3. *Let* $(a_1,b_1) \supseteq (a_2,b_2) \supseteq \ldots$ *be a nested sequence of intervals in* $\mathbb{R}$ *such that neither* $a_1,a_2,\ldots$ *nor* $b_1,b_2,\ldots$ *eventually stabilizes. Prove that* $\bigcap_{n=1}^{\infty}(a_n,b_n) \neq \emptyset$.

Proof. Let $A = \{a_1,a_2,\ldots\}$ and let $c = \sup A$. Then $c \geq a_n$ for all $n \in \mathbb{N}$. Since, for any n there is $l > n$ such that $a_l > a_n$, we have $c \geq a_l > a_n$. So, $c > a_n$ for all n.

Let $n \in \mathbb{N}$. For any $m \in \mathbb{N}$, if $n \leq m$ then $b_n \geq b_m > a_m$, and if $n > m$ then $b_n > a_n \geq a_m$; in any case, $b_n > a_m$, so b_n is an upper bound of A. Hence, $b_n \geq c$. Since, there is $l > n$ such that $b_l < b_n$, we have $c \leq b_l < b_n$. So, $c < b_n$ for all n.

We've proved that $a_n < c < b_n$ for all n, so $c \in (a_n,b_n)$ for all n. Hence, $c \in \bigcap_{n=1}^{\infty}(a_n,b_n) \neq \emptyset$, and this intersection is nonempty.

4. (Cf. Ex. 14,15 p. 25 in Browder) (a) *Let* $a,b \in \mathbb{R}$, $a < b$. *Find a bijective mapping from the interval* $[0,1]$ *onto the interval* $[a,b]$.

Solution. For $x \in [0,1]$, put $\varphi(x) = a + (b-a)x$, then φ maps $[0,1]$ to $[a,b]$. φ is invertible (φ^{-1} is given by $\varphi^{-1}(y) = \frac{y-a}{b-a}$), so is a bijection.

(b) *Find a bijective mapping from the interval* $(0,1)$ *onto* $\mathbb{R}$.

Solution. I define a mapping $\varphi\colon \mathbb{R} \longrightarrow (0,1)$ by $\varphi(x) = \begin{cases} \frac{1}{2} + \frac{1}{2+x}, & x > 0 \\ 0, & x = 0 \\ \frac{1}{2} - \frac{1}{2-x} \end{cases}$. Let us check that φ is one-to-one and onto. (And then $\varphi^{-1}\colon [0,1] \longrightarrow \mathbb{R}$ is defined and is also a bijection.) Clearly, $\varphi(x) \in \left(\frac{1}{2},1\right)$ if $x > 0$ and $\varphi(x) \in \left(0,\frac{1}{2}\right)$ if $x < 0$. For any $y \in \left(\frac{1}{2},1\right)$, $y = \varphi(x)$ iff $y = \frac{1}{2} + \frac{1}{2+x}$ iff $x = \frac{1}{y-1/2} - 2$; since $\frac{1}{2} < y < 1$, $x > 0$ and is uniquely defined. Similarly, if $y \in \left(0,\frac{1}{2}\right)$, then $y = \varphi(x)$ iff $y = \frac{1}{2} - \frac{1}{2-x}$ iff $x = \frac{-1}{1/2-y} + 2$; since $0 < y < \frac{1}{2}$, $x < 0$ and is uniquely defined.

(c) *Find a bijective mapping from* $[0,1]$ *onto* $(0,1)$.

Solution. Define $\psi\colon [0,1] \longrightarrow (0,1)$ by $\psi(x) = x$ if $x \neq 0,1,\frac{1}{2},\frac{1}{3},\ldots$, $\psi(0) = \frac{1}{2}$, $\psi(1) = \frac{1}{3}$, $\psi\left(\frac{1}{n}\right) = \frac{1}{n+2}$ for $n = 2,3,\ldots$

5. *Prove that* $|\mathbb{R}| = |\mathbb{R}^2|$.

Solution. Let $S = \{0,1\}^{\mathbb{N}}$, the set of 0,1-sequences. I'll prove that $|S^2| = |S|$; since there exists a bijection between S and $\mathbb{R}$, this will solve the problem. Given a pair of sequences (s,r), $s = s_1s_2\ldots$, $r = r_1r_2\ldots$, I form the new sequence $t = s_1r_2s_2r_2\ldots$. Since we can always reconstruct the sequences s and r from t, the mapping $(s,r) \mapsto t$ is a 1-1 correspondence between S^2 and S.

Expert's proof. $|\mathbb{R}^2| = |\mathbb{R}|^2 = |2^{\mathbb{N}}|^2 = (2^{|\mathbb{N}|})^2 = 2^{2|\mathbb{N}|} = 2^{|\mathbb{N}|} = |\mathbb{R}|$.

