in \mathbb{N}$. The norms $\|L\|$ and $\|L_n\|$ are well defined because $\ell_n, \ell \in \mathcal{A}$.

If $\|L\| = \infty$, then (3.56) is trivial. Assume that $\|L\| < \infty$.

For small $\varepsilon > 0$ we define

$$(3.57) \quad D_\varepsilon = \begin{cases} \|L_n\| - \varepsilon & \text{if } \|L_n\| < \infty \text{ and } \varepsilon < \|L_n\|; \\ \frac{1}{\varepsilon} & \text{if } \|L_n\| = \infty \text{ and } \|L\| < \frac{1}{\varepsilon} - 2\varepsilon. \end{cases}$$

By the definition of $\|L_n\|$, there exists a function f_0 , $\|f_0\|_{x,v} \leq 1$, and a set $M \subset J$, $|M| > 0$, such that

$$(3.58) \quad D_\varepsilon \leq \left| \int_I \ell_n(x,t) f_0(t) dt \right| \quad \text{for } x \in M.$$

Let $J_1 \subset J$, $J_2 \subset J$ be measurable sets such that

$$(3.59) \quad \begin{cases} \left| \int_I \ell(x,t) f_0(t) dt \right| < \infty & \text{for } x \in J_1, \\ \left| \int_I \ell(x,t) f_0(t) dt \right| = \infty & \text{for } x \in J_2. \end{cases}$$

If $|J_2| > 0$, then

$$\|L\| \geq \operatorname{ess\,sup}_{x \in J_2} \left| \int_I \ell(x,t) f_0(t) dt \right| = \infty,$$

which is a contradiction. Thus, $|I \setminus J_1| = |J \setminus J_1| = 0$.

Now, (3.59) implies

$$(3.60) \quad \int_I |\ell(x,t)| |f_0(t)| dt < \infty \quad \text{for } x \in J_1.$$

Let $\delta > 0$. Set $A_\delta = \{x \in J_1, \sup_{|A| < \delta} \int_A |\ell(x,t)| |f_0(t)| dt < \varepsilon\}$. Observe that $A_{\delta_2} \subset A_{\delta_1}$ for $0 < \delta_1 < \delta_2$.

We will show now that A_δ is a measurable set. For $x \in J_1$ we define the function

$$H_\delta(x) = \sup_{|A| < \delta} \int_A |\ell(x, t)| |f_0(t)| dt.$$

By (3.60), $H_\delta(x) < \infty$ for $x \in J_1$. Let \mathcal{C} be a countable system of sets from Lemma 3.9. Let $\lambda > 0$ and fix $x \in J_1$. By the definition of $H_\delta(x)$, there is a set A_0 , such that $|A_0| < \delta$ and

$$(3.61) \quad H_\delta(x) - \lambda \leq \int_{A_0} |\ell(x, t)| |f_0(t)| dt \leq H_\delta(x).$$

The absolute continuity of the Lebesgue integral and (3.60) now give the existence of $\eta > 0$ such that

$$(3.62) \quad \int_B |\ell(x, t)| |f_0(t)| dt < \lambda \quad \text{for } |B| < \eta.$$

Obviously, we can choose $\eta < \delta$. Let $N_0 \in \mathcal{C}$ such that $|A_0 \div N_0| < \eta$. Then (3.61) and (3.62) yield

$$\begin{aligned} H_\delta(x) &\geq \sup_{N \in \mathcal{C}} \int_N |\ell(x, t)| |f_0(t)| dt \geq \int_{N_0} |\ell(x, t)| |f_0(t)| dt \\ &= \int_{A_0} |\ell(x, t)| |f_0(t)| dt - \int_{A_0 \setminus N_0} |\ell(x, t)| |f_0(t)| dt \\ &\quad + \int_{N_0 \setminus A_0} |\ell(x, t)| |f_0(t)| dt \geq H_\delta(x) - 2\lambda. \end{aligned}$$

On letting $\lambda \rightarrow 0_+$ we have $H_\delta(x) = \sup_{N \in \mathcal{C}} \int_N |\ell(x, t)| |f_0(t)| dt$. Since the function $x \mapsto \int_N |\ell(x, t)| |f_0(t)| dt$ is measurable for any fixed $N \in \mathcal{C}$, the function H_δ is measurable as a supremum of countably many measurable functions. Moreover, as $A_\delta = H_\delta^{-1}((0, \varepsilon))$, the set A_δ is measurable for any $\delta > 0$.

Now, the absolute continuity of the Lebesgue integral and (3.60) give $\bigcup_{\delta > 0} A_\delta = J_1$. Hence there is a δ_0 such that $|A_{\delta_0} \cap M| > 0$. By Lemma 3.6 there is a set $M_1 \subset A_{\delta_0} \cap M$, $|(A_{\delta_0} \cap M) \setminus M_1| = 0$ and $M_1 = \mathfrak{D}(M_1)$, such that for any $x \in M_1$ there is a set $N_{\delta_0, x} \subset M_1$ such that $N_{\delta_0, x} = \mathfrak{D}(N_{\delta_0, x})$, $x \in N_{\delta_0, x}$ and $|B_{n, y} \div B_{n, z}| < \delta_0$ for $y, z \in N_{\delta_0, x}$. Let $x_0 \in M_1$ be fixed. Then N_{δ_0, x_0} satisfies

$$(3.63) \quad N_{\delta_0, x_0} = \mathfrak{D}(N_{\delta_0, x_0}),$$

$$(3.64) \quad x_0 \in N_{\delta_0, x_0},$$

$$(3.65) \quad |B_{n, y} \div B_{n, z}| < \delta_0 \quad \text{for } y, z \in N_{\delta_0, x_0}.$$

The properties (3.63) and (3.64) guarantee $|N_{\delta_0, x_0}| > 0$. Set $f_1(t) = f_0(t) \chi_{B_{n, x_0}}(t)$. Clearly, $\|f_1\|_{X, v} \leq 1$. Fix $x \in N_{\delta_0, x_0}$. Note that (3.64) and (3.65) give

$$(3.66) \quad |B_{n, x_0} \div B_{n, x}| < \delta_0.$$

Obviously,

$$\begin{aligned}
\int_I \ell(x, t) f_1(t) dt &= \int_{B_{n, x_0}} \ell(x, t) f_0(t) dt \\
&= \int_{B_{n, x_0} \cap B_{n, x}} \ell(x, t) f_0(t) dt + \int_{B_{n, x_0} \setminus B_{n, x}} \ell(x, t) f_0(t) dt \\
&= \int_I \ell_n(x, t) f_0(t) dt - \int_{B_{n, x} \setminus B_{n, x_0}} \ell(x, t) f_0(t) dt + \int_{B_{n, x_0} \setminus B_{n, x}} \ell(x, t) f_0(t) dt.
\end{aligned}$$

By (3.58), (3.66) and $N_{\delta_0, x_0} \subset A_{\delta_0}$, we have

$$D_\varepsilon - 2\varepsilon \leq \left| \int_I \ell(x, t) f_1(t) dt \right| \quad \text{for any } x \in N_{\delta_0, x_0}.$$

Since $|N_{\delta_0, x_0}| > 0$, we have

$$\|L\| \geq D_\varepsilon - 2\varepsilon.$$

If $\|L_n\| = \infty$, then $\|L\| \geq \frac{1}{\varepsilon} - 2\varepsilon$, which is a contradiction with (3.57). Thus, $\|L_n\| < \infty$ and $\|L\| \geq \|L_n\| - 3\varepsilon$. On letting $\varepsilon \rightarrow 0_+$ we obtain (3.56).

