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JORDAN STRUCTURES OF STRICTLY LOWER TRIANGULAR COMPLETIONS OF NILPOTENT MATRICES

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We prove the following theorem, conjectured by Rodman and Shalom: Let A be an $n \times n$ nilpotent matrix with Jordan blocks of sizes $q_1 \geq \ldots \geq q_s$. If $p_1 \geq \ldots \geq p_r$ is a sequence of positive integers such that $\{p_i\}_{i=1}^r$ majorizes $\{q_j\}_{j=1}^s$, then there exists a strictly lower triangular matrix T such that $A+T$ is nilpotent and $p_1, \ldots, p_r$ are the sizes of Jordan blocks of $A+T$.

1 INTRODUCTION

A partial triangular matrix is a matrix in which the upper triangular part (including the main diagonal) is specified, and the strictly lower triangular part is unspecified and considered as a set of free independent variables.

All matrices in this paper are over a field $\mathcal{F}$.

A *completion* of a partial matrix is any matrix which is obtained by replacing the unspecified entries with elements from $\mathcal{F}$. A matrix completion problem is a problem of finding all completions with specific properties of a given partial matrix.

Various matrix completion problems for partial triangular matrices have been studied in [1, 3, 5, 6, 7, 8, 10, 11, 12], including the problems concerning ranks, eigenvalues, Jordan forms, singular values, as well as applications to controllability of linear systems.

In [1, 11] the problem of the existence of a strictly lower triangular completion with given characteristic polynomial of the completed matrix has been completely solved. Generally speaking, such a completion is not unique (if exists). In [7], the possible geometric multiplicities of the eigenvalues of a completed matrix was studied. The Jordan forms of strictly lower triangular completions were investigated for different particular cases in [8, 10].

In this paper it is more convenient for us to consider strictly lower triangular completions as additive perturbations of full (not partial) matrices. The goal of this paper is to prove a proposition describing the general sufficient condition on Jordan structures of strictly lower triangular additive perturbations of a nilpotent matrix. These conditions were conjectured by Rodman and Shalom in [10].

Given two nonincreasing sequences of positive integers $\{p_i\}_{i=1}^r$ and $\{q_j\}_{j=1}^s$, the sequence $\{p_i\}_{i=1}^r$ *majorizes* $\{q_j\}_{j=1}^s$ if $r \leq s$, $\sum_{i=1}^t p_i \geq \sum_{j=1}^t q_j$ for $t = 1, \ldots, r$, and $\sum_{i=1}^r p_i = \sum_{j=1}^s q_j$ (see, for example, [9]).

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THEOREM **1.1** (Conjecture of Rodman and Shalom) *Let A be an $n \times n$ nilpotent matrix and let $q_1 \geq \ldots \geq q_s$ be the sizes of its Jordan blocks. If $\{p_i\}_{i=1}^r$ is a nonincreasing sequence of positive integers majorizing $\{q_j\}_{j=1}^s$, then there exists a strictly lower triangular matrix T such that $A+T$ is nilpotent with Jordan blocks of sizes $p_1, \ldots, p_r$.*

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2 ADMISSIBLE CORRECTIONS AND TRANSFORMATIONS

Two matrices A, A' are called *upper equivalent* if there exist a nondegenerate lower triangular matrix S and a strictly lower triangular matrix R such that $A' = S^{-1}AS + R$ (see [4]).

Given a nilpotent matrix A, we will look for a strictly lower triangular matrix T with a specified Jordan structure of $A+T$. But we may do it not for A itself, but for any nilpotent matrix from the class of matrices upper equivalent to A. Indeed, let A' be upper equivalent to A, namely $A' = S^{-1}AS + R$, where S is a nondegenerate lower triangular matrix and R is strictly lower triangular. Then the matrix $T' = S^{-1}TS - R$ is strictly lower triangular and

$$A' + T' = S^{-1}(A+T)S$$

has the same Jordan structure as $A+T$.

We will call the transformation $S^{-1}AS$ of a matrix A for a nondegenerate lower triangular matrix S *an admissible transformation* of A. The additive perturbation $A+T$ of A with a strictly lower triangular matrix T will be called *an admissible correction* of A.

Let $p \geq q$ be two elements of a (nonincreasing) sequence of positive integers. Replacing p by $p+1$ and q by $q-1$ (or deleting q in the case $q=1$) we obtain, after ordering its elements, a new nonincreasing sequence of positive numbers, which majorizes the previous one. It is easy to see that any sequence majorizing a given one can be obtained by a finite number of such operations.

