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FIBER BUNDLES WITH DEGENERATIONS AND THEIR APPLICATIONS TO COMPUTING FUNDAMENTAL GROUPS

ABSTRACT. An analogue of the initial segment of the exact sequence of the homotopy groups of a fiber-bundle is written out for the map which is fiberbundle over some large subset of the base and has local sections over all points of the base. As an application, presentations of the fundamental groups of the complements of the arrangements of complexified reflection hyperplanes of the root systems D_n and B_n in terms of generators and relations are computed.

0. INTRODUCTION

0.1. We will compute presentations of the pure braid groups corresponding to the Weyl groups of the root systems D_n, B_n, that is the fundamental groups of the complements in $\mathbf{C}^n$ of the arrangements of the complexified reflecting hyperplanes of the Weyl groups of the corresponding root systems. We could use the Zariski–van Kampen Theorem for this purpose, but we have preferred to obtain a more general theorem. There is a series of papers of Falk and Randell ([F], [F, R(1)], [F, R(2)]) devoted to the fundamental groups of the complement in $\mathbf{C}^n$ of hyperplane arrangements of fiber type. In the fiber type case, one has a tower of locally trivial fiber bundles

$$\mathbf{C}^n\backslash S_n \xrightarrow{p_n} \mathbf{C}^{n-1}\backslash S_{n-1} \xrightarrow{p_{n-1}} \cdots \xrightarrow{p_2} \mathbf{C}\backslash\{0\}$$

where each S_k is a union of hyperplanes, and the maps p_k are induced by a linear projection $\mathbf{C}^k \to \mathbf{C}^{k-1}$. Fibers of p_k are the complex affine line with finitely many punctures $\mathbf{C}^1\backslash\{a_1, \dots, a_{r_k}\}$; it is easy to see, that the p_k admit global sections $s_k: \mathbf{C}^{k-1}\backslash S_{k-1} \to \mathbf{C}^k\backslash S_k$, $p_k \circ s_k = \mathrm{id}$. This structure gives rise to a series of split exact sequences of fundamental groups:

$$(1) \qquad 1 \to \mathscr{F}_{r_k} \to \pi_1(\mathbf{C}^k\backslash S_k) \rightleftarrows \pi_1(\mathbf{C}^{k-1}\backslash S_{k-1}) \to 1,$$

where $\mathscr{F}_{r_k}$ is free of rank r_k, which leads us to an inductive treatment of $\pi_1(\mathbf{C}^n\backslash S_n)$ as a multigrade semidirect product of free groups.

In this context, the following natural question arises: to which extent can one generalize this approach to the case when the maps p_k are not locally

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trivial fiber bundles, that is in certain fibers $p_k^{-1}(t)$, $t \in \mathbf{C}^{k-1} \backslash S_{k-1}$, some of the punctures a_i stick together?

The classical arrangement of the complexified reflecting hyperplanes of the Weyl group of the root system D_n with the natural projection forgetting the last coordinate is such an example (see 0.7), whereas the root systems A_n, B_n provide fiber type examples.

In the work we give an analogue of exact sequence (1) for a topological locally non-trivial bundle. We are going to describe the result in the case of the bundle induced by a morphism of smooth manifolds (for the case of topological spaces, see Section 1).

0.2. Let $p: E \to B$ be a surjective morphism of connected smooth manifolds. Suppose that there exists an open subset $V \subset B$ such that $p|_{p^{-1}(V)}$ is a locally trivial bundle with a connected fiber and $B \backslash V$; $E \backslash p^{-1}(V)$ are unions of locally closed submanifolds of B, E, respectively, of codimension $\geqslant 2$. Suppose in addition that p has local sections over every point of B. For every component of $B \backslash V$ of codimension 2 choose a small loop around it and join this loop with a reference point by all possible continuous paths in V considered modulo homotopy. Lift the obtained set of generators of $\ker(\pi_1(V) \to \pi_1(B))$ into U so that every component of $E \backslash p^{-1}(V)$ of codimension 2 would get its loop. Denote the obtained subset of $\pi_1(p^{-1}(V))$ by $\mathring{H}$.

PROPOSITION. *There exists the exact sequence of groups*

$$(2) \qquad 1 \to R \to \pi_1(F)/\mathrm{Im}(\pi_2(V)) \to \pi_1(E) \to \pi_1(B) \to 1,$$

where R is the normal subgroup which is generated by the following two sets of elements:

(i) *the elements of the form $x_i^{-1} h_j^{-1} x_i h_j$, the x_i running over any set of generators of $\pi_1(F)/\mathrm{Im}(\pi_2(V))$ and h_j over the set $\mathring{H}$;*
(ii) *the powers of elements of $\mathring{H}$ projected trivially into $\pi_1(V)$, that is the elements of $\ker((p|_{p^{-1}(V)})_*: (\mathring{H}) \to \pi_1(V))$, where $(\mathring{H})$ is the subgroup of $\pi_1(p^{-1}(V))$ generated by $\mathring{H}$.*

0.3. This proposition is a rather technical result and it is not easy to use it (though a simple example is described in 1.12). We provide a proof for a general version (Theorem 1.4) of the Zariski–van Kampen Theorem (see [Zar], [vKam]), which implies, in particular, the following result in the case of manifolds:

PROPOSITION. *Under the assumptions of* (0.2), *suppose that the map p admits a global section which intersects every component of $E \backslash p^{-1}(V)$ of* *codimension 2. Then (0.2) is reduced to*

$$(3) \qquad 1 \to R \to \pi_1(F) \to \pi_1(E) \rightleftarrows \pi_1(B) \to 1,$$

where R is generated as a normal subgroup of $\pi_1(F)$ by the elements of the form $x_i^{-1}h_j^{-1}x_ih_j$, the x_i running over any set of generators of $\pi_1(F)$ and h_j running over any set of generators of $\ker(\pi_1(V) \to \pi_1(B))$.

In other words, only those loops in $\pi_1(F)$ coincide in $\pi_1(E)$, which can be obtained from one another by the action of the monodromies of the elements of $\pi_1(V)$ vanishing in $\pi_1(B)$.

0.4. Note, that the conditions imposed on the morphism p, except the condition of existence of sections, are naturally satisfied for the morphisms of smooth complex algebraic varieties. Thus, as an application, we obtain a proof of the following known fact:

PROPOSITION. *Let $p: S \to \mathbf{C}^{n-1}$ be a projection of an algebraic hypersurface S in $\mathbf{C}^n$ onto a generic hyperplane, $K \in \mathbf{C}^{n-1}$ the divisor consisting of the singularities of the projection p in codimension 1 (including both the singular points of p and the ramification of p). Then the fundamental group $\pi_1(\mathbf{C}^n \backslash S)$ has a presentation in terms of generators and relations with* $\deg S$ *generators and* $(\deg K)(\deg S)$ *basic relators. The relators are defined modulo conjugation of the generators only by topological types of the singularities of the projection p.*

0.5. We prove also the following proposition (see [K]):

PROPOSITION. *Let S', S'' be two hypersurfaces in $\mathbf{C}^n$ without common irreducible components. Assume that S', S'' are in generic position in codimension 1, that is the sets $(\bar{S}'\backslash S') \cup (\bar{S}''\backslash S'') \cup \operatorname{sing}(S') \cup \operatorname{sing}(S'')$ and $\bar{S}' \cap \bar{S}''$ do not have any common components of codimension 1. (Here $\bar{S}'$, $\bar{S}''$ denote the closure of S', S'' respectively in $\mathbf{P}^n$). Then*

$$\pi_1(\mathbf{C}^n\backslash(S' \cup S'')) = \pi_1(\mathbf{C}^n\backslash S') \times \pi_1(\mathbf{C}^n\backslash S'').$$

0.6. As a particular case of Proposition 0.4 we have

PROPOSITION. *Let $\{L_i,\ i = 1,\dots,d\}$ be a finite set of hyperplanes in $\mathbf{C}^n$; $\{M_j, j = 1,\dots,k\}$ be the set of the planes of codimension 2 which are their intersections. Then $\pi_1(\mathbf{C}^1\backslash\bigcup_1^d L_i)$ has a presentation of the form*

$$\left\{a_i,\ 1 \leqslant i \leqslant d \,\middle|\, \left[\prod_{j\in I_q} \overset{q}{\tilde{a}}_j,\ \overset{q}{\tilde{a}}_i\right],\ i \in \mathrm{I}_q,\ 1 \leqslant q \leqslant k\right\}$$

where $\mathrm{I}_q = \{i \leqslant d \mid M_q \subset L_i\}$ and $\overset{q}{\tilde{a}}_i$ is an appropriate conjugate of a_i.

0.7. Exact sequences (2) and (3) are used for the computation of concrete examples. Example 3.4 is that of the complement of the arrangement of complexified reflecting hyperplanes of the Weyl group of the root system D_n: $E_n = \mathbf{C}^n \backslash L_n$, $L_n = \bigcup_{1 \leqslant i < j \leqslant n} \{x_i \pm x_j = 0\}$. The projection $p: E_n \to E_{n-1}$ defined by $p(x_1, \dots, x_{n-1}, x_n) = (x_1, \dots, x_{n-1})$ has a non-constant fiber $p(x_1, \dots, x_{n-1}) = \mathbf{C}^1 \backslash \{\pm x_1, \dots, \pm x_{n-1}\}$. Thus, the fundamental group of the generic fiber is the free group $\mathscr{F}_{2n-2}$ with $2n-2$ generators $a_1, \dots, a_{n-1}$, $b_1, \dots, b_{n-1}$ where a_i, respectively b_i, corresponds to a loop around the puncture x_i, respectively $-x_i$. In [Mar] it is proved (from a purely algebraic reasoning), that each degeneration of the fiber occurring at $y_i = 0$ for some $i = 1, \dots, n-1$, provides the relation of commutation $a_i b_i a_i^{-1} b_i^{-1}$ in the group $\mathscr{F}_{2n-2}/R = \ker p_*$. We prove here that these are not all basic relations and provide an infinite complete generating set of the group of relations R as a normal subgroup of $\mathscr{F}_{2n-2}$ for $n = 3$.