Step 4. Denote $P_n = \text{ess sup}_{x \in I} F_n(x)$ for $n \in \mathbb{N}$, where $F_n(x)$ is defined by (3.55). Recall that $P = \text{ess sup}_{x \in \mathbb{N}} F(x)$. By Step 2, F_n and F are measurable, and, consequently, P and P_n are well defined. We shall prove

$$(3.67) \quad \liminf_{n \rightarrow \infty} P_n \geq P.$$

Denote for $x \in J$ and $f \in (X, v)$

$$F_n(x, f) = \left| \int_I \ell_n(x, t) f(t) dt \right| \quad \text{and} \quad F(x, f) = \left| \int_I \ell(x, t) f(t) dt \right|.$$

Fix $x \in J$ and $f \in (X, v)$. Let

$$I^+ = \{t; \ell(x, t) f(t) > 0\}, \quad I^- = \{t; \ell(x, t) f(t) < 0\}.$$

Obviously, the definition of ℓ_n gives

$$\ell_n(x, t) f(t) \geq 0 \quad \text{on } I^+, \quad \ell_n(x, t) f(t) \leq 0 \quad \text{on } I^-,$$

and

$$\begin{aligned}
\ell_n(x, t) f(t) &\nearrow \ell(x, t) f(t) \quad \text{a.e. in } I^+ \quad \text{for } n \rightarrow \infty, \\
\ell_n(x, t) f(t) &\searrow \ell(x, t) f(t) \quad \text{a.e. in } I^- \quad \text{for } n \rightarrow \infty.
\end{aligned}$$

Of course, $\ell_n(x, t) = \ell(x, t) = 0$ in $I \setminus (I^+ \cup I^-)$. Then

$$\begin{aligned}
(\ell_n(x, t) f(t))^+ &\nearrow (\ell(x, t) f(t))^+ \quad \text{for } n \rightarrow \infty, \\
(\ell_n(x, t) f(t))^- &\nearrow (\ell(x, t) f(t))^- \quad \text{for } n \rightarrow \infty,
\end{aligned}$$

and, consequently,

$$\int_I \ell_n(x, t) f(t) dt \rightarrow \int_I \ell(x, t) f(t) dt \quad \text{for } n \rightarrow \infty.$$

The last relation implies

$$(3.68) \quad F_n(x, f) \rightarrow F(x, f) \quad \text{for any } x \in J \text{ and } f \in (X, v).$$

Let $\varepsilon > 0$. Fix $x \in J$. By the definition of $F(x)$, there exists a function f_0 , $\|f_0\|_{X, v} \leq 1$, such that

$$(3.69) \quad F(x) - \varepsilon \leq F(x, f_0) \leq F(x).$$

By (3.68), it is possible to choose n_0 such that for any $n \geq n_0$

$$F(x, f_0) - \varepsilon \leq F_n(x, f_0) \leq F(x, f_0) + \varepsilon,$$

which together with (3.69) gives

$$F(x) - 2\varepsilon \leq F_n(x, f_0) \leq F(x) + \varepsilon,$$

and, consequently,

$$F(x) - 2\varepsilon \leq F_n(x, f_0) \leq \sup_{\|f\|_{X, v} \leq 1} F_n(x, f) = F_n(x).$$

Thus, for any $\varepsilon > 0$ there is an n_0 such that

$$F(x) - 2\varepsilon \leq F_n(x) \quad \text{for } n \geq n_0,$$

which in turn yields

$$(3.70) \quad F(x) \leq \liminf_{n \rightarrow \infty} F_n(x) \quad \text{for any } x \in J.$$

Let $I_n \subset J$, $|J \setminus I_n| = 0$ such that

$$(3.71) \quad P_n = \operatorname{ess\,sup}_{x \in I} F_n(x) = \sup_{x \in I_n} F_n(x).$$

Let $J_1 = \bigcap_{n=1}^{\infty} I_n$. Clearly, $|J \setminus J_1| = 0$. By (3.70) and (3.71),

$$\begin{aligned} P &\leq \operatorname{ess\,sup}_{x \in I} \liminf_{n \rightarrow \infty} F_n(x) \sup_{x \in J_1} \liminf_{n \rightarrow \infty} F_n(x) \\ &\leq \liminf_{n \rightarrow \infty} \sup_{x \in J_1} F_n(x) \leq \liminf_{n \rightarrow \infty} \sup_{x \in I_n} F_n(x) \\ &= \liminf_{n \rightarrow \infty} P_n, \end{aligned}$$

which proves (3.67).

Step 5. From Step 1 we know that $\ell_n \in \mathcal{A}$. Moreover, $|\ell_n(x, t)| \leq n$ in I^2 . Hence, ℓ_n satisfies the assumptions of Lemma 3.8, and thus $\|L_n\| \geq P_n$ for any $n \in \mathbb{N}$. Using (3.56) and (3.67), we get

$$\|L\| \geq \limsup_{n \rightarrow \infty} \|L_n\| \geq \limsup_{n \rightarrow \infty} P_n \geq \liminf_{n \rightarrow \infty} P_n \geq P$$

which completes the proof. \square

3.11 Lemma. *Let I be an arbitrary interval, $\ell \in \mathcal{A}$, and let (X, v) be a Banach function space on I . Then F is measurable and then $\|L\| \geq P$.*

Proof. Set $I_1 = (0, 1)$. Let $\varphi : I_1 \rightarrow I$ be a one-to-one increasing mapping, $\varphi \in C^1$. Denote by ψ the inverse mapping of φ . Note that the assumption $\varphi \in C^1$ guarantees that the sets $\varphi(M), \psi(N)$ are measurable for M, N being measurable and $|\varphi(M)| = 0, |\psi(N)| = 0$ if and only if $|M| = 0, |N| = 0$, respectively.

Set

$$(3.72) \quad v_1(y) = v(\varphi(y))\varphi'(y) \quad \text{for } y \in I_1.$$

Since $\varphi' > 0$, v_1 is a weight. Let us define for $g \in \mathfrak{M}(I_1)$ the norm $\|g\|_{X_1, v_1}$ by

$$(3.73) \quad \|g\|_{X_1, v_1} = \|g \circ \psi\|_{X, v},$$

where $(g \circ \psi)(x) = g(\psi(x))$.