Now, we may reformulate Theorem 1.1 in the following way:

THEOREM **2.1** *Let A be a nilpotent square matrix, let $p, q, h_1, \ldots, h_t$, with $p \geq q$, be the sizes of the Jordan blocks of A. Then a nilpotent matrix whose Jordan blocks have sizes $p+1, q-1, h_1, \ldots, h_t$ (or $p+1, h_1, \ldots, h_t$ in the case $q=1$) can be obtained from A by a sequence of admissible transformations and admissible corrections.*

Let $A = [\alpha_{m,l}]_{m,l=1}^n$ be an $n \times n$ nilpotent matrix. We fix an n-dimensional vector space L over $\mathcal{F}$ and an ordered basis $V = \{v_1, \ldots, v_n\}$ in L with $n(v)$ being the ordinal number in V of $v \in V$ (we do not suppose $n(v_k) = k$). We denote by $V(l)$ the l-th element of V: $n(V(l)) = l$, $l = 1, \ldots, n$. Then we can associate with A an endomorphism $\mathcal{A}$ of L:

$$\mathcal{A}(V(m)) = \sum_{l=1}^{n} \alpha_{m,l} V(l), \; m = 1, \ldots, n$$

(A acts on the rows of coordinates of vectors of L from the right). When we reconstruct A from $\mathcal{A}$, we have to follow the ordering $V(1), \ldots, V(n)$ of the elements of V; another ordering gives a matrix obtained from A by a permutation of its rows and the same permutation of its columns.

Now, admissible corrections of A correspond to changes of $\mathcal{A}$ of the form

$$\mathcal{A}'(V(m)) = \mathcal{A}(V(m)) + \sum_{l<m} \varepsilon_{m,l} V(l),\ m = 1, \dots, n,$$

for some $\varepsilon_{m,l} \in \mathcal{F}$. Admissible transformations of A correspond to changes of the basis V of the form

$$V'(m) = \sum_{l \le m} \lambda_{m,l} V(l),\ \lambda_{m,m} \neq 0,\ m = 1, \dots, n;$$

we will call such changes *admissible changes of basis.*

PROPOSITION **2.2** *Let $L_1 \subset L_2 \subset \dots \subset L_n = L$, $\dim L_k = k$, $k = 1, \dots, n$, be a sequence of linear subspaces of L. Then there exists an admissible change of basis $V \mapsto V' = \{v'_1, \dots, v'_n\}$ such that $v'_k \in L_k$, $k = 1, \dots, n$. In particular, for any $1 \le k \le n$, the set $\{v'_1, \dots, v'_k\}$ is a basis of L_k.*

PROOF: It is enough to prove that there exists an admissible change of basis $V \mapsto V'$ such that $v'_1 \in L_1$; then the statement of the proposition can be obtained by the factorization L/L_1 and an induction process on $\dim L$.

Let $u = \sum_{l=1}^{n} \beta_l V(l) \in L_1$ and let $k = \max\{l : \beta_l \neq 0\}$. Then the admissible change of basis

$$v'_1 = V'(k) = u,\quad V'(l) = V(l),\ l \neq k,$$

gives the desired result. ∎

COROLLARY **2.3** *Let Y be a c-dimensional linear space over $\mathcal{F}$, let $\mathcal{B} : Y \to L$ be a linear embedding. Let $U = \{u_1, \dots, u_c\}$ be a basis of Y and let $1 \le d_1 < \dots < d_c \le n$ be arbitrary integers. Then there exists an admissible change of basis $V \mapsto V' = \{v'_1, \dots, v'_n\}$ in L such that*

$$\mathcal{B}(u_i) = v'_{d_i} + \sum_{j<d_i} \beta_{i,j} v'_j,\quad i = 1, \dots, c,$$

for some $\beta_{i,j} \in \mathcal{F}$.

PROOF: Denote $w_{d_i} = \mathcal{B}(u_i)$, $i = 1, \dots, c$, and complete the set $\{w_{d_1}, \dots, w_{d_c}\}$ up to a basis $\{w_1, \dots, w_n\}$ of L. Put $L_k = \mathrm{Span}\,(w_1, \dots, w_k)$, $k = 1, \dots, n$. Using Proposition 2.2 finishes the proof. ∎

Under the notation of Corollary 2.3, the matrix $B = [\beta_{i,j}]_{\substack{i=1,\dots,c \\ j=1,\dots,n}}$ of $\mathcal{B}$ in the bases U, taken in the ordering $u_1, \dots, u_c$, and V', taken in the ordering $v'_1, \dots, v'_n$, has the form

$$\begin{array}{c} \begin{array}{ccccc} & d_1 & d_2 & \dots & d_p \\ & \downarrow & \downarrow & & \downarrow \end{array} \\ \begin{pmatrix} * \dots * & 1 & 0 & & \dots & & 0 \\ * & \dots & & * & 1\ 0 & \dots & 0 \\ \vdots & & & & & & \vdots \\ * & & \dots & & & * & 1\ 0 \dots 0 \end{pmatrix} \end{array} \tag{2.1}$$

(we put $\beta_{i,d_i} = 1$, $\beta_{i,j} = 0$ for $j > d_i$, $i = 1, \ldots, p$). (B acts on the rows of coordinates of vectors in L' from the right.) We will call the matrix of the form (2.1) *a* $(d_1, \ldots, d_p)$*-lower-triangular matrix.*

Now we are ready to describe a relatively simple form to which any nilpotent matrix can be reduced by admissible changes of basis. Let A be nilpotent, let $\mathcal{A}$ be the corresponding endomorphism of a space L with an ordered basis V. Let $r \in \mathbb{N}$ be such that $A^r = 0$, $A^{r-1} \neq 0$.