Exact sequence (3) is then applied to computing a presentation in terms of generators and relations of the fundamental groups of the complements of the unions of the complexified reflecting hyperplanes of the root systems D_n and B_n, that is of the corresponding generalized pure braid groups. For A_n such presentation is known, this is the classical Burau presentation (see, e.g., [M,K,S]); we obtain another one. For D_n there is another simpler presentation, obtained in [Mar] by algebraic means, and we could not establish an isomorphism of our presentation with that of [Mar].

0.8. Finally, we investigate the case when the hyperplanes of the arrangement are defined by real equations (*real arrangement*) to obtain the following result.

Call the broken line $P_0 P_1 \dots P_r$ consisting of segments of straight lines in $\mathbf{R}^n$ *monotone* with respect to a real arrangement S if for every hyperplane of S, defined by an equation $L(x) = 0$, we have $L(P_i) \neq 0$, $L(P_i - P_{i-1}) \neq 0 \; \forall i$ and $\operatorname{sgn} L(P_i - P_{i-1})$ is independent of i.

PROPOSITION. *Let us join a reference point $O \in \mathbf{R}^n \backslash S$ with every hyperplane of a real hyperplane arrangement S by broken lines which are monotonic with respect to S: they end in a small neighborhood of a point on one of these hyperplanes away from the intersections with the remaining ones. For each of these broken lines define a loop as a path which goes along the broken line, bypassing the points of intersection with S in the counterclockwise direction by small arcs in the complexification of the corresponding rectilinear segments of the broken line; then passes around the target hyperplane and returns along the same path.*

Then, thus obtained set of loops generates $\pi_1(\mathbf{C}^n \backslash (\mathbf{C}S))$.

0.9. In Section 1 we formulate the topological conditions for the hypotheses of Theorem 1.8 and prove it. As these conditions and the proof look too technical and obscure, we first investigate a more clear case assuming the existence of a global section; so, we start with Theorem 1.4, which is formally a corollary of Theorem 1.8. Then we apply these theorems to prove Propositions 0.2 and 0.3.

In Section 2 we consider the case in which the bundle space E is the complement of an affine algebraic subvariety of $\mathbf{C}^n$. We prove Propositions 0.4 and 0.5, give a prescription for the practical computation of the fundamental group in this case and investigate some examples.

Section 3 is devoted to the arrangements of hyperplanes and, particularly, to the presentation of the fundamental groups of the complements of the D_n and B_n arrangements. We prove also Proposition 0.8 here.

0.10. *Notation*

$A \times B$, the direct product of the groups A, B
$A \rtimes B$, the semidirect product of groups
$a^b = b^{-1}ab$
$[a, b] = a^{-1}b^{-1}ab$, the commutator of elements a, b of a group
(A), the subgroup generated by the elements of the subset A
$(A)^G$, the normal closure of (A) in a bigger group G
$[A, B]$, the commutator of two subsets A and B, i.e. the subgroup generated by the elements $[a, b]$, $a \in A$, $b \in B$
$\pi_1(X)$, the fundamental group of a topological space X
$\pi_n(X)$, its nth homotopical group
φ_*, the mapping corresponding to a continuous mapping φ of topological spaces and acting on the corresponding fundamental groups
$\mathring{D}$, the set of the inner points of a subset D of a topological space
∂D, the set of its boundary points
(a, b), the relation $ab = ba$ in a group
$(a_1, a_2, \ldots, a_k)$, the $k-1$ relations $a_1 a_2 \cdots a_k = a_2 \cdots a_k a_1 = \cdots = a_k a_1 \cdots a_{k-1}$
$\mathbf{1}_G$, the identity element of a group G.

1. Topological theorems

1.1. Let $p: E \to B$ be a continuous map of topological spaces. Suppose there exists a pathwise connected topological subspace $V \subset B$ such that $p|_{p^{-1}(V)}$ is a bundle in the sense of Serre. Points of V and fibers over them will be called generic; points of the complement $K = B \backslash V$ and fibers over them will be called singular. Denote $U = p^{-1}(V)$, $L = p^{-1}(K)$. Suppose we have fixed such a subspace V and introduce the following conditions on p:

(ε) p has a continuous section $s: B \to E$, $p \circ s = \mathrm{id}_B$.
(α) The generic fibers of p are pathwise connected.
(β) Every path in E with the endpoints in U can be moved off the singular fibers L by a continuous homotopy constant on its ends.
(γ) Every surface in E can be made 'transversal' to L at points of the section $s(B)$. That is, for every continuous map of the two-dimensional disk $\varphi: D \to E$ such that $\varphi(\partial D) \subset U$, there exists a continuous map $\psi: D \to E$ with the following properties:

1. $\psi|_{\partial D} = \varphi|_{\partial D}$
2. there exist a finite number of subdisks $d_i: D \hookrightarrow D$, $\mathrm{Im}(d_i) = D_i$, such that $D_i \cap \partial D = \varnothing$, $D_i \cap D_j = \varnothing$ for all $i \neq j$, and
3. $\psi^{-1}(L) \subset \bigcup_i \mathring{D}_i$, where $\mathring{D}_i$ denotes the interior of D_i,
4. $\psi(D_i) \subset s(B)$ for any i.

Our first goal is to find an analogue of an initial segment of the exact sequence of the homotopy groups of a bundle. Choose reference points $o' \in V$, $o'' = s(o')$ and define the fiber $F = p^{-1}(o')$ with its embeddings $f = i_F: F \hookrightarrow U$, $\tilde{f} = i_U \circ i_F: F \hookrightarrow E$.

Consider the commutative diagram of the fundamental groups:

$$\begin{tikzcd} 1 \arrow[r] & \pi_1(F) \arrow[r, "f_*"] \arrow[dr, "\tilde{f}_*"'] & \pi_1(U) \arrow[r, "p_*"] \arrow[d, "i_*"] & \pi_1(V) \arrow[r] \arrow[d, "j_*"] & 1 \\ & & \pi_1(E) \arrow[r, "p_*"] & \pi_1(B), & \end{tikzcd} \tag{4}$$

where $i_* = i_{U*}, j_* = i_{V*}$.

Condition (ε) gives us the existence of a splitting mapping s_*, condition (α) gives the exact sequence of the bundle; together they imply the exactness of the first row of the diagram.

Condition (β) means that $i_*: \pi_1(U) \to \pi_1(E)$ is surjective.

Let us now understand the algebraic meaning of condition (γ). Let $\alpha \in \ker(i_*)$, that is a loop which represents α is contractible in E. The contraction process is a continuous mapping of a two-dimensional disk with a distinguished point into E:

$$\varphi: D \to E,\ o \in \partial D,\ \varphi(o) = o'',\ \varphi_*(\partial D) = \alpha.$$

Using (γ), we will suppose that φ is transversal (in the sense described in the statement of condition (γ)) to the singular fibers at points of the section.

Join the point o with the subdisks D_i by paths w_i, whch do not meet $\partial D \backslash \{o\}$ and one another and renumber the pairs (D_i, w_i) in the order in which the paths w_i occur in a small neighborhood of the starting point o when moving in the counterclockwise direction (see Figure 1). Define the loops $\alpha_i = \varphi_*(w_i \circ \partial D_i \circ w_i^{-1}) \in \pi_1(U)$, where all ∂D_i and ∂D are the simple loops oriented in the counterclockwise direction.

The loop $(\Pi_i w_i \circ \partial D_i \circ w_i^{-1}) \circ (\partial D)^{-1}$ can be contracted in D without touching the interior parts of all the D_i. Hence $\Pi_i \alpha_i = \alpha$ in $\pi_1(U)$.

Look at α_i for some i. Let $\beta_i = \varphi_*(w_i)$, $\delta_i = \varphi_*(\partial D_i)$, $\gamma_i = s_* p_*(\beta_i)$. Then

$$\alpha_i = \beta_i \delta_i \beta_i^{-1} = (\gamma_i \beta_i^{-1})^{-1} (\gamma_i \delta_i \gamma_i^{-1})(\gamma_i \beta_i^{-1}).$$

The loop $\gamma_i \beta_i^{-1}$ lies in U, and its image $p_*(\gamma_i \beta_i^{-1})$ in V can be contracted along itself, so $p_*(\gamma_i \beta_i^{-1}) = \mathbf{1}_{\pi_1(V)}$ and $\gamma_i \beta_i^{-1}$ is represented by the loop $f_*(x_i)$ for some $x_i \in \pi_1(F)$.

The loop $\gamma_i \delta_i \gamma_i^{-1}$ lies entirely in $s(V)$ and can be contracted in $s(B)$; therefore, it is represented by $s_*(h_i)$ for some $h_i \in \ker(\pi_1(V) \xrightarrow{i} \pi_1(B))$. So, for every $\alpha \in \ker(i_*)$ we have two sets $\{x_i \in F\}$, $\{h_i \in \ker(j_*)\}$, such that $\alpha = \Pi_i f_*(x_i)^{-1} s_*(h_i) f_*(x_i)$.