Further,

$$(X_1, v_1) = \{g \in \mathfrak{M}(I_1), \|g\|_{X_1, v_1} < \infty\}.$$

We shall prove that (X_1, v_1) is a Banach function space. Clearly, the assumption $\varphi \in C^1$ gives $\|g\|_{X_1, v_1} = \|h\|_{X_1, v_1}$ for $g_2 = h$ a.e. in I_1 . The easy facts $(g+h) \circ \psi = g \circ \psi + h \circ \psi$ and $(cg) \circ \psi = c(g \circ \psi)$ with (3.73) give that (X_1, v_1) is a normed linear subspace of $\mathfrak{M}(I_1)$ and that the property (2.1) from Definition 2.1 is satisfied. The relation $|f| \circ \psi = |f \circ \psi|$ yields (2.2). Let $f_n, f \in \mathfrak{M}(I_1)$, $0 \leq f_n \nearrow f$. Then $\|f_n\|_{X_1, v_1} = \|f_n \circ \psi\|_{X, v} \nearrow \|f \circ \psi\|_{X, v} = \|f\|_{X_1, v_1}$ which gives (2.3). Now, let $E_1 \subset I_1$, $v_1(E_1) = \int_{E_1} v_1(y)dy < \infty$. Set $E = \varphi(E_1)$. Clearly, by (3.72) and the substitution $y = \varphi(x)$ we obtain

$$v(E) = \int_I v(x)dx = \int_{I_1} v_1(y)dy = v_1(E_1) < \infty.$$

Since $\chi_{E_1} = \chi_{\varphi(E_1)} \circ \varphi$, we have, according to (2.4) and (3.73),

$$\|\chi_{E_1}\|_{X_1, v_1} = \|\chi_E \circ \varphi\|_{X_1, v_1} = \|\chi_E\|_{X, v} < \infty.$$

Thus, we have proved the property (2.4) for (X_1, v_1) . Set $g \in (X_1, v_1)$ and $f = g \circ \psi$. Using (2.5), (3.72), and an appropriate substitution, we obtain

$$\begin{aligned} \int_{E_1} g(y)v_1(y)dy &= \int_E g(\psi(x))v_1(\psi(x))\psi'(x)dx = \int_E f(x)v(x)dx \\ &\leq C_E \|f\|_{X, v} = C_E \|g \circ \psi\|_{X, v} = C \|f\|_{X_1, v_1} \end{aligned}$$

which guarantees (2.5) for (X_1, v_1) . Consequently, (X_1, v_1) is a Banach function space.

Define for

$$(3.74) \quad \ell_1(y, s) = \ell(\varphi(y), \varphi(s))\varphi'(s), \quad s, y \in I_1,$$

and

$$(L_1g)(y) = \int_I \ell_1(y, s)g(s)ds \quad \text{for } g \in (X_1, v_1).$$

We claim that $\ell_1 \in \mathcal{A}$. Let $J \subset I$ be a set from the property $\ell \in \mathcal{A}$. Let $\varphi(J_1) = J$. Then, $|I_1 \setminus J_1| = 0$ due to $\varphi \in C^1$. Fix $y \in J_1$ and $g \in (X_1, v_1)$. By from (3.73) we have $f = g \circ \psi \in (X, v)$ and $\varphi(y) = x \in J$. Using the change of variables $t = \varphi(s)$ and (3.74) we obtain

$$(3.75) \quad \begin{aligned} A_1(y, g) &= \int_{I_1} \ell_1(y, s)g(s)ds = \int_{I_1} \ell(\varphi(y), \varphi(s))\varphi'(s)f(\varphi(s))ds \\ &= \int_I \ell(x, t)f(t)dt = A(x, f). \end{aligned}$$

Since, by $\ell \in \mathcal{A}$, the integral $A(x, f)$ exists, $A_1(y, g)$ exists as well, and therefore $\ell_1 \in \mathcal{A}$. For $y \in I_1$, let $F_1(y) = \sup_{\|g\|_{X_1, v_1} \leq 1} |\int_{I_1} \ell_1(y, s)g(s)ds|$. By (3.73), $\|g\|_{X_1, v_1} \leq 1$ if and only if $\|f\|_{X, v} \leq 1$. By (3.74) and the change of variables $t = \varphi(s)$,

$$\begin{aligned} F(x) &= \sup_{\|g\|_{X_1, v_1} \leq 1} \left| \int_I \ell(\varphi(y), \varphi(s))\varphi'(s)f(\varphi(s)) ds \right| \\ &= \sup_{\|g\|_{X_1, v_1} \leq 1} \left| \int_{I_1} \ell_1(y, s)g(s)ds \right| = F_1(y). \end{aligned}$$

So, $F(\varphi(y)) = F_1(y)$, b for any $y \in I_1$, and, consequently,

$$(3.76) \quad P = \operatorname{ess\,sup}_{x \in I} F(x) = \operatorname{ess\,sup}_{y \in I_1} F(\varphi(y)) = \operatorname{ess\,sup}_{y \in I_1} F_1(y) = P_1.$$

By (3.75),

$$\|L\| = \sup_{\|f\|_{X, v} \leq 1} \operatorname{ess\,sup}_{x \in I} |A(x, f)| = \sup_{\|g\|_{X_1, v_1} \leq 1} \operatorname{ess\,sup}_{y \in I_1} |A_1(y, g)| = \|L_1\|.$$

Now, Lemma 3.10 and (3.76) give $\|L\| = \|L_1\| \geq P_1 = P$, which completes the proof. \square

3.12 Lemma. *Let I be an arbitrary interval, let (X, v) be a Banach function space and $\ell \in \mathcal{A}$. Then the corresponding operator L satisfies*

$$\|L\| \geq \|\ell\|_{L_\infty(X', v)}.$$

Proof. Let $\varepsilon > 0$. Fix $x \in I$. Then, by Lemma 2.4, there is a function f , $\|f\|_{X, v} = 1$, such that

$$(3.77) \quad \left| \int_I \ell(x, t)f(t)dt \right| \geq (1 - \varepsilon) \left\| \frac{\ell(x, \cdot)}{v(\cdot)} \right\|_{X', v}$$

Let $J \subset I$ be the set from the definition of $\ell \in \mathcal{A}$. Assume that $x \in J$. Then (3.77) gives

$$F(x) \geq (1 - \varepsilon) \left\| \frac{\ell(x, \cdot)}{v(\cdot)} \right\|_{X', v}.$$

This inequality gives together with Lemma 3.11

$$\|L\| \geq P = \operatorname{ess\,sup}_{x \in I} F(x) \geq (1 - \varepsilon) \operatorname{ess\,sup}_{x \in I} \left\| \frac{\ell(x, \cdot)}{v(\cdot)} \right\|_{X', v} = (1 - \varepsilon) \cdot \|\ell\|_{L_\infty(X', v)}$$

On letting $\varepsilon \rightarrow 0_+$ we get $\|L\| \geq \|\ell\|_{L_\infty(X', v)}$, which proves the lemma. \square

3.13 Remark. Let I be an arbitrary interval, (X, v) a Banach function space and $\ell \notin \mathcal{A}$. Recall that $\ell \in \mathfrak{M}(I^2)$. Then

$$\|\ell\|_{L_\infty(X', v)} = \infty.$$

Proof. Since $\ell \in \mathfrak{M}(I^2)$ the norm $\|\ell\|_{L_\infty(X', v)}$ exists. By Lemma 3.1, $\|\ell\|_{L_\infty(X', v)} < \infty$ implies $\ell \in \mathcal{A}$, a contradiction. \square

From Lemma 3.1, 3.12 and Remark 3.13 follows the following theorem.