6. *Let* A *be a set of pairwise disjoint intervals in* $\mathbb{R}$. *Prove that* A *is countable (or finite).*

Proof. For any $I \in A$, find a rational number $r_I \in I$; we get a mapping $A \longrightarrow \mathbb{Q}$, $I \mapsto r_I$. This mapping is injective since the intervals in A are disjoint, and so $I,J \in A$, $I \neq J$ implies $r_I \neq r_J$. Hence, $|A| \leq |\mathbb{Q}|$.

7. *Prove that the set of irrational numbers is uncountable.*

Proof. If the set S of irrational numbers were countable, then the set $\mathbb{R} = \mathbb{Q} \cup S$ of real numbers would be countable as the union of two countable sets. Since this is not the case, S is uncountable.

9. *Let* $a \in \mathbb{R}$, $a \neq 0$. *Prove that the set* $a\mathbb{Q} = \{ar,\ r \in \mathbb{Q}\}$ *is dense in* $\mathbb{R}$.

Proof. Assume that $a > 0$. Let (b,c) be an interval in $\mathbb{R}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, the interval $\left(\frac{b}{a},\frac{c}{a}\right)$ contains a rational number r, $\frac{b}{a} < r < \frac{c}{a}$, so $b < ar < c$, so the interval (b,c) contains the point ar.

If $a < 0$ and an interval (b,c) is given, find $r \in \mathbb{Q}$ such that $(-a)r \in (-c,-b)$, then $ar \in (b,c)$.

10. *Prove that the set of transcendental numbers is dense in* $\mathbb{R}$.

Proof. Let I be any interval in $\mathbb{R}$. Since the set of algebraic numbers is countable and I is uncountable, I contains transcendental numbers.

11. *Prove that the number* $\alpha = \sqrt{2} + \sqrt{3}$ *is irrational and algebraic.*

Proof. We have $\alpha^2 = 2 + 3 + 2\sqrt{6}$; since $\sqrt{6}$ is irrational, α^2 is irrational, and so, α is irrational. We then have $(\alpha^2 - 5)^2 = 24$, so α satisfies the equation $\alpha^4 - 10\alpha^2 + 1 = 0$ and is, therefore, algebraic.

12. *For $a \in \mathbb{R}$ we define $a^0 = 1$, and for $n \in \mathbb{N}$ we define a^n inductively by $a^0 = 1$ and $a^{n+} = a^n \cdot a$. For $a \neq 0$ and $n \in \mathbb{N}$ we also define $a^{-n} = (a^n)^{-1}$. Prove that*
(i) *$a^{n+m} = a^n a^m$ for all $a \neq 0$, $n, m \in \mathbb{Z}$;*
(iv) *if $a > 1$, then $a^n > 1$ for all $n \in \mathbb{N}$; if $0 < a < 1$ then $0 < a^n < 1$ for all $n \in \mathbb{N}$;*
(v) *if $a, b > 0$ and $n \in \mathbb{N}$, then $a^n > b^n$ iff $a > b$;*

Proof. (i) If $n = 0$, then $a^{n+m} = a^m = 1 \cdot a^m = a^n a^m$ for all $m \in \mathbb{N}$.

Let $n \in \mathbb{N}$. We have $a^{n+0} = a^n = a^n \cdot 1 = a^n a^0$ and $a^{n+1} = a^n \cdot a = a^n \cdot a^1$. Now, if $a^{n+m} = a^n a^m$ for some m, then $a^{n+(m+1)} = a^{(n+m)+1} = a^{n+m} a = a^n a^m a = a^n a^{m+1}$, so we have $a^{n+m} = a^n a^m$ for all $m \in \mathbb{N}$ by induction on m.

If $n \in \mathbb{N}$, $m \in -\mathbb{N}$ and $n > -m$, let $k = n - (-m) = n + m$. Then $a^n = a^k a^{-m} = a^k (a^m)^{-1}$, so $a^n a^m = a^k = a^{n+m}$.

If $n \in \mathbb{N}$, $m \in -\mathbb{N}$ and $n < -m$, or if $n, m \in -\mathbb{N}$, then $a^{-n-m} = a^{-n} a^{-m}$, so $a^{n+m} = a^n a^m$.
(iv) $a^1 = a > 1$, and if $a^n > 1$ for some $n \in \mathbb{N}$ then $a^{n+1} = a^n a > a^n \cdot 1 = a^n > 1$, so $a^n > 1$ for all $n \in \mathbb{N}$ by induction on n.
(v) if $a > b > 0$, then $ab^{-1} > 1$, so $a^n (b^n)^{-1} = a^n (b^{-1})^n = (ab^{-1})^n > 1$, so $a^n > b^n$. If $a < b$ then, similarly, $a^n < b^n$. So, $a = b$.