Thus we have arrived at the following algebraic situation (we replace π_1 of F, U, V, E, B by abstract groups denoted by F, U, V, E, B respectively):

$$\begin{tikzcd}
& & 1 \arrow[d] & 1 \arrow[d] & \\
& & G \arrow[r, shift left, "p"] \arrow[d] & H \arrow[l, shift left, "s"] \arrow[d] & \\
1 \arrow[r] & F \arrow[r, "f"] \arrow[dr, "\tilde{f}"'] & U \arrow[r, shift left, "p"] \arrow[d, "i"] & V \arrow[l, shift left, "s"] \arrow[r] \arrow[d, "j"] & 1 \\
& & E \arrow[r, shift left, "p"] \arrow[d] & B \arrow[l, shift left, "s"] & \\
& & 1 & &
\end{tikzcd} \tag{5}$$

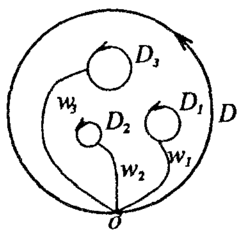

Fig. 1.

This is a commutative diagram of group homomorphisms in which the third line and both columns are exact (here $G = \ker(i)$, $H = \ker(j)$).

In addition, the normal closure of $s(H)$ in U contains (and, therefore, coincides with) G:

$$G = (s(H))^U$$

(formally, it is even weaker than (γ)).

We will identify F, H, V and G with their images in U.

1.2. LEMMA. *Under the above assumptions, the sequence*

$$1 \to R := ([F, H]) \to F \xrightarrow{\tilde{f}} E \xrightarrow{p} B \to 1$$

is exact.

Proof. 1. $p: E \to B$ is surjective, as $p \circ s(B) = B$.

2. (a) For any $x \in F$ $p \circ \tilde{f}(x) = p \circ i(x) = j \circ p(x) = \mathbf{1}_B$.

(b) Let $\alpha \in E$, $p(\alpha) = \mathbf{1}_B$. Then $\exists \tilde{\alpha} \in U$, $i(\tilde{\alpha}) = \alpha \Rightarrow p \circ i(\tilde{\alpha}) = j \circ p(\tilde{\alpha}) = \mathbf{1}_B$ $\Rightarrow h := p(\tilde{\alpha}) \in H$, $\exists x \in F$: $\tilde{\alpha} = hx$. But $H \subset G$, as $i(H) = s \circ j(H) = \mathbf{1}_E$, so $\alpha = i(\tilde{\alpha}) = i(h) \circ i(x) = \tilde{f}(x)$.

3. $R := ([F, H])$ is the subgroup of U generated by the elements of the form $[x, h]$ where $x \in F$, $h \in H$. As F is normal in U, $R \subseteq F$. So we have to prove only that $R = F \cap G$.

(a) $H \subseteq G$, G is normal in $U \Rightarrow (x^{-1}h^{-1}x)h \in G$, i.e. $R \subseteq G$

(b) Let $\alpha \in G \cap F$. As G is contained in $(H)^U$, $\alpha = \Pi_{i=1}^k \lambda_i^{-1}\tilde{h}_i\lambda_i$, where $\tilde{h}_i \in H$, $\lambda_i \in U$.

Denote $\alpha_i = \lambda_i^{-1}\tilde{h}_i\lambda_i$ and represent λ_i for every i in the form $\lambda_i = b_i x_i$, $x_i \in F$, $b_i \in V$. Then $\alpha_i = x_i^{-1}b_i^{-1}\tilde{h}_i b_i x_i = x_i^{-1}h_i^{-1}x_i$, where $h_i = b_i^{-1}\tilde{h}_i^{-1}b_i \in H$, as H is normal in V, and $\alpha_i = [x_i, h_i]h_i^{-1}$.

But $\forall g, h \in H, x \in F$ we have $g^{-1}[x, h] = [x^g, h^g]g^{-1}$, where $x^g = g^{-1}xg \in F$, $h^g = g^{-1}hg \in H$. Therefore

$$\alpha = \prod_{i=1}^{k} ([x_i, h_i]h_i^{-1}) = \prod_{i=1}^{k} [x_i', h_i'] \prod_{i=1}^{k} h_i^{-1},$$

where

$$x_i' = \left(\prod_{j=i-1}^{1} h_j\right)^{-1} x_i \left(\prod_{j=i-1}^{1} h_j\right) \in F, h_i' = \left(\prod_{j=i-1}^{1} h_j\right)^{-1} h_i \left(\prod_{j=i-1}^{1} h_j\right) \in H.$$

But $\alpha \in F$, so $\Pi_{i=1}^k h_i^{-1} \in F$ and, consequently, $\Pi_{i=1}^k h_i^{-1} = \mathbf{1}_V$ and $\alpha = \Pi_{i=1}^k [x_i', h_i'] \in R$.

1.3. Let us view the elements $[x, h]$, generating R, as relations in F of the form $x^h = x$, and try to reduce their number.

LEMMA. *Let* $\mathring{H} \subseteq H$ *be a subset generating* H, $\mathring{F} \subseteq F$ *be a subset generating* F. *Then* $R = ([\mathring{F}, \mathring{H}])^F$.

Proof.

$$[xy, h] = ([y, h]^y)[y, h]$$
$$[x, gh] = [x, g][x^g, h] \quad \text{for all } x, y \in F,\ h, g \in H.$$

1.4. Let us return to the bundle. The group $\pi_1(V)$ acts on $\pi_1(F)$ by monodromies:

$$x \in \pi_1(F),\ b \in \pi_1(V) \Rightarrow x^b = s_*(b)^{-1} x s_*(b).$$

The elements of $\pi_1(V)$ which are equal to $\mathbf{1}$ in $\pi_1(B)$ act on $\pi_1(F)$ in such a way that the restriction of their action onto $\pi_1(E)$ is trivial. We have proved that all the additional relations which should be imposed on $\pi_1(F)$ in order to get the image of $\pi_1(F)$ in $\pi_1(E)$ are obtained in this way. Hence, we have proved the following theorem.

THEOREM. *Under the conditions* (ε), (α), (β), (γ) *the sequence*

$$1 \to R \hookrightarrow \pi_1(F) \to \pi_1(E) \rightleftarrows \pi_1(B) \to 1$$

is split exact. Here R *is the normal subgroup of* $\pi_1(F)$ *generated by the elements of the form* $x_i^{-1} x_i^{h_j}$, x_i *running over a set of generators of* $\pi_1(F)$, *and* h_j *over a set of generators of the group* $H = \ker(\pi_1(V) \to \pi_1(B))$.

1.5. REMARK. The topological conditions (β)–(γ) are naturally satisfied when the map p is a morphism of smooth manifolds and the singular set L is a union of submanifolds of codimension $\geqslant 2$ (see 1.11) and, so, we will call this situation 'the bundle with degenerations in codimension 2'.

This notion can be reformulated for the case 'the bundle with degeneration in codimension n'. All the constructions are carried over almost without changes. As the result we obtain that the sequence

$$R \hookrightarrow \pi_{n-1}(F) \to \pi_{n-1}(E) \rightleftarrows \pi_{n-1}(B) \to \pi_{n-2}(F) \to \cdots \to 1$$

is split exact, where $R = ([\pi_1(F), \ker(\pi_{n-1}(V) \to \pi_{n-1}(B))])^{\pi_{n-1}(F)}$, i.e. R is the normal closure in $\pi_{n-1}(F)$ of the set of the elements of the form $h_j^{-1} h_j^{x_i}$, where $\{x_i\}$ generates $\pi_1(F)$, $\{h_j\}$ generates $\ker(\pi_{n-1}(V) \to \pi_{n-1}(B))$, and h^x is the result of the natural action of π_1 on π_{n-1}.

1.6. Now let us try to refuse from the assumption of the existence of a global section. To start with, we note, that we used only an algebraic section, i.e. a splitting homomorphism in the diagram (5). This section exists automatically, for example, in the case in which H and $\pi_1(B)$ are free groups; but it has also to satisfy the condition $(s(H))^U = G$. If not, change our conditions:

(β') The loops in E, and also in B, can be moved off the singular fibers/points.

(γ') A set of continuous mappings of the two-dimensional disk is singled out: $\Sigma_E = \{\varphi_\sigma : D \to E\}$, $\varphi_\sigma(\partial D) \subset U \,\forall\sigma$, so that for every continuous mapping $\varphi: D \to E$ there exists such continuous mapping $\psi: D \to E$ that $\varphi|_{\partial D} = \psi \mid \partial D$ and there exists a finite set of subdisks $\{d_i : D \hookrightarrow D\}$ which do not meet ∂D and one another and such that $\psi^{-1}(L) \subset \bigcup_i \mathring{D}_i$, where $D_i = \mathrm{Im}(d_i)$, and $\psi \circ d_i \in \Sigma_E \,\forall i$.

(δ') There exists a set of continuous mappings $\Sigma_B = \{\varphi_\delta : D \to B\}$, $\varphi_\delta(\partial D) \subset V \,\forall\delta$, such that the following properties hold:

1. for every continuous $\varphi: D \to B$, there exists a continuous mapping $\psi: D \to B$ such that $\varphi|_{\partial D} = \psi|_{\partial D}$ and $\psi^{-1}(K) \subset \bigcup_i (\mathring{D}_i)$, where $D_i = \mathrm{Im}(d_i)$, $\{d_i : D \to D\}$ being the finite set of subdisks which do not meet ∂D and one another, and $\psi \circ d_i \in \Sigma_B$;
2. for any $\varphi_\delta \in \Sigma_B$ a mapping $\tilde{\varphi}_\delta : D \to E$ exists, such that $p \circ \tilde{\varphi}_\delta|_{\partial D} = \varphi_\delta|_{\partial D}$.