3.14 Theorem. Let I be an arbitrary interval, (X, v) a Banach function space, and $\ell \in \mathfrak{M}(I^2)$. Then either $\|L\| = \|\ell\|_{L_\infty(X', v)}$ (when $\ell \in \mathcal{A}$), or $\|L\|_{L_\infty(X', v)} = \infty$ (when $\ell \notin \mathcal{A}$).

3.15 Remark Theorem 3.14 shows that L is bounded if and only if $\|\ell\|_{L_\infty(X', v)} < \infty$.

4. Compactness of a general kernel operator

In this section we investigate the distance of the operator L from the set of all compact operators $K : (X, v) \rightarrow L_\infty$. Define

$$D = \inf\{\|L - K\|, K \in \mathcal{K}\},$$

where \mathcal{K} is the set of all compact operators. Denote by \mathfrak{K} the set of all kernels $k \in \mathfrak{M}(I^2)$ that can be written as

$$k(x, t) = \sum_{i=1}^n \chi_{M_i}(x) \psi_i(t)$$

for some $n \in \mathbb{N}$, $\chi_{M_i} \in \mathfrak{M}(I)$, and $\frac{\psi_i}{v} \in (X', v)$.

Clearly, $k \in \mathfrak{K}$ implies $k \in L_\infty(X', v)$. Let \mathfrak{C} be the closure of \mathfrak{K} in $L_\infty(X', v)$. Define further

$$d := \inf\{\|\ell - k\|_{L_\infty(X', v)}, k \in \mathfrak{C}\} = \inf\{\|\ell - k\|_{L_\infty(X', v)}, k \in \mathfrak{K}\},$$

4.1 Lemma. *Let $k \in \mathfrak{K}$. Then the operator $(Kf)(x) = \int_I k(x, t)f(t)dt$ is a finite dimensional bounded operator. Consequently, K is compact.*

Proof. Let $k(x, t) = \sum_{i=1}^n \chi_{M_i}(x)\psi_i(t)$, $\chi_{M_i} \in \mathfrak{M}(I)$ and $\frac{\psi_i}{v} \in (X', v)$. Since $k \in L_\infty(X', v)$, the operator K is bounded by Remark 3.15. Moreover,

$$Kf(x) = \sum_{i=1}^n \int_I \psi_i(t)f(t)dt \chi_{M_i}(x) = \sum_{i=1}^n A_i \chi_{M_i}(x).$$

Now, (2.6) gives $|A_i(f)| \leq \|\frac{\psi_i}{v}\|_{X', v} \|f\|_{X, v}$ which implies $A_i \in (X, v)^*$ (the dual space) and, consequently, K is a bounded operator. \square

4.2 Theorem. *The inequality $D \leq d$ holds.*

Proof. By Lemmas 2.6 and 4.1, we can write

$$\begin{aligned} D &\leq \inf_{k \in \mathfrak{K}} \sup_{\|f\|_{X, v} \leq 1} \operatorname{ess\,sup}_{x \in I} \int_I |\ell(x, t) - k(x, t)| |f(t)| dt \\ &\leq \inf_{k \in \mathfrak{K}} \sup_{\|f\|_{X, v} < 1} \operatorname{ess\,sup}_{x \in I} \left\| \frac{\ell(x, \cdot) - k(x, \cdot)}{v(\cdot)} \right\|_{X', v} \|f\|_{X, v} \\ &= \inf_{k \in \mathfrak{K}} \|\ell - k\|_{L_\infty(X', v)} = d \end{aligned}$$

which proves the assertion. \square

In the rest of this section we shall prove that $\frac{d}{2} \leq D$.

4.3 Definition. We say that a finite system of sets $\mathcal{A} = \{\Omega_j, j = 1, 2, \dots, n\}$ is a *partition* of I if $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$, and $\bigcup_{j=1}^n \Omega_j = I$.

4.4 Lemma. *Let n, N be positive integers. Let $\mathcal{A}_i = \{\Omega_j^i, j = 1, 2, \dots, N\}$, $i = 1, 2, \dots, n$ be partitions of I . Then there is a positive integer m and a partition $\mathcal{A} = \{E_k, k = 1, 2, \dots, m\}$ of I such that*

$$(4.1) \quad \text{for any } i \in \{1, 2, \dots, n\} \text{ and } k \in \{1, 2, \dots, m\} \text{ there exists} \\ \text{a } j \in \{1, 2, \dots, N\} \text{ such that } E_k \subset \Omega_j^i.$$

Proof. We use the induction on n . Let $n = 1$. Then the assertion is obvious.

Assume that $\mathcal{A}_i = \{\Omega_j^i, j = 1, 2, \dots, N\}$, $i = 1, 2, \dots, n+1$, are partitions of I . By the induction assumption, there is a partition of I , $\tilde{\mathcal{A}} = \{\tilde{E}_k, k = 1, 2, \dots, \tilde{m}\}$ such that

$$(4.2) \quad \text{for any } i \in \{1, 2, \dots, n\} \text{ and } k \in \{1, 2, \dots, \tilde{m}\} \text{ there exists} \\ \text{a } j \in \{1, 2, \dots, N\} \text{ such that } \tilde{E}_k \subset \Omega_j^i.$$

Set $F_{kj} = \tilde{E}_k \cap \Omega_j^{n+1}$, $k \in \{1, 2, \dots, \tilde{m}\}$, and $j \in \{1, 2, \dots, N\}$. Define a system of sets \mathcal{A} by

$$\mathcal{A} = \{F_{kj}, k \in \{1, 2, \dots, \tilde{m}\} \text{ and } j \in \{1, 2, \dots, N\}\}.$$

Let $k_1 \in \{1, 2, \dots, \tilde{m}\}$ and $j_1, j_2 \in \{1, 2, \dots, N\}$. If $k_1 \neq k_2$, then $F_{k_1 j_1} \subset \tilde{E}_{k_1}$, $F_{k_2 j_2} \subset \tilde{E}_{k_2}$ and, consequently, $F_{k_1 j_1} \cap F_{k_2 j_2} = \emptyset$. Let $j_1 \neq j_2$. Then, using $F_{k_1 j_1} \subset \Omega_{j_1}^{n+1}$ and $F_{k_2 j_2} \subset \Omega_{j_2}^{n+1}$, we get $F_{k_1 j_1} \cap F_{k_2 j_2} = \emptyset$. Hence, for $(k_1, j_1) \neq (k_2, j_2)$ we have $F_{k_1 j_1} \cap F_{k_2 j_2} = \emptyset$. Further, let $x \in I$. Since $\tilde{\mathcal{A}}$ and $\{\Omega_j^{n+1}, j = 1, 2, \dots, N\}$ are partitions of I , there are $k \in \{1, 2, \dots, \tilde{m}\}$, $j \in \{1, 2, \dots, N\}$ such that $x \in \tilde{E}_k \cap \Omega_j^{n+1} = F_{kj}$, and, consequently, $\bigcup_{k=1}^{\tilde{m}} \bigcup_{j=1}^N = I$. Thus $\{F_{jk}, k \in \{1, 2, \dots, \tilde{m}\}, j \in \{1, 2, \dots, N\}\}$, is a partition of I .