Denote $L_k = \operatorname{Ker} \mathcal{A}^k \subseteq L$; then

$$L_1 \subset L_2 \subset \ldots \subset L_r = L, \text{ and } \mathcal{A}(L_{k+1}) \subseteq L_k, \; k = 1, \ldots, r-1.$$

Using Proposition 2.2, we may assume that, for $e_k = \dim L_k$, $k = 1, \ldots, r$, the set $\{v_1, \ldots, v_{e_k}\}$ is a basis of L_k. For every $k = 1, \ldots, r$, denote

$$\begin{gathered} c_k = e_k - e_{k-1}, \\ v_j^k = v_{e_{k-1}+j}, \; j = 1, \ldots, c_k, \\ V_k = \{v_1^k, \ldots, v_{c_k}^k\}, \\ Y_k = \operatorname{Span} V_k. \end{gathered}$$

For $k = 1, \ldots, r-1$ denote

$$\mathcal{A}_k = \operatorname{Proj}_{Y_k} \circ \mathcal{A}|_{L_{k+1}},$$

where $\operatorname{Proj}_{Y_k}$ is the projection of L onto Y_k agreed with V. Then, for any $1 \leq k \leq r$, $c_k = \dim Y_k$ is the number of the Jordan blocks of A whose sizes are not less than k, $L_k = Y_1 \oplus \ldots \oplus Y_k$, and, for $k = 1, \ldots, r-1$, $\mathcal{A}(Y_{k+1}) \subseteq L_k$ and $\mathcal{A}_k$ is an embedding of Y_{k+1} into Y_k.

Hence, in the basis V taken in the ordering $v_1^r, \ldots, v_{c_r}^r, v_1^{r-1}, \ldots, v_{c_{r-1}}^{r-1}, \ldots, v_1^1, \ldots, v_{c_1}^1$ A has the form

$$\begin{pmatrix} O_r & A_{r-1} & & & * \\ & O_{r-1} & A_{r-2} & & \\ & & O_{r-2} & \ddots & \\ & 0 & & \ddots & A_1 \\ & & & & O_1 \end{pmatrix}, \tag{2.2}$$

where O_k, $k = 1, \ldots, r$, is the $c_k \times c_k$ zero matrix and A_k, $k = 1, \ldots, r-1$, is the $c_{k+1} \times c_k$ matrix of rank c_{k+1} which corresponds to $\mathcal{A}_k$.

We will now describe some possible forms of the matrices A_k, $k = 1, \ldots, r-1$, which they can take under further admissible transformations. Let, for $k = 1, \ldots, r-1$, one have some integers $1 \leq d_1^k < \ldots < d_{c_{k+1}}^k \leq c_k$. Using Corollary 2.3, one can make an admissible change of the basis V_{r-1} of Y_{r-1} such that

$$\mathcal{A}_{r-1}(v_i^r) = v_{d_i^{r-1}}^{r-1} + \sum_{j < d_i^{r-1}} \alpha_{i,j}^{r-1} v_j^{r-1}, \; i = 1, \ldots, c_r, \; \alpha_{i,j}^{r-1} \in \mathcal{F},$$

that is the matrix A_{r-1} of $\mathcal{A}_{r-1}$ in this new basis is $(d_1^{r-1}, \ldots, d_{c_r}^{r-1})$-lower-triangular.

In the same way, one can make an admissible change of the basis V_{r-2} in Y_{r-2} such that the matrix A_{r-2} of $\mathcal{A}_{r-2}$ in the basis V_{r-1} of Y_{r-1} and the new basis V_{r-2} is

$(d_1^{r-2}, \ldots, d_{c_{r-1}}^{r-2})$-lower-triangular. Continuing this process, we obtain as a result a new basis $V = \{v_j^k,\ j = 1, \ldots, c_k,\ k = 1, \ldots, r\}$ such that

$$\mathcal{A}_k(v_i^{k+1}) = v_{d_i^k}^k + \sum_{j<d_i^k} \alpha_{i,j}^k v_j^k,\ i = 1, \ldots, c_{k+1},\ \alpha_{i,j}^k \in \mathcal{F},$$

for every $k = 1, \ldots, r-1$. We have proved the following theorem:

THEOREM 2.4 *Let A be a square nilpotent matrix, let $r \in \mathbb{N}$ be such that $A^r = 0$, $A^{r-1} \neq 0$, and let c_k for $k = 1, \ldots, r$ be the number of the Jordan blocks of A whose sizes are not less than k. Then, for any integers $1 \leq d_1^k < \ldots < d_{c_{k+1}}^k \leq c_k$, $k = 1, \ldots, r-1$, there exists an admissible transformation of A which has, up to a permutation of its rows and columns, the form (2.2) with every A_k, $k = 1, \ldots, r-1$, being $(d_1^k, \ldots, d_{c_{k+1}}^k)$-lower-triangular.*

3 GRAPHS OF MATRICES

Let $A = [\alpha_{m,l}]_{m,l=1}^n$ be an $n \times n$ matrix over $\mathcal{F}$; we now associate with A its *graph* $\Gamma(A)$. It is a directed graph whose set of vertices $V = \{v_1, \ldots, v_n\}$ is ordered; we denote by $n(v)$ the ordinal number in V of $v \in V$. Two vertices $u, w \in V$ of $\Gamma(A)$ are joined by a directed edge (or *an arrow*) $u \mapsto w$ if and only if $\alpha_{n(u),n(w)} \neq 0$; *the weight* of the arrow is $\alpha_{n(u),n(w)} \in \mathcal{F}$. Now the statement "there is an arrow $u \overset{\alpha}{\mapsto} w$" means just that $\alpha_{n(u),n(w)} = \alpha \neq 0$. We say that the arrow $u \mapsto w$ *passes from u to w*.