The algebraic meaning of these conditions is:

(β') The homomorphisms $\pi_1(V) \to \pi_1(B)$ and $\pi_1(U) \to \pi_1(E)$ are surjective.

(γ') For every mapping $\varphi_\sigma \in \Sigma_E$ join some point on the boundary $p \circ \varphi_\sigma(\partial D)$ with the reference point o' by all possible paths in V modulo homotopy. Lift every such path into U so that it would join the boundary $\varphi_\sigma(\partial D)$ with the point o'' in an arbitrary way; mind the fact that the generic fiber is pathwise connected. Consider the set of the loops obtained by passing from o'' to $\varphi_\sigma(\partial D)$, then along $\varphi_\sigma(\partial D)$ and back along the same path for all $\varphi_\sigma \in \Sigma_E$ and both orientations of ∂D. Denote the obtained subset of $\pi_1(U)$ by $\mathring{H}$. The paths of $\mathring{H}$ can be contracted in E and represent, therefore, elements of

$$G := \ker(\pi_1(U) \to \pi_1(E)).$$

On the other hand, from considerations similar to those in (γ), we will obtain that every loop of $\pi_1(U)$ which can be contracted in E, that is every element $g \in G$, can be represented in the form $g = \Pi_i x_i^{-1} h_i x_i$, where $x_i \in \pi_1(F)$, $h_i \in \mathring{H}$.

(δ') There exists a set of the generators in $H := \ker(\pi_1(V) \to \pi_1(B))$ which lift to G; it means that the mapping $G \to H$ is surjective.

Thus, the algebraic situation under the assumptions that the conditions (α), (β'), (γ'), (δ') are satisfied, is the following:

$$
\text{(6)}\qquad
\begin{tikzcd}
 & & 1 \arrow[d] & 1 \arrow[d] & \\
 & & G \arrow[r, "p"] \arrow[d] & H \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & F \arrow[r, "f"] \arrow[dr, "\tilde{f}"'] & U \arrow[r, "p"] \arrow[d, "i"] & V \arrow[r] \arrow[d, "j"] & 1 \\
 & & E \arrow[r, "p"] \arrow[d] & B \arrow[d] & \\
 & & 1 & 1 &
\end{tikzcd}
$$

This is a commutative diagram of group homomorphisms, in which the second and third lines and both columns are exact. (F denotes here $\pi_1(F)/\pi_2(V)$: because of the absence of a global section the homotopical sequence does not break into exact triples.) In addition, we have a subset $\mathring{H} \subset G$ such that $G = (\mathring{H})^F$; we will assume that $\mathring{H} = \mathring{H}^{-1}$.

1.7. LEMMA. *Under the above conditions the sequence*

$$1 \to R := ([F, \mathring{H}])((\mathring{H}) \cap F)) \to F \xrightarrow{\tilde{f}} E \to B \to 1$$

is exact.

Proof. 1. Exactness at B: $E \xrightarrow{p} B$ is surjective, because $\forall b \in B \, \exists \lambda \in V$, $\alpha \in U$ such that $j(\lambda) = b$, $p(\alpha) = \lambda$ and, therefore, $p(i(\alpha)) = b$.

2. Exactness at E:

(a) $\forall x \in F \; p \circ \tilde{f}(x) = p \circ i \circ f(x) = j \circ p \circ f(x) = 1$

(b) Let $\alpha \in E$, $p(\alpha) = 1$. Then $\exists \tilde{\alpha} \in U$ such that $i(\tilde{\alpha}) = \alpha$ and, so, $j \circ p(\tilde{\alpha}) = p \circ i(\tilde{\alpha}) = 1$ and $h := p(\tilde{\alpha}) \in H$.

Let $g \in G$ such that $p(g) = h$; then $\tilde{\alpha} = gx$ for some $x \in F$, and $\alpha = i(gx) = \tilde{f}(x)$.

3. Exactness at F: the only thing we have to prove is that $F \cap G = R$. As F and G are normal in U, we have $R \subset F \cap G$. Let $\alpha \in F \cap G$. Then, for some $x_i \in F$, $h_i \in \mathring{H}$, $\alpha = \Pi_i x_i^{-1} h_i x_i = \Pi_i [x_i, h_i^{-1}] h_i$.

But $\forall g_1, g_2 \in \mathring{H}$, $x \in F$

$$
\begin{aligned}
g_2[x, g_1] &= g_2 x^{-1} g_2^{-1} g_2 g_1^{-1} g_2^{-1} g_2 x g_2^{-1} g_2 g_1 = y_1^{-1} g_2 g_1^{-1} g_2^{-1} y_1 g_2 g_1 g_2^{-1} g_2 \\
&= y_1^{-1} y_1^{g_2 g_1 g_2^{-1}} g_2 = y_1^{-1} y_1^{g_2} y_1^{-g_2} y_1^{g_2 g_1} y_1^{-g_2 g_1} y_1^{g_2 g_1 g_2^{-1}} g_2 \\
&= [y_1, g_2][y_2, g_1][y_3, g_2^{-1}] g_2,
\end{aligned}
$$

where $y_1 = x^{g_2^{-1}}$, $y_2 = y_1^{g_2}$, $y_3 = y_2^{g_1} \in F$. Therefore, in the expression for α, we can pull all h_i through it to the right, and obtain $\alpha = rh$, $r \in R$, $h = \Pi\, h_i$. As $\alpha \in F$, so $p(h) = \mathbf{1}_V$; hence $h \in (\mathring{H}) \cap F$.

1.8. We can restrict ourselves in the definition of R to generators of F, and thus we obtain the following theorem:

THEOREM. *Under the conditions* (α), (β'), (γ'), (δ') *the sequence*

$$1 \to R \to \pi_1(F)/\pi_2(V) \to \pi_1(E) \to \pi_1(B) \to 1$$

is exact, where R is the normal subgroup generated by the elements $x_i^{-1}x_i^{h_j}$, x_i running over a set of generators of $\pi_1(F)/\pi_2(V)$, and h_j over the set $\mathring{H}$; and by the elements of the form $h = \Pi_i\, h_i$, where $h_i \in \mathring{H}$, $p_*(h) = \mathbf{1}_{\pi_1(V)}$.

1.9. REMARK. In the 'codim n' case (see Remark 1.5) R is generated by $[\pi_1(F), \mathring{H}]$ and $\pi_{n-1}(F) \cap (\mathring{H})$; $\mathring{H} \subset \pi_{n-1}(U)$.

1.10. INTERPRETATION. (γ') and (δ') together imply, that $p(\mathring{H})$ generates H, that is $\mathring{H}$ is a lifting of a set of generators of H. The words of these generators which are trivial in H, may be non-trivial in E (it is not so if there exists a global section); from this the subgroup $(\mathring{H}) \cap F$ arises. Theorem 1.7 says, that the only addition to $([F, \mathring{H}])$, which is known to be a part of the kernel of $\tilde{f}$ by Theorem 1.4, consists in this subgroup.

1.11. The topological conditions (β')–(δ') become more clear in the smooth situation. Let now $p: E \to B$ be a surjective morphism of smooth manifolds, as in 0.2. First, every element of $\pi_1(E)$ can be represented by a smooth path transversal to L by the Transversality Theorem, and hence, by a path which does not intersect L, as the codimension of components of L is $\leqslant 2$. This provides condition (β').

Choose one arbitrary point on every component of $L = E\backslash p^{-1}V$ of codimension 2 and small disks with centers at these points and transversal to the corresponding component. These disks just form the set Σ_E introduced in (γ'). Indeed, by the same theorem, we can apply homotopy to every surface in E to make it smooth and transversal to the components of L of codimension 2 at a discrete set of points of intersection. Every such intersection, say at a point q, can be moved into one of the chosen points: join q by a continuous path lying in the smooth part of L with the corresponding chosen point, say q_0. Take the disk in Σ_E with center in q_0 and bring it into q along this path so that we get a disk in the surface under consideration (Figure 2). Now, let us replace this surface by the new one obtained by adding the lateral surface of the tube and its upper base instead of the lower one. As the disk D is compact, after a finite number of such modifications we will fulfill condition (γ').

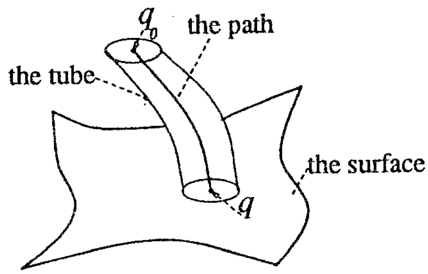

Fig. 2.

In exactly the same way, choose a little disk for every component of $K = B \backslash V$ of codimension 2 with the center on this component and transversal to it. This set of disks forms the set Σ_B from condition (δ'); we need in addition that each disk from Σ_B could be lifted into E. This holds since there exists a local section over every point of B. We find ourselves under the hypothesis of Theorem 1.8; this proves Proposition 0.2.

If p admits a global section intersecting every component of L of codimension 2, we can locate Σ_E in the image of this section; this leads us to the conditions of Theorem 1.4 and we get Proposition 0.3 as a corollary.

1.12. Now we provide a very simple example of a computation of a fundamental group. This is the single example without global section intersecting every component of L of codimension 2 in this work.

Let $\sigma: E' \to \mathbf{C}^2$ be the blow up with the center $(0,0), p': \mathbf{C}^2 \to \mathbf{C}^1$, $p'(x, y) = x$, $E = E' \backslash (u \cup v)$, where u (respectively, v) is the proper transform of the straight line $x = y$ (respectively, $x = -y$); $p = p' \circ \sigma|_E$.