Now, choose $i \in \{1, 2, \dots, n+1\}$, $k_1 \in \{1, 2, \dots, \tilde{m}\}$, and $j_1 \in \{1, \dots, N\}$. If $1 \leq i \leq n$, then, by (4.2), there is a $j \in \{1, 2, \dots, N\}$ such that $\tilde{E}_{k_1} \subset \Omega_j^i$. Therefore, $F_{k_1 j_1} \subset \Omega_j^i$. Now, $F_{k_1 j_1} \subset \Omega_{j_1}^{n+1}$, provided that $i = n+1$. We have thus proved that for every fixed i, k_1, j_1 there exists a $j \in \{1, 2, \dots, N\}$ such that $F_{k_1 j_1} \subset \Omega_j^i$.

Let $m := \tilde{m}N$ and let $b : \{1, \dots, m\} \rightarrow \{(k, j); j = 1, 2, \dots, N, k = 1, 2, \dots, \tilde{m}\}$ be a bijection. Set $E_k = F_{b(k)}$ and $\mathcal{A} = \{E_k, k = 1, 2, \dots, m\}$. Then \mathcal{A} is a partition of I satisfying (4.1) for $i = 1, 2, \dots, n+1$, which completes the proof. \square

Let B be the unit ball in (X, v) . Let $M \subset L_\infty$ and $\eta > 0$. We say that $N \subset L_\infty$ is a η -net in M if for every $f \in M$ there is a $g \in N$ with $\|f - g\|_{L_\infty} \leq \eta$.

4.5 Lemma. *Let*

$$\sigma = \inf\{\eta; \text{there exists a finite } \eta\text{-net of } L(B)\}.$$

Then $\sigma \leq D$.

Proof. Let $\varepsilon > 0$. Take $K \in \mathcal{K}$ such that

$$\|L - K\| \leq D + \varepsilon.$$

Since $K \in \mathcal{K}$, there exists a finite ε -net $\{g_1, g_2, \dots, g_n\}$ of $K(B)$. Let $g \in L(B)$. Then there is a function $f \in B$ such that $Lf = g$. Choose g_i with $\|Kf - g_i\|_{L_\infty} \leq \varepsilon$. Then

$$\|g - g_i\|_{L_\infty} = \|Lf - g_i\|_{L_\infty} \leq \|Lf - Kf\|_{L_\infty} + \|Kf - g_i\|_{L_\infty} \leq D + 2\varepsilon.$$

Thus, $\{g_1, \dots, g_n\}$ is a finite $(D + 2\varepsilon)$ -net of $L(B)$ and, consequently, $\sigma \leq D + 2\varepsilon$. On letting $\varepsilon \rightarrow 0_+$ we obtain the assertion. \square

It is worth noting that Lemma 4.5 remains true under more general assumptions, namely, for Banach spaces X, Y , a bounded linear operator $T : X \rightarrow Y$, and σ, D defined in an analogous way.

4.6 Lemma. *Let λ be a measure on I such that λ -measurable sets coincide with the Lebesgue measurable sets, and $\lambda(E) = 0$ if and only if $|E| = 0$. Let $h(x, t) \in \mathfrak{M}(I^2)$, such that $h(x, t)v(t) \in \mathcal{A}$. Then the function $x \mapsto \|h(x, \cdot)\|_{X', v}$ is λ -measurable. Moreover, for $E \subset I$ measurable, $0 < \lambda(E) < \infty$, we have*

$$\left\| \frac{1}{\lambda(E)} \int_I h(x, \cdot) d\lambda(x) \right\|_{X', v} \leq \frac{1}{\lambda(E)} \int_I \|h(x, \cdot)\|_{X', v} d\lambda(x).$$

Proof. Define $\Phi(x) = \|h(x, \cdot)\|_{X',v}$. Clearly,

$$F(x) = \sup_{\|f\|_{X,v} \leq 1} \int_I |h(x,t)f(t)v(t)dt| = \sup_{\|f\|_{X,v} \leq 1} \left| \int_I h(x,t)f(t)v(t)dt \right|.$$

By 3.11, the last expression is a Lebesgue measurable function, whence F is Lebesgue measurable. Due to the assumptions on λ , F is λ -measurable, which proves the first part of the lemma.

Now, using the Fubini theorem, we have

$$\left\| \frac{1}{\lambda(E)} \int_I h(x, \cdot) d\lambda(x) \right\|_{X',v} = \frac{1}{\lambda(E)} \sup_{\|f\|_{X,v} \leq 1} \left| \int_I \int_I h(x,t)f(t)v(t) dt d\lambda(x) \right| = A,$$

say. Let $\varepsilon > 0$. Then there is an $f_0 \in B$ such that

$$\begin{aligned} A - \varepsilon &\leq \frac{1}{\lambda(E)} \left| \int_I \int_I h(x,t)f_0(t)v(t) dt d\lambda(x) \right| \\ &\leq \frac{1}{\lambda(E)} \int_I F(x) d\lambda(x) = \frac{1}{\lambda(E)} \int_I \|h(x, \cdot)\|_{X',v} d\lambda(x). \end{aligned}$$

On letting $\varepsilon \rightarrow 0_+$ we obtain the assertion. \square

The main idea of the proof of the following lemma is taken from [W].

4.7 Lemma. *The inequality $\frac{d}{2} \leq \sigma$ holds.*

Proof. Let $\varepsilon > 0$. Let $\{g_1, g_2, \dots, g_n\}$ be a finite $(\sigma + \varepsilon)$ -net of $L(B)$. Since $L(B)$ is bounded in L_∞ , the set $\{g_1, g_2, \dots, g_n\}$ is bounded in L_∞ , too. Hence there exists an $A > 0$ such that $\text{ess sup}_{x \in I} |g_i(x)| \leq A$, $i = 1, 2, \dots, n$. We can even assume that $\sup_{x \in I} |g_i(x)| \leq A$ because in the opposite case we simply change every function g_i on a set of measure zero.

Let $\{I_j, j = 1, 2, \dots, N\}$ be a partition of $[-A, A]$ such that I_j are intervals and $|I_j| \leq \varepsilon$. Let $\Omega_j^i = g_i^{-1}(I_j)$, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, N$. Then the systems $\mathcal{A}_i = \{\Omega_j^i, j = 1, 2, \dots, N\}$ are partitions of I . By Lemma 3.4, there is a partition of I , say, $\mathcal{A} = \{E_k, k = 1, 2, \dots, m\}$, such that (4.1) holds.