It is clear that the graph $\Gamma(A)$ is simply another description of the matrix A, and A can be easily reconstructed from $\Gamma(A)$. Moreover, we can consider the set V of the vertices of the graph $\Gamma(A)$ as an ordered basis of an n dimensional space. Now, admissible transformations of A lead to some transformations of $\Gamma(A)$; hence, we may regard changes of the basis V as transformations of $\Gamma(A)$.

In Section 4 we shall use admissible changes of basis of the form $u' = \alpha u + \beta w$, $u, w \in V$ with $n(u) > n(w)$. In terms of graph $\Gamma(A)$, this change deals with only arrows passing from and to u and w. Namely, if $v \overset{\lambda}{\mapsto} u$ and $v \overset{\mu}{\mapsto} w$ are the arrows passing from a vertex $v \in V$ to u and w, then in the new graph $\Gamma(A')$ obtained from $\Gamma(A)$ by the admissible change of basis $u' = \alpha u + \beta w$, there will be arrows $v \overset{\lambda/\alpha}{\mapsto} u'$ and $v \overset{\mu-\lambda\beta/\alpha}{\longmapsto} w$ (for convenience, in this paragraph we allow to arrows to have zero weights). If the arrows passing from u and w to a vertex $z \in V$ in $\Gamma(A)$ are $u \overset{\zeta}{\mapsto} z$ and $w \overset{\xi}{\mapsto} z$, in $\Gamma(A')$ there will be $u' \overset{\lambda\zeta+\mu\xi}{\longmapsto} z$ and $w \overset{\xi}{\mapsto} z$ (see Fig. 1).

The admissible corrections of A correspond in the graph $\Gamma(A)$ to changing the weight of arrows passing from vertices with greater numbers to vertices with smaller ones: for $u, w \in V$ satisfying $n(u) > n(w)$ we may change the weight α of the arrow $u \overset{\alpha}{\mapsto} w$, up to deleting or adding this arrow.

To make the geometric form of the graph $\Gamma(A)$ more clear, we embed its vertices into the Cartesian product $\mathbb{R}_x \times \mathbb{N}_y$. The first coordinate $x = x(v)$ of this coordinate system will be called *the position*, the second coordinate $y = y(v)$ will be called *the level*. We are completely free to make *geometric transformations* of $\Gamma(A)$, that is to change the coordinates of vertices of $\Gamma(A)$ (we will say also that we *move* the vertices); it is clear that geometric

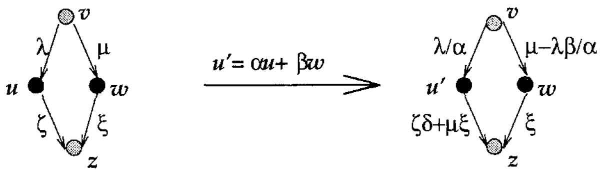

Figure 1: An admissible change of basis: $u' = \alpha u + \beta w$

transformations of a graph preserve the matrix corresponding to the graph. The embedded graph will be still called *the graph of* A and denoted by $\Gamma(A)$.

We will say that:

a vertex $u \in V$ *is on the level* k (or, simple saying, *of level* k) if $y(u) = k$;

a vertex $u \in V$ *is higher* (*lower*) than a vertex $w \in V$ if $y(u) > y(w)$ (respectively, $y(u) < y(w)$);

a vertex u *is on the left* (*on the right*) of w if $x(u) < x(w)$ (respectively, $x(u) > x(w)$);

a vertex u *is above* (*under*) w if $x(u) = x(w)$ and $y(u) = y(w) + 1$ (respectively, $y(u) = y(w) - 1$);

an arrow $u \mapsto w$ *goes down* if $y(u) > y(w)$; *goes directly down* if $y(u) > y(w)$ and $x(u) = x(w)$; *goes down-left* if $y(u) > y(w)$ and $x(u) \geq x(w)$.

If U is a subset of V, we denote by $\#U$ the number of elements of U, and by U_k the subset of U consisting of the vertices of level k: $U_k = \{u \in U : y(u) = k\}$. For $U, W \subseteq V$, *the arrows passing from* U *to* W are the arrows $u \mapsto w$ with $u \in U$, $w \in W$.

Let us give now some more technical definitions. We say that $\Gamma(A)$ is *downward* if all its arrows go down (Fig. 2). Note that such a graph contains no loops.

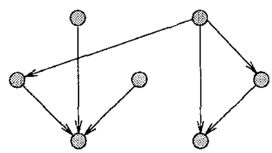

Figure 2: A downward graph

Let $U, W \subseteq V$. We say that $U \xrightarrow{A} W$ *is properly downward* if the arrows passing from U to W go down, for any $k \geq 1$ the arrows passing from U_{k+1} to W_k go down-left and for every $u \in U_{k+1}$ there is $w \in W_k$ under u with an arrow $u \mapsto w$.