The mapping p degenerates only over $0 \in \mathbf{C}^1$; L consists of the two components (Figure 3) which are the proper transform of $x = 0$ and the exceptional line. The generic fiber is $\mathbf{C}^1$ with two punctures u and v; so, we

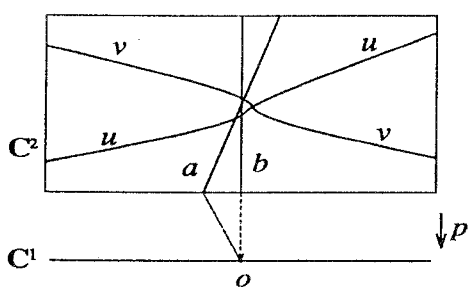

Fig. 3.

have the exact sequence

$$1 \to R \to \mathscr{F}_2 \to \pi_1(E) \to 1,$$

and $\pi_1(E) = \mathscr{F}_2/R$; $\mathscr{F}_2$ is free with generators u and v.

Proposition 0.2 shows a way to obtain the relators. The set $\mathring{H}$ is generated by only two elements a and b which correspond to the proper transform of the line $x = 0$ and the exceptional curve respectively. As a and b both are projected by p_* into the same element of $\pi_1(\mathbf{C}^1 \setminus 0)$, the simple loop around 0, we see that all the relators in $\pi_1(E)$ are $u^{-1}u^a$, $v^{-1}v^a$, and ab^{-1}.

The first two can be written out immediately since the monodromy with respect to a is easily calculated. Its action on the basic loops u, v is given by the formulas $u^a = uvuv^{-1}u^{-1}$, $v^a = uvu^{-1}$ (see 2.3 for details). It gives us the single relation $uv = vu$, and $\pi_1(E)$ is commutative.

To obtain the last relation, we have to contract the loop ab^{-1} into the fiber. To this end, let us unfold the bundle over the elementary loop in $\mathbf{C}^1$ around 0 (Figure 4).

We obtain Figure 5 and see that $ab^{-1} \sim u + v$; so, $u = -v$ in $\pi_1(E)$, and $\pi_1(E) \simeq \mathbf{Z}$.

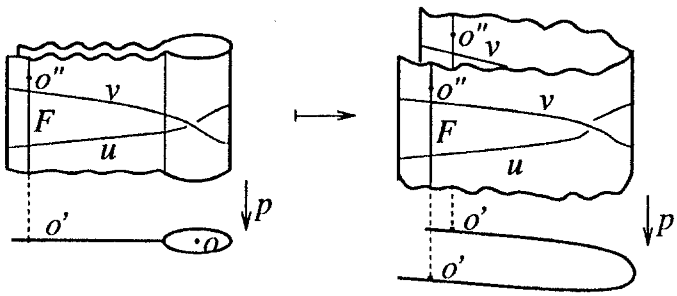

Fig. 4.

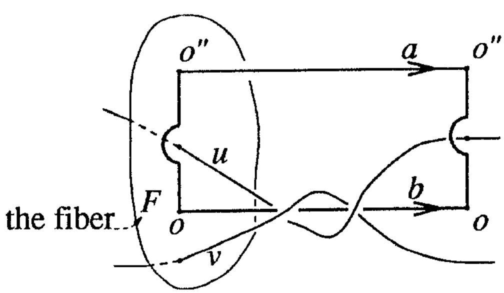

Fig. 5.

1.13. To conclude this section, we will give an example of a mapping, which satisfies the assumptions (ε)–(γ) and the base of which is not a manifold and, moreover, is not Hausdorff.

Consider a closed braid ϕ, that is a continuous mapping of a finite number of copies of S^1 into $\mathbf{R}^3_{x,y,z}\backslash\{x = y = 0\}$ for which, in the standard cylinder coordinates r, θ, z: $x = r\cos\theta$, $y = r\sin\theta$, the phase θ is a monotonic function of the coordinate of every circumference ([Bir]). Define the topological space B in the following way: $B = S^1 \cup \{O\}$ as a set, its topology coincides with the Borel one on S^1 and the only open subset containing the point O is B itself. Let $\tilde{p}$ be the mapping from $\mathbf{R}^3$ onto B of the form

$$\tilde{p}(r, \theta, z) = \left|\begin{array}{ll} \phi \in S^1 & \text{for } r \neq 0 \\ O & \text{for } r = 0. \end{array}\right.$$

It is immediately seen that $\tilde{p}$ is continuous.

Let p be the restriction of $\tilde{p}$ to the complement E in $\mathbf{R}^3$ of the image of the braid: $E = \mathbf{R}^3\backslash\mathrm{Im}(\phi)$, $p = \tilde{p}|_E$; the conditions (ε)–(γ) can be easily verified. Its generic fiber is the (topological) plane ($\theta = \text{const}$, $r > 0$) with a finite number k of punctures; its fundamental group is, therefore, free with k generators. We can, consequently, write out the exact sequence (3) in this situation and obtain the following known presentation of the group $\pi_1(E)$:

$$\pi_1(E) = \{a_i,\, i = 1, \ldots, k \,|\, a_i^{-1} a_i^{\phi} = 1,\, i = 1, \ldots, k\},$$

where a_i^{ϕ} is the result of the action of the braid ϕ on the generator a_i of the free grope $\mathscr{F}_k$ with the set of generators $\{a_i, i = 1, \ldots, k\}$ (see [Bir]).

2. Application to the case of an algebraic hypersurface in $\mathbf{C}^n$

2.1. Let us now turn to the case in which the bundle space E is the complement of an algebraic hypersurface S of degree d in $\mathbf{C}^n$; let $p'\colon \mathbf{C}^n \to \mathbf{C}^{n-1}$ be the projection along a generic direction, $p = p'|_{\mathbf{C}^n\backslash S}$. As was mentioned above, in the algebraic case we have only to check the existence of sections; all other assumptions of our theorems are automatically satisfied.

But p has a section; for example, in the coordinates in which p is the mapping forgetting the last coordinate, it can be defined by the formula $s(z) = (z, \max_{w \in (p')^{-1}(z) \cap S}|w| + 1)$, $z \in \mathbf{C}^{n-1}$.

The set K of singular points consists of the image under the projection into $\mathbf{C}^{n-1}$ of the set of singularities of S and the ramification locus of $p'|_S$. The closure in $\mathbf{C}^n$ of the union of singular fibers is $\bar{L} = (p')^{-1}(K) \simeq K \times \mathbf{C}^1$; every time its component meets $s(\mathbf{C}^{n-1})$ at a non-singular point.

Proposition 0.3 says that $\pi_1(E)$ is generated, because of the simple connectedness of the base $B = \mathbf{C}^{n-1}$, by the group $\pi_1(F)$, which is free with d generators. These generators, say $a_1, \dots, a_d$, are the elementary loops passing around d points in different sheets of S in some fiber.

Let us pass on to the relations. Let $\tilde{K}$ be the union of divisorial components of K, $V = \mathbf{C}^{n-1} \backslash K$, $\tilde{V} = \mathbf{C}^{n-1} \backslash \tilde{K}$. Then $\pi_1(V) = \pi_1(\tilde{V})$, and hence, we can replace K by $\tilde{K}$ and V by $\tilde{V}$ in our further considerations. So, we can suppose that K is a divisor in $\mathbf{C}^{n-1}$; we will call it *the branch divisor of p*.

Thus, we have arrived at the same situation in $\mathbf{C}^{n-1}$: a hypersurface is cut out, and so, $\pi_1(V)$ is generated by $k = \deg K$ generators $b_1, \dots, b_k$ which are elementary loops around the sheets of K lying in any complex line which is in generic position with respect to K.

So, $\pi_1(E)$ is the group with d generators $a_1, \dots, a_d$ and dk relations $a_i^{b_j} = a_i$, $i \leqslant d, j \leqslant k$, where a^b is the result of the action of b on a by monodromy. This proves Proposition 0.4.

In particular, we have obtained the fact, that $\pi_1(E)$ coincides with $\pi_1(E \cap$ a generic complex two-dimensional plane), because all computations were made in such a plane generated by the generic fibers of the two bundles $\mathbf{C}^n \backslash S \to \mathbf{C}^{n-1}$ and $\mathbf{C}^{n-1} \backslash K \to \mathbf{C}^{n-2}$. Hence, we can assume that $n = 2$ and K is the set of singular points $\{b_1, \dots, b_k\} \subset \mathbf{C}^1$.

2.2. Of course, the number of relations is in general much less since a part of those written above are trivial. Every point b_j corresponds to a singularity, in which only few sheets meet, and only the relations which involve the loops around these sheets are non-trivial among those generated by the action of b_j.

Now, we will describe in more detail the structure of the relations. The fiber over the reference point o' is $\mathbf{C}^1 \backslash \{a_1, \dots, a_d\}$. We will assume that $\operatorname{Re} a_i$ are different (if necessary, apply the rotation of the fiber). Choose $o'' \in p^{-1}(o')$ such that $\operatorname{Re} o'' = 0$, $\operatorname{Im} o''$ big enough, and the loops starting at o'' and passing around the points a_i in the counterclockwise direction as a set of generators of $\pi_1(F)$ (see Figure 6). We will use the same notation for points and the

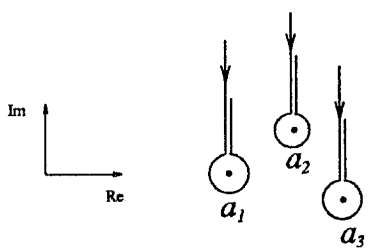

Fig. 6.

corresponding loops. We move on the base to the point b_j along the segment $[o', b_j]$ (bypassing other singular points). Meanwhile the deleted points a_i move on the fiber and entangle the loops a_i, as shown e.g. on Figure 7. Every time when $\operatorname{Re} a_{i_1} = \operatorname{Re} a_{i_2}$ and the corresponding loops become entangled we will change the generators to disentangle them. (In the situation depicted on Figure 7, $a_2 \mapsto a_2' = a_1 a_2 a_1^{-1}$.)