Let $B = \{E_k \in \mathcal{A}, |E_k| > 0\}$. Then we can write $B = \{E_k, k = 1, 2, \dots, m_1\}$ where $m_1 \leq m$. Clearly,

$$(4.3) \quad E_{k_1} \cap E_{k_2} = \emptyset, \quad k_1, k_2 \in \{1, \dots, m_1\}, \quad k_1 \neq k_2,$$

$$(4.4) \quad \left| I \setminus \bigcup_{k=1}^{m_1} E_k \right| = 0,$$

and

$$(4.5) \quad \text{for every } i \in \{1, 2, \dots, n\} \text{ and } k \in \{1, 2, \dots, m_1\} \text{ there is a } j \in \{1, 2, \dots, N\} \text{ such that } E_k \subset \Omega_j^i.$$

We define the operator

$$(P_\varepsilon f)(x) = \sum_{k=1}^{m_1} \chi_{E_k}(x) \frac{\int_{E_k} f(t) e^{-t^2} dt}{\int_{E_k} e^{-t^2} dt}.$$

Then $(P_\varepsilon f)(x)$ is defined on $\bigcup_{k=1}^{m_1} E_k$ and therefore, by (4.4), it is defined a.e. on I . It is not difficult to see that $P_\varepsilon : L_\infty \rightarrow L_\infty$ is a bounded linear finite dimensional operator. Moreover, using (4.3), we obtain

$$\begin{aligned} (P_\varepsilon^2 f)(x) &= \sum_{k=1}^{m_1} \chi_{E_k}(x) \frac{1}{\int_{E_k} e^{-t^2} dt} \int_{E_k} (P_\varepsilon f)(t) e^{-t^2} dt \\ &= \sum_{k=1}^{m_1} \frac{\chi_{E_k}(x)}{\int_{E_k} e^{-t^2} dt} \int_{E_k} \sum_{\ell=1}^{m_1} \chi_{E_\ell}(t) \frac{\int_{E_\ell} f(s) e^{-s^2} ds}{\int_{E_\ell} e^{-s^2} ds} e^{-t^2} dt \\ &= \sum_{k=1}^{m_1} \chi_{E_k}(x) \frac{\int_{E_k} f(s) e^{-s^2} ds}{\int_{E_k} e^{-s^2} ds} = (P_\varepsilon f)(x), \end{aligned}$$

which proves

$$(4.6) \quad P_\varepsilon^2 = P_\varepsilon.$$

In other words, P_ε is a projection. Further, due to (4.3),

$$\begin{aligned} \|P_\varepsilon f\|_{L_\infty} &\leq \operatorname{ess\,sup}_{x \in I} \sum_{k=1}^{m_1} \chi_{E_k}(x) \frac{\int_{E_k} |f(t)| e^{-t^2} dt}{\int_{E_k} e^{-t^2} dt} \\ &\leq \|f\|_{L_\infty} \operatorname{ess\,sup}_{x \in I} \sum_{k=1}^{m_1} \chi_{E_k}(x) = \|f\|_{L_\infty}, \end{aligned}$$

which gives

$$(4.7) \quad \|P_\varepsilon\| \leq 1.$$

Let Z be the finite dimensional subspace of L_∞ defined by

$$Z = \left\{ f = \sum_{k=1}^{m_1} a_k \chi_{E_k}(x), (a_1, \dots, a_{m_1}) \in \mathbb{R}^{m_1} \right\}.$$

In fact, $P_\varepsilon : L_\infty \rightarrow Z$. Moreover, let $f = \sum_{k=1}^{m_1} a_k \chi_{E_k}(x) \in Z$. Then, by (4.3), we can write

$$\begin{aligned} (P_\varepsilon f)(x) &= \sum_{k=1}^{m_1} \chi_{E_k}(x) \frac{1}{\int_{E_k} e^{-t^2} dt} \int_{E_k} a_k e^{-t^2} dt \\ &= \sum_{k=1}^{m_1} a_k \chi_{E_k}(x) = f(x), \end{aligned}$$

which shows that

$$(4.8) \quad P_\varepsilon f = f \quad \text{for any } f \in Z.$$

We claim that $\text{dist}(g_i, Z) \leq \varepsilon$ for any $i \in \{1, 2, \dots, n\}$. Fix $i \in \{1, 2, \dots, n\}$. By (4.5), for every $k \in \{1, 2, \dots, m_1\}$ there is a set $\Omega_{j_k}^i$ such that $E_k \subset \Omega_{j_k}^i$. Consequently, $g_i(E_k) \subset I_{j_k}$. Choose $\gamma_k \in I_{j_k}$, $k = 1, 2, \dots, m_1$ and define the function \bar{g}_i by

$$\bar{g}_i(x) = \sum_{k=1}^{m_1} \gamma_k \chi_{E_k}(x).$$

Then $\bar{g}_i \in Z$ and, moreover, $|I_{j_k}| \leq \varepsilon$ implies that

$$(4.9) \quad \|g_i - \bar{g}_i\|_{L_\infty} = \sup_{k \in \{1, 2, \dots, m_1\}} \text{ess sup}_{x \in E_k} |g_i(x) - \gamma_k| \leq \varepsilon.$$

Let $f \in B$. We shall estimate $\|Lf - P_\varepsilon Lf\|_{L_\infty}$. Choose g_i such that $\|Lf - g_i\|_{L_\infty} \leq \sigma + \varepsilon$. Then

$$\begin{aligned} \|Lf - P_\varepsilon Lf\|_{L_\infty} &\leq \|Lf - g_i\|_{L_\infty} + \|P_\varepsilon(Lf - g_i)\|_{L_\infty} + \|g_i - \bar{g}_i\|_{L_\infty} + \|\bar{g}_i - P_\varepsilon g_i\|_{L_\infty} \\ &\leq \sigma + \varepsilon + \|P_\varepsilon\|(\sigma + \varepsilon) + \|g_i - \bar{g}_i\|_{L_\infty} + \|\bar{g}_i - P_\varepsilon g_i\|_{L_\infty}. \end{aligned}$$

Using (4.7)–(4.9) and $\bar{g}_i = P\bar{g}_i$, we get

$$\|Lf - P_\varepsilon Lf\|_{L_\infty} \leq 2\sigma + 3\varepsilon + \|P_\varepsilon(\bar{g}_i - g_i)\|_{L_\infty} \leq 2\sigma + 4\varepsilon,$$

that is,

$$(4.10) \quad \|L - P_\varepsilon L\| \leq 2\sigma + 4\varepsilon.$$

Now let us deal with $(P_\varepsilon Lf)(x)$. Clearly,

$$\begin{aligned} (P_\varepsilon Lf)(x) &= \sum_{j=1}^{m_1} \chi_{E_j}(x) \frac{\int_{E_j} (Lf)(t) e^{-t^2} dt}{\int_{E_j} e^{-t^2} dt} \\ &= \sum_{j=1}^{m_1} \chi_{E_j}(x) \frac{1}{\int_{E_j} e^{-t^2} dt} \int_{E_j} \int_I \ell(t, s) f(s) ds e^{-t^2} dt \\ &= \int_I \left(\sum_{j=1}^{m_1} \chi_{E_j}(x) \frac{\int_{E_j} \ell(t, s) e^{-t^2} dt}{\int_{E_j} e^{-t^2} dt} \right) f(s) ds \\ &= \int_I k_\varepsilon(x, s) f(s) ds, \end{aligned}$$

where

$$(4.11) \quad k_\varepsilon(x, s) = \sum_{j=1}^{m_1} \chi_{E_j}(x) \frac{\int_{E_j} \ell(t, s) e^{-t^2} dt}{\int_{E_j} e^{-t^2} dt} = \sum_{j=1}^{m_1} \chi_{E_j}(x) \psi_j(t) \quad \text{say.}$$

Thus, $P_\varepsilon L$ is a kernel operator with the kernel $k_\varepsilon(x, s)$. Now, $\ell \in \mathfrak{M}(I^2)$ implies $\psi_j(s) \in \mathfrak{M}(I)$, and, consequently, $k_\varepsilon \in \mathfrak{M}(I^2)$.