We say that the graph $\Gamma(A)$ *is properly downward* if $V \xrightarrow{A} V$ is properly downward. In particular, the properly downward graph is downward and under every its vertex whose level is greater than one there is another vertex (Fig. 3).

The column is a subset of V consisting of all vertices having the same position. In the properly downward graph the levels of the elements of a column pass over all positive integers from 1 up to *the height* of the column. If b is a column, b_k denotes the element of b of level k.

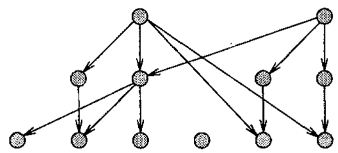

Figure 3: A properly downward graph

For $U \subseteq V$, let $\Xi_U = \{n(u) : u \in U\}$. For $U, W \subseteq V$, we denote

$$A(U,W) = [\alpha_{m,l}]_{\substack{m\in\Xi_U\\ l\in\Xi_W}};$$

$A(U,W)$ is a $(\#U)\times(\#W)$ submatrix of A.

Given $U, W \subseteq V$, we say that $U \xrightarrow{A} W$ *is nondegenerately downward* if

$$\operatorname{rank} A(U,W) = \#U.$$

The graph $\Gamma(A)$ *is nondegenerately downward* if it is downward and $V_{k+1} \xrightarrow{A} V_k$ is nondegenerately downward for any $k \geq 1$.

LEMMA **3.1** *Let $U \subseteq V_{k+1}$, $W \subseteq V_k$ for some $k \geq 1$. If $U \xrightarrow{A} W$ is properly downward then it is nondegenerately downward. In particular, if $\Gamma(A)$ is properly downward then it is nondegenerately downward.*

PROOF: Let W' be the set of the elements of W located under elements of U. Then, up to a permutation of its rows and columns, the square submatrix $A(U,W')$ of $A(U,W)$ is upper triangular with all nonzero elements on its main diagonal. Hence, the rank of $A(U,W')$ is equal to its size $\#U$. ∎

To start the proof of Theorem 2.1 we need two more statements, the first of which is trivial and the second being a simple corollary of the results of the previous section.

PROPOSITION **3.2** *If the graph $\Gamma(A)$ of a matrix A is downward then A is nilpotent. If $\Gamma(A)$ is nondegenerately downward then the number of Jordan blocks of A of size k is $(\#V_k) - (\#V_{k+1})$.*

PROPOSITION **3.3** *Let A be a nilpotent matrix, let the sizes of its Jordan blocks be $h_1, \ldots, h_t$ in an arbitrary ordering. Then there exist an admissible transformation A' of A and an embedding of $\Gamma(A')$ into $\mathbb{R}\times\mathbb{N}$ such that $\Gamma(A')$ is properly downward and the heights of its columns from the left to the right are $h_1, \ldots, h_t$.*

PROOF: Let $r = \max\{h_l,\ l = 1,\ldots,t\}$. For every $k = 1,\ldots,r$, let c_k be the number of the Jordan blocks of A whose sizes are not less than k, and, for $1 \leq k \leq r-1$, let $l_1^k < \ldots < l_{c_{k+1}}^k$ be all the integers satisfying $h_{l_i^k} \geq k+1$. We define a set of positive integers $\{d_i^k,\ i = 1,\ldots,c_{k+1},\ k = 1,\ldots,r-1\}$ by

$$d_i^k = \#\{l \in \mathbb{N} :\ 1 \leq l \leq l_i^k,\ h_l \geq k\}.$$

Using Theorem 2.4, we make an admissible change of basis in L in order to obtain a basis $\{v_j^k,\ j=1,\dots,c_k,\ k=1,\dots,r\}$ in which A has the form (2.2) such that every A_k, $k=1,\dots,r-1$, is $(d_1^k,\dots,d_{c_{k+1}}^k)$-lower-triangular. For $k=1,\dots,r$, we place $v_1^k,\dots,v_{c_k}^k$ onto the level k from the left to the right in such a way that $v_{d_i^k}^k$ is under v_i^{k+1}, $i=1,\dots,c_{k+1}$, $k=1,\dots,r-1$. The obtained graph satisfies the demands of the proposition. ■

4 PROOF OF RODMAN-SHALOM CONJECTURE

In Section 2 we gave several propositions, which allowed us to prove Theorem 2.1 for matrices of some special form. In Section 3 we associated with every nilpotent matrix an ordered directed multigraph, embedded in $\mathbb{R}\times\mathbb{N}$, which contains all the information on matrix A. This graph has an especially suitable structure for the matrices constructed in Section 2. We interpreted admissible corrections and admissible transformations of a matrix in terms of the corresponding graph (we will call such transformations *admissible corrections* and *admissible transformations of the graph*, respectively). Now we are ready to reformulate Theorem 2.1 in terms of graphs.

Let A be a nilpotent matrix and let $\Gamma(A)$ be a corresponding graph. According to Proposition 3.3, we may and will assume that, up to some admissible corrections of A, $\Gamma(A)$ is properly downward and that the heights of its columns from the left to the right are $q,h_1,\dots,h_t,p$.