2.3. The exact prescription is the following: if at some moment we have $\operatorname{Re} a_{i_1} = \cdots = \operatorname{Re} a_{i_l}$ simultaneously, we order them according to the value of the imaginary part, $\operatorname{Im} a_{i_1} > \operatorname{Im} a_{i_2} > \cdots > \operatorname{Im} a_{i_l}$, and define the change of the generators by

$$a'_{i_1} = a_{i_1},\ a'_{i_2} = a_{i1}^{\varepsilon_{2,1}} a_{i_2} a_{i_1}^{-\varepsilon_{2,1}},\ \ldots,\ a'_{i_l} = \left(\prod_{p=1}^{l-1} a_{i_p}^{\varepsilon_{l,p}}\right) a_{i_l} \left(\prod_{p=1}^{l-1} a_{i_p}^{\varepsilon_{l,p}}\right)^{-1},$$

where $\varepsilon_{p,q} = \operatorname{sgn}(\operatorname{Re} a_{i_p} - \operatorname{Re} a_{i_q})$ right before the coincidence. It is clear that the loops a_i' will form a system of generators of $\pi_1(E)$ as well. So, when we approach b_j we will have in its neighborhood a natural system of generators which are conjugate to the original ones.

Over b_j some of the deleted points coincide; without loss of generality we can assume that $a_1 = a_2 = \cdots = a_{l_j} = 0$ and $\operatorname{Re} a_i \neq 0$ for $i > l_j$. Then passing around b_j along a small loop will produce a movement of the points $a_1, \ldots, a_{l_j}$ in a small neighborhood of 0 entangling the corresponding loops, and all the a_i with $i > l_j$ will remain too far to intervene into this process. Identifying the loops a_i', $i = 1, \ldots, l_j$, with the result of its entangling around b_j, we obtain the relation $(a_i')^{b_j} = a_i'$, the exact form of $(a_i')^{b_j}$ depending on the topological type of the germ of the branch divisor of p near 0. Thus, we have in total $\Sigma_{j=1}^k l_j$ relations.

2.4. To illustrate the described procedure, we will consider an easy example of a simple ramification point of order l: $a = \sqrt[l]{b}$. The result of the action of the loop around the singular point 0 is the cyclic permutation of the deleted

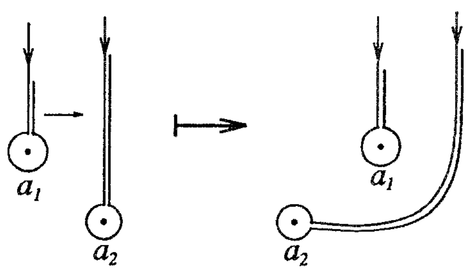

Fig. 7.

points $a_1, \ldots, a_l$ ordered by the decrease of their complex argument. If we choose generators as shown on Figure 8, we will obtain the relations

$$a_l = a_{l-1}, \ldots, a_i = a_{i-1}, \ldots, a_1 = a_1 a_2 \cdots a_{l-1} a_l a_{l-1}^{-1} \cdots a_1^{-1}.$$

REMARK. The last relation is obviously redundant, as it can be reduced to $a_1 = a_l$ by using the remaining ones. This is a general rule, so that the number of the relations decreases to $\Sigma_{b_j}(l_j - 1)$.

We have obtained, that if S has a simple ramification, then certain conjugates of the generators corresponding to the sheets giving this ramification are equal. However, this is evident *a priori*: all generators corresponding to the same irreducible component of the hypersurface S are conjugate. More complicated relations can be obtained over the singular points corresponding to intersections of the components of S.

2.5. If the base B is not simply connected, the set of the relations becomes, as a rule, infinite. Let us study the same example of deleted hypersurface in the following case: $E = \mathbf{C}^n \backslash S$, L is a complex line, $S = S' \cup (S'' \times L)$, $S'' \subset \mathbf{C}^{n-1}$, $S' \subset \mathbf{C}^n = \mathbf{C}^{n-1} \times L$ and is in generic position with respect to L. Then, considering the projection $p: E \to \mathbf{C}^{n-1} \backslash S''$ along L, we again find ourselves in the situation of Proposition 0.3, but base $B := \mathbf{C}^{n-1} \backslash S''$ now is not simply connected.

The general fiber is $\mathbf{C} \backslash \{a_i\}_1^d$, where $d = \deg S'$. Define again the branch divisor $K \subset \mathbf{C}^{n-1}$ of singularities and ramification of S' and denote $V = B \backslash K$. The group $\pi_1(V)$ is generated by the simple loops $b_1, \ldots, b_k$ surrounding the sheets of K and $v_1, \ldots, v_c$ surrounding the sheets of S'' in an appropriate complex line. Proposition 0.3 implies that $\pi_1(E)$ is generated by the loops $a_i|_1^d$ and the loops $v_k|_1^c$ lift into a section of the bundle:

$$\pi_1(E) = (\mathscr{F}_d / R) \rtimes \pi_1(\mathbf{C}^{n-1} \backslash S'')$$

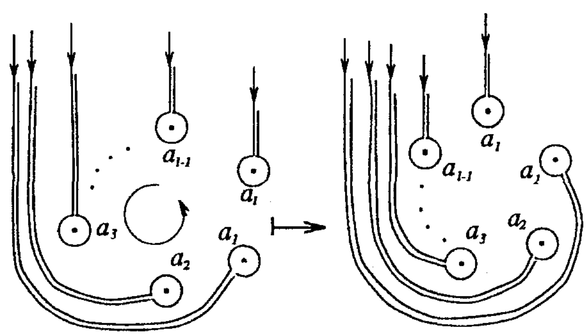

Fig. 8. The simple ramification of order l.

where $\mathscr{F}_d$ is free with the set $\{a_i\}$ of generators and R is generated by the elements of the form $a_i^{vb_jv^{-1}} a_i^{-1}$, where $i \leqslant d$, $j \leqslant k$, and v runs over all the group $\pi_1(\mathbf{C}^{n-1}\backslash S'')$.

This means, that the relators can be obtained by the prescription 2.3 in an appropriate one-dimensional section of the base, but now one has to write out these relators not only for the generators, but also for all their images under the action of all elements v of the infinite group $\pi_1(\mathbf{C}^{n-1}\backslash S'')$:

$$(a_i^v)^{b_j} = a_i^v.$$

2.6. The following is the simplest example to illustrate the considerations of Section 2.5. Look at the complexification of the real configuration including four straight lines x, y, u, a with three intersections (x, y, a), (u, x), (u, a); the map p is the projection along a (see Figure 9). We have the exact sequence

$$1 \to G \to \pi_1(\mathbf{C}^2\backslash(x \cup y \cup u \cup a)) \to \pi_1(\mathbf{C}^1\backslash\{A\}) \to 1.$$

We are going to write out a presentation of G in terms of generators and relations, and show that G is not finitely presented.

The base of the bundle is $\mathbf{C}^1\backslash\{A\}$, and the singular set consists of one point B. The fundamental group of the fiber is generated by three elements: x, u, y. The generators of the fundamental group of $\mathbf{C}^1\backslash\{A, B\}$ act as follows:

$$\begin{array}{llll} A\colon & x \mapsto xyxy^{-1}x^{-1} & y \mapsto xyx^{-1} & u \mapsto u \\ B\colon & x \mapsto uxu^{-1} & y \mapsto y & u \mapsto uxux^{-1}u^{-1} \end{array}$$

B gives only the relations of commutation of u and x, and all relators of G have the form $[x^A, u^A]$. So,

$$G = \{x, y, u \mid [x^{(xy)^n}, u], n \in \mathbf{Z}\}.$$

Apply the change of the variables $z = xy$, $x = x$. Then the relators are $[z^{-n}xz^n, u]$, $n \in \mathbf{Z}$. The group $\{\Pi_{i=1}^k x^{n_i} z^{m_i} \mid \Sigma_{i=1}^k m_i = 0\}$ is freely generated by

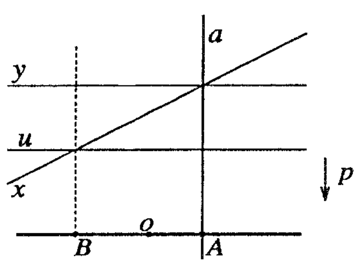

Fig. 9.

the elements $x_n = z^{-n}xz^n$, $n \in \mathbf{Z}$. So, G has the presentation

$$\{z, u, x_n, n \in \mathbf{Z} \mid zx_nz^{-1}x_{n-1}^{-1}, [x_n, u]\},$$

and $G \simeq \mathscr{F}_\infty(x_n) \rtimes \mathscr{F}_2(u, z)$. Now it is clear, that all these relations are independent.