Define the measure λ on I by $\lambda(E) = \int_E e^{-t^2} dt$. It is not difficult to prove that λ satisfies the assumptions of Lemma 4.6. Moreover, we have for any $j \in \{1, 2, \dots, m_1\}$ $0 < \lambda(E_j) \leq \lambda(I) = \int_I e^{-t^2} dt \leq \int_{-\infty}^{\infty} e^{-t^2} dt < \infty$. Setting $h(x, t) = \frac{\ell(x, t)}{v(t)}$, we have $h(x, t)v(t) \in \mathcal{A}$, and, using also Lemma 4.6, we can write

$$\begin{aligned} \left\| \frac{\psi_j(s)}{v(s)} \right\|_{X', v} &= \left\| \int_{E_j} \frac{\ell(t, s) e^{-t^2}}{v(s) \int_{E_j} e^{-t^2} dt} dt \right\|_{X', v} = \left\| \frac{1}{\lambda(E_j)} \int_{E_j} k(t, \cdot) d\lambda \right\|_{X', v} \\ &\leq \frac{1}{\lambda(E_j)} \int_{E_j} \|k(t, \cdot)\|_{X', v} d\lambda(t) = \frac{1}{\lambda(E_j)} \int_{E_j} \left\| \frac{\ell(t, \cdot)}{v(\cdot)} \right\|_{X', v} d\lambda(t) \\ &\leq \operatorname{ess\,sup}_{x \in E_j} \left\| \frac{\ell(x, \cdot)}{v(\cdot)} \right\|_{X', v} \leq \|\ell\|_{L_\infty(X', v)}, \end{aligned}$$

which gives

$$(4.12) \quad \left\| \frac{\psi_j}{v} \right\| \leq \|\ell\|_{L_\infty(X', v)}.$$

This implies

$$\begin{aligned} \|\ell - k_\varepsilon\|_{L_\infty(X', v)} &\leq \|\ell\|_{L_\infty(X', v)} + \left\| \sum_{j=1}^{m_1} \chi_{E_j}(x) \psi_j(s) \right\|_{L_\infty(X', v)} \\ &\leq \|\ell\|_{L_\infty(X', v)} + \sum_{j=1}^{m_1} \|\chi_{E_j}\|_{L_\infty} \left\| \frac{\psi_j}{v} \right\|_{X', v}. \end{aligned}$$

From (4.12) we obtain $\|\ell - k_\varepsilon\|_{L_\infty(X', v)} \leq (1 + m_1) \|\ell\|_{L_\infty(X', v)}$ and, consequently, using also Lemma 3.1, we have $\ell - k_\varepsilon \in \mathcal{A}$. Now, Theorem 3.14 yields

$$\|\ell - k_\varepsilon\|_{L_\infty(X', v)} = \|L - P_\varepsilon L\|.$$

Together with (4.10) this implies

$$\|\ell - k_\varepsilon\| \leq 2\sigma + 4\varepsilon.$$

By (4.11) and (4.12), $k_\varepsilon \in \mathfrak{K}$ and, consequently,

$$d = \inf_{k \in \mathfrak{K}} \|\ell - k\|_{L_\infty(X', v)} \leq \|\ell - k_\varepsilon\|_{(X', v)} \leq 2\sigma + 4\varepsilon.$$

On letting $\varepsilon \rightarrow 0_+$ we obtain $d \leq 2\sigma$, and the proof is complete. \square

4.8 Corolary. $\frac{d}{2} \leq D \leq d$.

4.9 Remark. Corolary 4.8 shows that an operator L is compact if and only if its kernel ℓ can be approximated in $L_\infty(X', v)$ by kernels $k_n \in \mathfrak{K}$.

5. APPLICATION TO THE HARDY OPERATOR

Let $I = [a, b]$, $-\infty \leq a < b \leq +\infty$. We define the Hardy operator by $Hf(x) = \int_a^x f(t)dt$. Further, let

$$U(x, \varepsilon) = \begin{cases} (x - \varepsilon, x + \varepsilon) \cap [a, b] & \text{if } -\infty < x < \infty \\ (-\infty, -\frac{1}{\varepsilon}) \cap [a, b] & \text{if } x = -\infty \\ (\frac{1}{\varepsilon}, \infty) \cap [a, b] & \text{if } x = \infty. \end{cases}$$

We also denote

$$B(x) = \lim_{\varepsilon \rightarrow 0^+} \|\chi_{U(x, \varepsilon)} \frac{1}{v}\|_{X', v}, \quad \text{and} \quad B = \sup_{a \leq x \leq b} B(x).$$

In [LP], a characterization of the boundedness and compactness of the Hardy operator was characterized for $I = [0, \infty]$. It was shown that H is bounded if and only if $\frac{1}{v} \in (X', v)$, and that H is compact if and only if $B = 0$. We will apply the results of Sections 3 and 4 to the Hardy operator and $I = [a, b]$. Observe that the Hardy operator is given by the kernel $h(x, t) = \chi_{(a, x)}(t)$, i.e. $\int_a^x f(t)dt = \int_I \chi_{(a, x)}(t)f(t)dt$.

5.1 Theorem ([LP]). *The operator H is bounded from (X, v) into L_∞ if and only if $\|\frac{1}{v}\|_{X', v} < \infty$.*

Proof. By Remark 3.15, H is bounded if and only if $\|h\|_{L_\infty(X', v)} < \infty$. Moreover, $\|H\| = \|h\|_{L_\infty(X', v)}$. Then

$$\|H\| = \operatorname{ess\,sup}_{x \in I} \left\| \frac{h(x, \cdot)}{v(\cdot)} \right\|_{X', v} = \operatorname{ess\,sup}_{x \in I} \left\| \frac{\chi_{(a, x)}(t)}{v(t)} \right\|_{X', v} = \left\| \frac{1}{v} \right\|_{X', v},$$

which completes the proof. \square

5.2 Lemma. *The inequality $d \leq B$ holds.*

Proof. Let $\varepsilon > 0$. From the definition of B we know that for every $x \in [a, b]$ there is an $\eta(x) > 0$ such that

$$\left\| \chi_{U(x, \eta(x))}(t) \frac{1}{v(t)} \right\|_{X', v} \leq B + \varepsilon.$$