THEOREM **4.1** *There exists a sequence of admissible corrections, admissible transformations and geometric transformations of* $\Gamma(A)$ *giving as a result a nondegenerately downward graph* $\Gamma(A')$ *such that the heights of its columns from the left to the right are* $q-1,h_1,\dots,h_t,p+1$ *(or* $h_1,\dots,h_t,p+1$*, in the case* $q=1$*).*

Let A' be the matrix corresponding to the graph $\Gamma(A')$ from the statement of Theorem 4.1. Then A' is obtained from A by a sequence of admissible corrections and admissible transformations. Furthemore, from Proposition 3.2 follows that A' is nilpotent with the sizes of Jordan blocks $q-1,h_1,\dots,h_t,p+1$ (or $h_1,\dots,h_t,p+1$ in the case $q=1$). It follows that Theorem 4.1 implies Theorem 2.1 and, so, gives an affirmative answer to Conjecture of Rodman and Shalom (Theorem 1.1).

In order to prove Theorem 4.1 we will first describe a geometric transformation of the graph $\Gamma(A)$ of a matrix A with a marked vertex s, which will be called *the insertion of* s *into the right column of* $\Gamma(A)$. The insertion can be done under the assumption that $\Gamma(A)$ satisfies some *conditions of insertion*, which will be formulated now.

Let $\Gamma(A)$ be a graph, let s be a vertex of $\Gamma(A)$ of level $k\in\mathbb{N}$ and let D be the set of all vertices of $\Gamma(A)$ excluding s. Let there be a vertex under every vertex $v\in D$, let a be the extreme right column of $\Gamma(A)$ and let $p\geq k-1$ be its height. Denote $H=D\setminus a$.

The conditions of insertion of s **into** a**.** If $\Gamma(A)$ does not contain the vertex a_k (that is, if $p=k-1$), then $\Gamma(A)$ is downward. If $\Gamma(A)$ does contain a_k (that is, if $p\geq k$), then $\Gamma(A)$ contains an arrow $a_k\mapsto s$, and the graph obtained from $\Gamma(A)$ by deleting this arrow is downward. In addition, $D_{i+1}\xrightarrow{A}D_i$ for $1\leq i\leq k-2$ and $H_k\bigcup\{s\}\xrightarrow{A}D_{k-1}$ are nondegenerately downward, and $D_{i+1}\xrightarrow{A}D_i$ for $i\geq k$ is properly downward.

In the assumption that the conditions of insertion are satisfied, *the insertion of* s *into the column* a is the following procedure: the level of each a_i, $i \geq k$, increases by 1 and s is placed into a on the level k (Fig. 4).

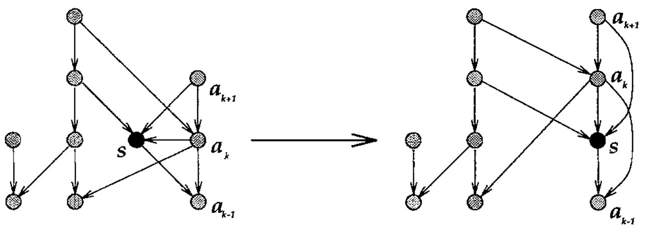

Figure 4: The insertion of s into the column a

LEMMA **4.2** *Let* $\Gamma(A)$ *satisfy the conditions of insertion. Then the graph* $\Gamma(A')$ *obtained from* $\Gamma(A)$ *by the insertion of* s *into* a *is nondegenerately downward.*

Note that the insertion of s into a is a geometric transformation of $\Gamma(A)$ and, hence, the matrix A' corresponding to $\Gamma(A')$ coincides with A.

PROOF: Let V' be the set of vertices of $\Gamma(A')$. We have $V_i' = D_i$ for $1 \leq i \leq k-1$ and $i > p+1$, $V_k' = H_k \bigcup\{s\}$, $V_i' = H_i \bigcup\{a_{i-1}\}$ for $k+1 \leq i \leq p+1$.

Since for $k \leq i \leq p$ there are no arrows passing from H_{k+1} to a_k (as $D_{i+1} \xrightarrow{A} D_i$ is properly downward), $\Gamma(A')$ is downward. We also have:

1. $V_{i+1}' \xrightarrow{A'} V_i'$ for $1 \leq i \leq k-2$ and $i > p+1$ is the same as $D_{i+1} \xrightarrow{A} D_i$ and, so, nondegenerately downward.
2. $V_k' \xrightarrow{A'} V_{k-1}'$ is nondegenerately downward by the assumptions.
3. $H_{k+1} \xrightarrow{A'} H_k$ is properly downward and, so, is nondegenerately downward by Lemma 3.1; if $p \geq k$, there are no arrows passing from a_k to H_k and there is an arrow $a_k \overset{\lambda}{\mapsto} s$. Therefore, the submatrix $A(H_{k+1} \bigcup\{a_k\}, H_k \bigcup\{s\})$ of matrix A has (up to a permutation of its rows and columns) the following form:

$$\begin{pmatrix} A(H_{k+1}, H_k) & * \\ 0 & \lambda \end{pmatrix}$$

and, so, its rank is equal to $\operatorname{rank} A(H_{k+1}, H_k) + 1 = \#\big(H_k \bigcup\{s\}\big)$. This shows that $V_{k+1}' = H_{k+1} \bigcup\{a_k\} \xrightarrow{A'} V_k' = H_k \bigcup\{s\}$ is nondegenerately downward.
4. The same argument shows that $V_{i+1}' = H_{i+1} \bigcup\{a_i\} \xrightarrow{A'} V_i' = H_i \bigcup\{a_{i-1}\}$, $k+1 \leq i \leq p+1$, and $V_{p+2}' = H_{p+2} \xrightarrow{A'} V_{p+1}' = H_{p+1} \bigcup\{a_p\}$ are nondegenerately downward.

Thus, $V_{i+1}' \xrightarrow{A'} V_i'$ is nondegenerately downward for any $i \geq 1$ and, so, $\Gamma(A')$ is nondegenerately downward. ■

PROOF OF THEOREM 4.1: We denote by a the right column of $\Gamma(A)$, by s the upper vertex of the left column of $\Gamma(A)$, $D = V \setminus \{s\}$ and $H = D \setminus a$.

We will describe now a finite algorithm, consisting of admissible transformations, admissible corrections and some geometric transformations of $\Gamma(A)$, giving as a result a new graph, satisfying the conditions of insertion of the vertex s into the right column a. We will move s to the right (in a level) and up (from a level to the next one). At the same time all other vertices of A, up to changes of notation, will stay at their former places. The final insertion will increment the height of the right column of $\Gamma(A)$ and the obtained graph will have the heights of columns $q-1, h_1, \ldots, h_t, p+1$ and be nondegenerately downward by Lemma 4.2. These imply the desired result.

At every step of the algorithm, the following conditions will hold:

I. $\Gamma(A)$ is downward;

II. s is on the left of a;

and, furtermore, when s is on the level k, $k \leq p$,

III. $D_{i+1} \xrightarrow{A} D_i$ is properly downward for any $i \geq 1$, $i \neq k-1$;

IV. $D_{k+1} \xrightarrow{A} D_k \bigcup \{s\}$ is properly downward;

V. $D_k \bigcup \{s\} \xrightarrow{A} D_{k-1}$ depends on a parameter $\lambda_k \in \mathcal{F}$ in such a way that

a) when $\lambda_k = 0$, $D_k \xrightarrow{A} D_{k-1}$ is properly downward and the arrows passing from s to D_{k-1} go down-left;

b) when $\lambda_k \neq 0$, $H_k \bigcup \{s\} \xrightarrow{A} D_{k-1}$ is nondegenerately downward.

At the beginning, s is on the level q and the conditions I–V hold independently of the value of a formal parameter λ_q.

Step 1. The movement of s to the right in a level.

Let s be on the level k for some $q \leq k \leq p$. Assume that on the right of s there is the vertex b_k of a column $b \neq a$. We have one of the following two cases:

1. The height of b is k, or the arrow $b_{k+1} \mapsto s$ does not exist. We move s to the right of b_k; this does not fail the conditions I–V (Fig. 5).

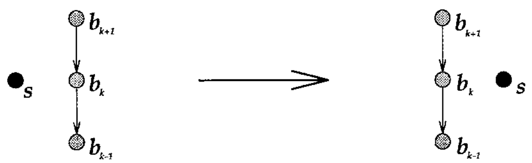

Figure 5: The movement of s to the right, case 1

2. There is an arrow $b_{k+1} \overset{\alpha}{\mapsto} s$. Since $D_{k+1} \xrightarrow{A} D_k$ is assumed to be properly downward, there is also some $b_{k+1} \overset{\beta}{\mapsto} b_k$.

a) Let $n(s) < n(b_k)$. We make an admissible change of basis: $b'_k = \beta b_k + \alpha s$. In the obtained graph $\Gamma(A')$ there does not already exist the arrow $b_{k+1} \mapsto s$ and there is $b_{k+1} \overset{1}{\mapsto} b'_k$. Move s to the right of b'_k (Fig. 6).

The conditions I–V hold for $\Gamma(A')$: it is downward, s is on the left of a. We will use the notation D_i, H_i, $i = 1, \ldots, r$, for subsets of $\Gamma(A')$ as well. Doing the above change of basis, we dealt with the vertices of the level k of $\Gamma(A)$ only and, so, $D_{i+1} \xrightarrow{A'} D_i$ for $i \neq k-1, k$ remain properly downward. All the arrows passing from D_{k+1} to $D_k \bigcup \{s\}$ go still down-left,

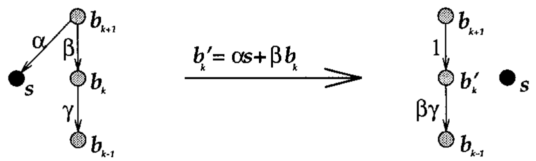

Figure 6: The movement of s to the right, case 2a

and for any $u \in D_{k+1}$ there is an arrow passing to D_k and going directly down, that is $D_{k+1}\xrightarrow{A'}D_k\bigcup\{s\}$ and $D_{k+1}\xrightarrow{A'}D_k$ are properly downward.