2.7. Our next goal is to prove Proposition 0.5.

LEMMA. *Let S', S'' be two hypersurfaces in $\mathbf{C}^n$ without common irreducible components, S'' contain the affine line $(x + L)$ with every its point x for some complex line L, i.e. $S'' = \tilde{S} \times L$, $\tilde{S} \subset \mathbf{C}^{n-1}$, and the direction L be in generic position with respect to S'. Denote $S = S' \cup S''$.*

1. *Then there exists the group G, such that $G_S := \pi_1(\mathbf{C}^n \backslash S)$ is a semidirect product $G_S \simeq G \rtimes G_{S''}$, where $G_{S''} := \pi_1(\mathbf{C}^n \backslash S'')$, and $G_{S'} := \pi_1(\mathbf{C}^n \backslash S')$ is a quotient of G.*
2. *If, in addition, the branch divisor of S with respect to the projection along L has no components lying in $\tilde{S}$, then $G_S \simeq G_{S'} \times G_{S''}$.*

Proof. Make the projection along L: $p: (\mathbf{C}^n \backslash S) \to (\mathbf{C}^{n-1} \backslash \tilde{S})$. Let $K \subset \mathbf{C}^{n-1}$ be the branch divisor of S. We have, by Proposition 0.2, the split sequence

$$1 \to R \to F \to G_S \rightleftarrows G_{\tilde{S}} \to 1,$$

where F is the free group generated by the loops around the sheets of S' in a generic fiber. But $G_{S'}$ is generated by the same set of loops, therefore there exists a natural surjection $F \to G_{S'} \to 1$.

R is generated, as a normal subgroup of F, by the relators $x^h x^{-1}$ for $x \in F$, $h \in \pi_1(\mathbf{C}^{n-1} \backslash \tilde{S} \backslash K)$; the corresponding relations hold also in $G_{S'}$. So, for $G := F/R$, there exists the surjection $G \to G_{S'} \to 1$. And, as $G_{\tilde{S}} \simeq G_{S''}$, we have $G_S \simeq G \rtimes G_{S''}$.

If, in addition, assumption 2 is true, then all the components of the branch divisor of S' in $\mathbf{C}^{n-1}$ are represented in $\mathbf{C}^{n-1} \backslash \tilde{S}$ as well. So, all relations of $G_{S'}$ are satisfied in G, that is $G = G_{S'}$. Furthermore, when passing around the sheets of $\tilde{S}$ in a generic plane, the different sheets of S' stay far from one another, and so, the action of $G_{\tilde{S}}$ on the generators of $G_{S'}$ is trivial. This shows that the semidirect product is indeed direct: $G_S \simeq G_{S'} \times G_{S''}$.

PROOF OF PROPOSITION 0.5. The general case is obtained from the lemma by an easy trick. Add one more dimension to $\mathbf{C}^n$,

$$\mathbf{C}^n \oplus \mathbf{C}u = \mathbf{C}^{n+1}$$

and define $\tilde{S}'' = S'' + \mathbf{C}u$.

Choose a vector v in $\mathbf{C}^n$ generic with respect to S', and span S' along the direction $w = v + u$: $\tilde{S}' := S' + \mathbf{C}w$. Then u is a generic direction for S'. Making v longer we can assume that all components of the branch divisor of S' are not contained in S''. Then $\tilde{S}'$, $\tilde{S}''$ and $\tilde{S} := \tilde{S}' \cup \tilde{S}''$ with $L = \mathbf{C}u$ satisfy all the assumptions of the lemma. But all the singularities and ramifications of $\tilde{S}$ in complex codimension 2 are those of S, where S is a generic section of $\tilde{S}$ in $\mathbf{C}^{n+1}$, and so, $G_S \simeq G_{\tilde{S}} \simeq G_{S'} \times G_{S''}$.

3. Arrangements of hyperplanes

3.1. Now, having in mind hyperplanes, we are interested in the case of the transverse intersection of l smooth branches; in this case the corresponding l punctures in the fiber make a complete revolution around a common center. If these points a_i are numbered according to the order of augmentation of their real parts, we obtain the following set of relations:

$$a_i = \left(\prod_1^l a_p\right) a_i \left(\prod_1^l a_p\right)^{-1}, \ i \leqslant l.$$

This is equivalent to the following:

$$a_1 a_2 \cdots a_l = a_2 a_3 \cdots a_l a_1 = \cdots = a_l a_1 a_2 \cdots a_{l-1}.$$

For $l = 2$ this gives the commutation $a_1 a_2 = a_2 a_1$.

In particular, for the case when S is a union of hyperplanes, we obtain Proposition 0.6. The singular set K in this case is the projection into $\mathbf{C}^{n-1}$ of the set of transverse intersections; the generic points of K correspond to the intersections of complex codimension 2 and the relations modulo conjugation of generators are given by the table of the intersections in codimension 2.

3.2. In the case of an arrangement of hyperplanes we can modify prescription 2.3 for computing the conjugations in using segments of (real) straight lines in $\mathbf{C}^{n-1}$ as paths $[o', b_j]$.

If at a moment the real parts of a set $\{a_{i_1}, \ldots, a_{i_l}\}$ of punctures are equal, divide this set into the subsets of points with equal imaginary parts, and order these points according to the decrease of the imaginary parts:

$$\{a_{i_1}, \ldots, a_{i_l}\} = \{a_1^1 = a_2^1 = \cdots = a_{l_1}^1\} \cup \{a_1^2 = a_2^2 = \cdots = a_{l_2}^2\} \cup \cdots$$
$$\operatorname{Im} a_1^1 > \operatorname{Im} a_1^2 > \cdots.$$

In every subset (say, qth) we order the points according to the augmentation of their real parts just before the coincidence, and write down the corresponding relations:

$$a_1^q a_2^q \cdots a_{l_q}^q = a_2^q \cdots a_{l_q}^q a_1^q = a_{l_q}^q a_1^q \cdots a_{l_q-1}^q$$

(here q is not an exponent, but a superscript).

Now, define the change of generators:

$$\begin{array}{lll} a_1^q \mapsto a_1^q & \text{or} & (a_{l_q}^q)^{-1} \cdots (a_2^q)^{-1} a_1^q (a_2^q) \cdots (a_{l_q}^q) \\ a_2^q \mapsto (a_1^q) a_2^q (a_1^q)^{-1} & \text{or} & (a_{l_q}^q)^{-1} \cdots (a_3^q)^{-1} a_2^q (a_3^q \cdots (a_{l_q}^q) \\ \vdots \quad \vdots & & \vdots \\ a_{l_q} \mapsto (a_1^q)(a_2^q) \cdots a_{l_q}^q \cdots (a_2^q)^{-1} (a_1^q)^{-1} & \text{or} & a_{l_q}^q. \end{array}$$

The next step is the change described in Section 2.3, corresponding to the points of the previous subsets:

$$a_j^q \mapsto (a_1^1)^{\varepsilon_1^1} (a_2^1)^{\varepsilon_2^1} \cdots (a_{l_{q-1}}^{q-1})^{\varepsilon_{l_{q-1}}^{q-1}} a_j^q (a_{l_{q-1}}^{q-1})^{-\varepsilon_{l_{q-1}}^{q-1}} \cdots (a_1^1)^{-\varepsilon_1^1}.$$

3.3. In this section we restrict ourselves with the real case, where the hyperplanes are defined by real equations. In this case K is real too, that is all the singular points b_j lie on the same real line, namely on the real axis. It is enough to pass along this line to obtain all relations. In addition, the deleted points a_i over this line are real, and the coincidence of their real parts is the same as the coincidence of the points themselves. So, in this case prescription 3.2 can be simplified as follows.

At the moment of the coincidence of $a_{i_1}, \ldots, a_{i_l}$, order these points according to the order of augmentation before the coincidence, say $a_1, \ldots, a_l$, write down the relations:

$$a_1 a_2 \cdots a_l = a_2 \cdots a_l a_1 = \cdots = a_l a_1 \cdots a_{l-1}$$

and make the change:

$$\begin{array}{lll} a_1 \mapsto a_1 & \text{or} & a_l^{-1} \cdots a_2^{-1} a_1 a_2 \cdots a_l \\ a_2 \mapsto a_1 a_2 a_1^{-1} & \text{or} & a_l^{-1} \cdots a_3^{-1} a_2 a_3 \cdots a_l \\ \vdots \quad \vdots & & \vdots \\ a_l \mapsto a_1 a_2 \cdots a_l \cdots a_2^{-1} a_1^{-1} & \text{or} & a_l. \end{array}$$

REMARK. One can introduce a non-canonical order on the set of hyperplanes by the slopes of the real straight lines in $\mathbf{R}^2$ which are traces of the hyperplanes in a generic plane section. This will provide the right order over all singular points.

3.4. As an application, we compute the fundamental group of the complement of the arrangement of hyperplanes $D_n := \{x_i \pm x_j, 1 \leqslant i, j \leqslant n, i \neq j\}$ in $\mathbf{C}^n$; we denote this group by $\mathscr{D}_n$. This is a real arrangement, and the computations are easy; but the result is non-canonical and cumbersome.

Denote the hyperplanes $x_i = x_j$ by the symbols $a_{i,j}\,(=a_{j,i})$, and $x_i = -x_j$ by $b_{i,j}\,(=b_{j,i})$. These symbols form a set of generators of the group; the relations will be obtained from the intersections. In codimension 2 we have four types of double intersections $(a_{i,j}, b_{i,j})$, $(a_{i,j}, b_{k,l})$, $(a_{i,j}, a_{k,l})$, $(b_{i,j}, b_{k,l})$, and two types of triple ones $(a_{i,j}, a_{j,k}, a_{i,k})$, $(a_{i,j}, b_{j,k}, b_{i,k})$; here i, j, k, l are pairwise distinct. We will code the relations by the same parentheses, e.g. $ab = ba$ is (a, b), and $abc = bca = cab$ is (a, b, c). This yields all the relations of the group, but only modulo conjugation of the generators. To find these conjugations is the main problem.