13. *Let $c > 0$. By induction on n, prove that for any $n \in \mathbb{N}$, $(1 + c)^n \geq 1 + nc$.*
Proof. This is trivial for $n = 1$. If this is true for some n, then for $n + 1$ we have

$$(1 + c)^{n+1} = (1 + c)^n (1 + c) \geq (1 + nc)(1 + c) = 1 + nc + c + nc^2 > 1 + (n + 1)c.$$

14. *For $b > 0$ and $m, n \in \mathbb{N}$, prove that $(b^m)^{1/n} = (b^{1/n})^m$.*
Proof. We have

$$\big((b^{1/n})^m\big)^n = (b^{1/n})^{mn} = \big((b^{1/n})^n\big)^m = b^m,$$

so $(b^{1/n})^m = (b^m)^{1/n}$ by definition of $x^{1/n}$.

15. *Let $b > 1$.*
(a) *If m, n, p, q are integers, $n, q > 0$, and $r = \frac{m}{n} = \frac{p}{q}$, prove that $(b^m)^{1/n} = (b^p)^{1/q}$.* (Hence it makes sense to define $b^r = (b^m)^{1/n}$.)
Proof. Let $c = (b^m)^{1/n}$ and $d = (b^p)^{1/q}$. Then $c^{nq} = (c^n)^q = (b^m)^q = b^{mq} = b^{pn} = (b^p)^n = (d^q)^n = d^{nq}$, so $c^{nq} = d^{nq}$, and so, $d = c$.
(b) *Prove that $b^{r+s} = b^r b^s$ for every $r, s \in \mathbb{Q}$.*
Proof. Let $r = \frac{m}{n}$ and $s = \frac{p}{q}$. Then $(b^r b^s)^{nq} = (b^r)^{nq} (b^s)^{nq} = (((b^m)^{1/n})^n)^q (((b^p)^{1/q})^q)^n = (b^m)^q (b^p)^n = b^{mq+np}$, so $b^r b^s = (b^{mq+np})^{1/nq} = b^{\frac{mq+pn}{nq}} = b^{r+s}$. In particular, $b^{-r} b^r = b^0 = 1$ and so, $b^{-r} = 1/b^r$.
(c) *For $x \in \mathbb{R}$, let $B(x) = \{b^t : t \in \mathbb{Q},\ t < x\}$. Prove that $b^r = \sup B(r)$ for any $r \in \mathbb{Q}$.* (Hence it makes sense to define $b^x = \sup B(x)$ for every real x.)
Proof. First, $b^t < b^r$ for all $t < r$, so b^r is an upper bound of $B(r)$. Now take $c < b^r$ and check that c is not an upper bound of $B(r)$. We may assume that $c > 0$. Then $d = b^r/c > 1$. Then there exists $n \in \mathbb{N}$ such that $d^n > b$. (If $d = 1 + u$, $u > 0$, then $d^n = (1 + u)^n \geq 1 + nu > b$ if n is large enough.) So, $b^r/c = d > b^{1/n}$ and so, $b^{r-\frac{1}{n}} > c$. But $r - \frac{1}{n} \in \mathbb{Q}$ and $r - \frac{1}{n} < r$; so, c is not an upper bound of $B(r)$.
(d) *Prove that $b^{x+y} = b^x b^y$ for every $x, y \in \mathbb{R}$.*
Proof. For any $r, s \in \mathbb{Q}$, $r < x$, $s < y$, we have $r + s \in \mathbb{Q}$, $r + s < x + y$, so $b^r b^s = b^{r+s} \leq b^{x+y}$; hence, b^{x+y} is an upper bound of the set $B(x)B(y)$, so $b^{x+y} \geq \sup(B(x)B(y)) = \sup B(x) \sup B(y) = b^x b^y$.

We need to prove that $b^x b^y \geq b^{x+y}$, for which it suffices to show that $b^x b^y$ is an upper bound of the set $B(x + y) = \{b^t : t \in \mathbb{Q},\ t < x + y\}$. Let $t \in \mathbb{Q}$, $t < x + y$. Find $r, s \in \mathbb{Q}$, $r < x$, $s < y$, such that $t < r + s$. (Put $\varepsilon = x + y - t$, and choose $r, s \in \mathbb{Q}$ s.t. $x - \varepsilon/2 < r < x$ and $y - \varepsilon/2 < s < y$; then $r + s > x + y - \varepsilon = t$.) Now, since $r + s \in \mathbb{Q}$, we have $b^t < b^{r+s} = b^r b^s \leq b^x b^y$. Hence, $b^x b^y$ is an upper bound of $B(x + y)$.