We have two cases, depending on the value of λ_k. If $D_k\xrightarrow{A}D_{k-1}$ is properly downward (in the case $\lambda_k = 0$), then all the arrows passing from b_k' and s to D_{k-1} in $\Gamma(A')$ hold going down-left, and there is an arrow, namely $b_k'\overset{\beta\gamma}{\mapsto}b_{k-1}$, passing from b_k' directly down, that is $D_k\xrightarrow{A'}D_{k-1}$ is properly downward. If $H_k\bigcup\{s\}\xrightarrow{A}D_{k-1}$ is nondegenerately downward (in the case $\lambda_k \neq 0$), then $H_k\bigcup\{s\}\xrightarrow{A'}D_{k-1}$ is nondegenerately downward as well, because our changes of basis dealt only with $H_k\bigcup\{s\}$ and, so, could not change the rank of the matrix $A(H_k\bigcup\{s\}, D_{k-1})$.

b) Let $n(s) > n(b_k)$. We make an admissible change of basis: $s'' = \beta b_k + \alpha s$, change notation: $b_k' = s''$, $s' = b_k$, and move s' and b_k' to the right in such a way that b_k' will be under b_{k+1} (Fig. 7). Using the argument involved in **a)**, we see that the conditions I–V hold for the obtained graph.

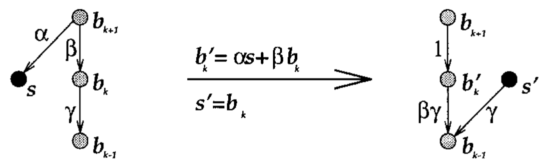

Figure 7: The movement of s to the right, case 2b

We repeat Step 1 till there are not any vertices of level k between s and a_k. Let us keep notation $\Gamma(A)$ and A for the obtained graph and the corresponding matrix.

Step 2. The insertion of s into a, or moving of s to the next level.
We have one of the following three cases:

1. $n(a_k) > n(s)$.
We add a new arrow $a_k\overset{\varepsilon}{\mapsto}s$, $\varepsilon \in \mathcal{F}$, $\varepsilon \neq 0$, (this is an admissible correction) and put $\lambda_k \neq 0$. The conditions of insertion of s into a hold; the insertion finishes the proof.

2. $n(a_k) < n(s)$ and either $p = k$, or there is an arrow $a_{k+1}\mapsto s$.
We lift s onto the level $k+1$, add an arrow $s\overset{\varepsilon}{\mapsto}a_k$, $\varepsilon \in \mathcal{F}$, $\varepsilon \neq 0$, (this is an admissible correction) and put $\lambda_k = 0$. In the obtained graph $\Gamma(A')$, $D_k\xrightarrow{A'}D_{k-1}$ is properly downward and the only arrow passing from s to D_k is $s\overset{\varepsilon}{\mapsto}a_k$. The submatrix $A'(H_{k+1}\bigcup\{s\}, H_k\bigcup\{a_k\})$ of the matrix A' corresponding to $\Gamma(A')$ has, up to a permutation of its rows and columns,

the form

$$\begin{pmatrix} A(H_{k+1}, H_k) & * \\ 0 & \varepsilon \end{pmatrix}$$

and, since $H_{k+1} \xrightarrow{A'} H_k$ is properly downward, $A(H_{k+1}, H_k)$ is of full rank, rank $A(H_{k+1}, H_k) = \#H_{k+1}$. Hence, rank $A'(H_{k+1} \bigcup\{s\}, H_k \bigcup\{a_k\}) = \#H_{k+1} + 1$ and, so, $H_{k+1} \bigcup\{s\} \xrightarrow{A'} D_k = H_k \bigcup\{a_k\}$ is nondegenerately downward. So, we may insert s into a and finish the proof.

3. $n(a_k) < n(s)$, $p > k$ and the arrow $a_{k+1} \mapsto s$ does not exist.

We put $\lambda_k = 0$, add the arrow $s \overset{\lambda_{k+1}}{\mapsto} a_k$ (the value of λ_{k+1} will be determinated during the further steps of the algorithm), and move s onto the level $k+1$ to the left of D_{k+1}. The conditions I–V hold: the obtained graph $\Gamma(A')$ remains downward, $D_k \xrightarrow{A'} D_{k-1}$ is properly downward. In case $\lambda_{k+1} = 0$ there are no arrows passing from s to D_k. In case $\lambda_{k+1} \neq 0$ the only arrow passing from s to D_k is $s \overset{\lambda_{k+1}}{\mapsto} a_k$, so, the matrix $A'(H_{k+1} \bigcup\{s\}, D_k)$ has the form

$$\begin{pmatrix} A(H_{k+1}, H_k) & * \\ 0 & \lambda_{k+1} \end{pmatrix}$$

and, hence, $H_{k+1} \bigcup\{s\} \xrightarrow{A} D_k$ is nondegenerately downward. Now, we restore the old notation $\Gamma(A)$ and A for $\Gamma(A')$ and A' respectively, and repeat the procedure of movement to the right, Step 1 of the algorithm, for s being on the level $k+1$. ∎

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