Since $\bigcup_{x \in I} U(x, \eta(x)) \supset I$ and $I = [a, b]$ is a compact set in the topology induced by $U(x, \varepsilon)$, we can choose $x_1, \dots, x_n \in I$ such that $\bigcup_{i=1}^n U(x_i, \eta(x_i)) \supset I$. Denote $\tilde{U}_i = U(x_i, \eta(x_i))$. Take α_i, β_i , $i = 1, 2, \dots, n$, such that

$$(5.1) \quad U_i := (\alpha_i, \beta_i) \subset \tilde{U}_i, \quad i = 1, 2, \dots, n,$$

and

$$(5.2) \quad |I \setminus \bigcup_{i=1}^n U_i| = 0.$$

Let us define $k(x, t) = \sum_{i=1}^n \chi_{U_i}(x) \chi_{(a, \alpha_i)}(t)$. Clearly, by (5.1) and (5.2), we have

$$\begin{aligned}
d &\leq \operatorname{ess\,sup}_{x \in I} \left\| \frac{1}{v(t)} \left(\sum_{i=1}^n \chi_{U_i}(x) (\chi_{(a, x)}(t) - \chi_{(a, \alpha_i)}(t)) \right) \right\|_{X', v} \\
&= \operatorname{ess\,sup}_{x \in I} \left\| \frac{1}{v(t)} \sum_{i=1}^n \chi_{U_i}(x) \chi_{(\alpha_i, x)}(t) \right\|_{X', v} \\
&\leq \operatorname{ess\,sup}_{x \in I} \left\| \frac{1}{v(t)} \sum_{i=1}^n \chi_{U_i}(x) \chi_{U_i}(t) \right\|_{X', v} \\
&\leq \operatorname{ess\,sup}_{x \in I} \sum_{i=1}^n \chi_{U_i}(x) \left\| \frac{\chi_{U_i}(t)}{v(t)} \right\|_{X', v} \leq B + \varepsilon.
\end{aligned}$$

Therefore, $d \leq B + \varepsilon$ for any $\varepsilon > 0$, and the assertion follows. \square

5.3 Lemma. *The inequality $B \leq 4d$ holds.*

Proof. Let $\varepsilon > 0$. Then, for some M_i and ψ_i , $i = 1, \dots, n$,

$$(5.3) \quad \left\| \chi_{(a, x)}(t) - \sum_{i=1}^n \chi_{M_i}(x) \psi_i(t) \right\|_{L_\infty(X', v)} \leq d + \varepsilon.$$

Let $x_0 \in [a, b)$. Then there is a $k \in \{1, 2, \dots, n\}$ such that $|(x_0, x_0 + \sigma) \cap M_k| > 0$ for any $\sigma > 0$. Set $x_1 = \operatorname{ess\,sup} M_k$, i.e., $x_1 = \inf\{y, |(y, b) \cap M_k| = 0\}$. Then (5.3) gives

$$\begin{aligned}
d + \varepsilon &\geq \operatorname{ess\,sup}_{x \in I} \left\| \frac{1}{v(t)} \left(\chi_{(a, x)}(t) - \sum_{i=1}^n \chi_{M_i}(x) \psi_i(t) \right) \chi_{M_k}(x) \chi_{(x_0, x_1)}(t) \right\|_{X', v} \\
&= \operatorname{ess\,sup}_{x \in I} \left\| \frac{1}{v(t)} \left(\chi_{(x_0, x)}(t) - \chi_{(x_0, x_1)}(t) \psi_k(t) \right) \chi_{M_k}(x) \right\|_{X', v} \\
&= \operatorname{ess\,sup}_{x \in M_k} \left\| \frac{1}{v(t)} \left(\chi_{(x_0, x)}(t) (1 - \psi_k(t)) - \chi_{(x, x_1)}(t) \psi_k(t) \right) \right\|_{X', v}.
\end{aligned}$$

Since $(x_0, x) \cap (x, x_1) = \emptyset$ for every $x \in M_k$, we have

$$\begin{aligned}
d + \varepsilon &\geq \operatorname{ess\,sup}_{x \in M_k} \left\| \frac{1}{v(t)} \chi_{(x_0, x)}(t) (1 - \psi_k(t)) \right\|_{X', v} \\
&= \left\| \frac{1}{v(t)} \chi_{(x_0, x_1)}(t) (1 - \psi_k(t)) \right\|_{X', v}
\end{aligned}$$

and

$$d + \varepsilon \geq \operatorname{ess\,sup}_{x \in M_k} \left\| \frac{1}{v(t)} \chi_{(x, x_1)}(t) \psi_k(t) \right\|_{X', v} = \left\| \frac{1}{v(t)} \chi_{(x_0, x_1)}(t) \psi_k(t) \right\|_{X', v}.$$

As a consequence we obtain

$$(5.4) \quad \left\| \frac{\chi_{(x_0, x_1)}(t)}{v(t)} \right\|_{X', v} \leq \left\| \chi_{(x_0, x_1)}(t) \frac{1 - \psi_k(t)}{v(t)} \right\|_{X', v} \\ + \left\| \chi_{(x_0, x_1)}(t) \frac{\psi_k(t)}{v(t)} \right\|_{X', v} \leq 2d + 2\varepsilon.$$

Let $B^+(x) = \lim_{\varepsilon \rightarrow 0_+} \left\| \frac{\chi_{(x, x+\varepsilon)}(t)}{v(t)} \right\|_{X', v}$ for $x \in [a, b]$ and, analogously, $B^-(x) = \lim_{\varepsilon \rightarrow 0_+} \left\| \frac{\chi_{(x-\varepsilon, x)}(t)}{v(t)} \right\|_{X', v}$ for $x \in (a, b]$. Then $B(a) = B^+(a)$, $B(b) = B^-(b)$ and $B(x) \leq B^+(x) + B^-(x)$ for $x \in (a, b)$ which together with (5.4) yields

$$B(x_0) \leq B^+(x_0) + B^-(x_0) \leq 4d + 4\varepsilon.$$

Letting $\varepsilon \rightarrow 0_+$, we obtain $B(x_0) \leq 4d$ and, consequently, $B \leq 4d$, which completes the proof. \square

Corollary. *The inequalities $\frac{B}{8} \leq D \leq B$ hold.*

Proof. This is an immediate consequence of Corollary 4.8 and Lemmas 5.2 and 5.3. \square

Corollary. *The Hardy operator is compact if and only if $B = 0$.*

REFERENCES

- [BS] C. Bennett and R. Sharpley, *Interpolation of Operators*, Pure and Appl. Math., vol. 129, Academic Press, New York, 1988.
- [EE] D.E. Edmunds and W.D. Evans, *Spectral Theory of Differential Operators*, Oxford Univ. Press, Oxford, 1987.
- [EEH] D.E. Edmunds, W.D. Evans and D.J. Harris, *Approximation numbers of certain Volterra integral operators*, J. London Math. Soc. **38** (1988), 471-489.
- [LP] Q. Lai and L. Pick, *The Hardy operator, L_∞ and BMO*, J. London Math. Soc. **48** (1993), 167-177.
- [L] W.A.J. Luxemburg, “*Banach Function Spaces*”, Thesis, Delft, 1955.
- [LZ] W.A.J. Luxemburg and A.C. Zaanen, *Compactness of integral operators in Banach function spaces*, Math. Annalen **149** (1963), 150-180.
- [O] B. Opic, *On the distance of the Riemann-Liouville operator from compact operators*, Proc. Amer. Math. Soc. **122** (1994), 495-501.
- [S] V.D. Stepanov, *Weighted inequalities for a class of Volterra convolution operators*, J. London Math. Soc. **45** (1992), 232-242.
- [W] P. Wojtaszczyk, *Banach Spaces for Analysts*, Cambridge Univ. Press, 1991.

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