Choose a real two-dimensional plane (s, t) in $\mathbf{C}^n$, defined by

$$x_k = p_k s + q_k t - r_k, \text{ where } p_i \ll p_j,\ q_i \lll q_j,\ r_i \llll r_j \text{ for } i < j. \tag{7}$$

In this plane our hyperplanes trace out the straight lines

$$a_{i,j} = \left\{ s = \frac{q_j - q_i}{p_j - p_i} t + \frac{r_j - r_i}{p_j - p_i} \right\},\quad b_{i,j} = \left\{ s = \frac{q_j + q_i}{p_j + p_i} t + \frac{r_j + r_i}{p_j + p_i} \right\}.$$

We will now move, according to prescription 3.3, along the t-axis watching the intersections of our lines.

We start from $t = -\infty$. Here our lines are arranged in the following order:

$$b_{1,2} > a_{1,2} > b_{2,3} > b_{1,3} > a_{1,3} > a_{2,3} > b_{3,4} > \cdots > a_{n-1,n},$$

that is

$$a_{i,j} > b_{k,l},\ b_{i,l} > a_{,l} \quad \text{for } i, j < l$$

$$b_{i,l} > b_{j,l},\ a_{i,l} > a_{j,l} \quad \text{for } i < j < l.$$

Divide the set of the lines into groups according to the augmentation of their greater subscript: the kth group contains $a_{i,k}$, $b_{i,k}$ for $i < k$. We see that when $t = -\infty$, the groups are widely separated, ordered according to the decrease of their numbers; inside every group the elements are ordered according to their smaller subscript: first $a_{i,k}$ according to the decrease of the subscript, and then $b_{i,k}$ according to its magnitude. Call this order *original*.

The binary intersections give us relations of commutation and do not involve any conjugations. The triple ones involve conjugations for the middle element (see Figure 10). Let us enumerate the triple intersections, watching

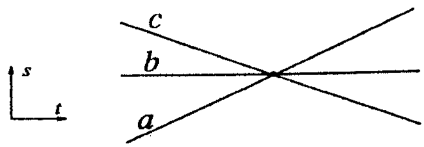

Fig. 10. $b \mapsto aba^{-1}$ or $c^{-1}bc$.

the order of the elements. The original order is following:

$(a_{j,k}, a_{i,k}, a_{i,j}), (b_{i,k}, b_{j,k}, a_{i,j}), (a_{j,k}, b_{i,k}, b_{i,j}), (a_{i,k}, b_{j,k}, b_{i,j})$ for $i < j < k$.

In every set of parentheses there is an element of a lesser group; we will make the change of generators given by prescription 3.3 by the conjugation by this element:

$$a_{i,k} \mapsto a_{i,j}^{-1} a_{i,k} a_{i,j},\ b_{i,k} \mapsto a_{i,j}^{-1} b_{i,k} a_{i,j},\ b_{i,k} \mapsto b_{i,j}^{-1} b_{i,k} b_{i,j}.$$

Call the conjugating element a junior element of the triple.

The lines $a_{i,j}$ (or $b_{i,j}$) and $a_{k,l}$ (or $b_{k,l}$) meet one another at the point

$$t = -\frac{\dfrac{r_1 \pm r_k}{p_l \pm p_k} - \dfrac{r_i \pm r_j}{p_i \pm p_j}}{\dfrac{q_l \pm q_k}{p_l \pm p_k} - \dfrac{q_i \pm q_j}{p_i \pm p_j}} \approx -\frac{r_l}{p_l},$$

where $l > i, j, k$. Therefore, we observe the following order of the intersections when we increase t: first the elements of junior groups intersect the senior groups, and only after this they meet one another. So, by the moment of the meeting with the senior groups, every element of the junior ones will not have any conjugations and their order will be original.

To find the order of the intersections of the elements of the same group one could make a long computation; but it is easier to record just the sequence of junior elements of triple intersections. The order of the triple intersections is obviously recovered from the sequence of junior elements.

We also have to find out the order of triple intersections which have occurred by the moment of every double one $(a_{k,l}, b_{k,l})$, $k < l$; this is an exceptional case, as there is no junior elements. Since the elements $a_{1,l}, b_{1,l}$ lie between $a_{k,l}, b_{k,l}$, our lines first have to meet them; the last such intersections are those with the junior element $b_{1,l}$: they are $(a_{k,l}, b_{1,l}, b_{1,k})$ and $(a_{1,l}, b_{k,l}, b_{1,k})$. So, by this moment the intersection $(a_{1,l}, b_{1,l})$ has not yet occurred. But the previous intersections, with the junior element $a_{1,k}$, are $(a_{k,l}, a_{1,l}, a_{1,k})$ and $(b_{k,l}, b_{1,l}, a_{1,k})$, so, the intersection $(a_{k,l}, b_{k,l})$ has not occurred (Figure 11). Hence, the intersection $(a_{k,l}, b_{k,l})$ takes place between those

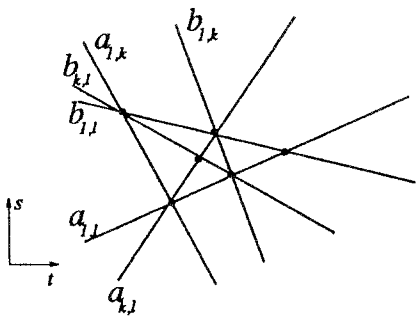

Fig. 11.

with $a_{1,k}$ and $b_{1,k}$. The last intersections of the lth group, which are with the junior element $b_{1,2}$, that is $(a_{2,l}, b_{1,l}, b_{1,2})$ and $(a_{1,l}, b_{2,l}, b_{1,2})$, show, that the intersection $(b_{1,l}, a_{1,l})$ has not yet happened.

Now, as we know the order of all intersections, we can describe the relations of $\mathscr{D}_n$. In all formulas we suppose $i \neq j \neq k \neq i$, $i, j, k < l$; $a_{i,j} = a_{j,i}$, $b_{i,j} = b_{j,i}$. The relations have the form:

$$(\tilde{a}_{j,l}, \tilde{a}_{i,l}, a_{i,j}) \quad \text{for } i < j, \ (\tilde{b}_{j,l}, \tilde{b}_{i,l}, a_{i,j}) \quad \text{for } j < i, \ (\tilde{a}_{i,l}, \tilde{b}_{j,l}, b_{i,j}),$$

$$(\tilde{a}_{k,l}, a_{i,j}), (\tilde{a}_{k,l}, b_{i,j}), (\tilde{b}_{k,l}, a_{i,j}), (\tilde{b}_{k,l}, b_{i,j}), (\tilde{a}_{k,l}, \tilde{b}_{k,l}).$$

Here the tilde denotes the conjugation of the corresponding generator; a generator without tilde is a junior element. To obtain the explicit expressions for these conjugations, we have to take the product of the elements of the previous groups which are lesser than the junior element in their original order, according to the following rule:

$$a_{i,j} \text{ for } \tilde{a}_{i,l} \ (j > i); \ b_{i,j} \ (j \neq i) \text{ and } a_{j,i} \ (j < i) \text{ for } \tilde{b}_{i,l}.$$

Here is the list of the relations of $\mathscr{D}_n$ (we assume $1 \leqslant i < j < k < l \leqslant n$):

$\langle 1 \rangle$ $(a_{k,l}^{a_{k,l-1}\cdots a_{k,k+1}}, b_{k,l}^{b_{k,l-1}\cdots b_{k,k+1}a_{k-1,k}\cdots a_{1,k}})$

$\langle 2 \rangle$ $(a_{k,l}^{a_{k,l-1}\cdots a_{k,k+1}}, a_{i,j})$

$\langle 3 \rangle$ $(a_{j,l}^{a_{j,l-1}\cdots a_{j,k}}, a_{i,k})$

$\langle 4 \rangle$ $(a_{i,l}^{a_{i,l-1}\cdots a_{i,k+1}}, a_{j,k})$

$\langle 5 \rangle$ $(a_{k,l}^{a_{k,l-1}\cdots a_{k,k+1}}, b_{i,j})$

$\langle 6 \rangle$ $(a_{j,l}^{a_{j,l-1}\cdots a_{j,k}}, b_{i,k})$

$\langle 7 \rangle$ $(a_{i,l}^{a_{i,l-1}\cdots a_{i,k}}, b_{j,k})$

$\langle 8\rangle$ $(b_{k,l}^{b_{k,l-1}\cdots b_{k,k+1}a_{k-1,k}\cdots a_{1,k}b_{1,k}\cdots b_{k-1,k}}, a_{i,j})$

$\langle 9\rangle$ $(b_{j,l}^{b_{j,l-1}\cdots b_{j,k+1}}, a_{i,k})$

$\langle 10\rangle$ $(b_{i,l}^{b_{i,l-1}\cdots b_{i,k+1}}, a_{j,k})$

$\langle 11\rangle$ $(b_{k,l}^{b_{k,l-1}\cdots b_{k,k+1}a_{k-1,k}\cdots a_{1,k}b_{1,k}\cdots b_{k-1,k}}, b_{i,j})$

$\langle 12\rangle$ $(b_{j,l}^{b_{j,l-1}\cdots b_{j,k+1}}, b_{i,k})$

$\langle 13\rangle$ $(b_{i,l}^{b_{i,l-1}\cdots b_{i,k}}, b_{j,k})$

$\langle 14\rangle$ $(a_{j,l}^{a_{j,l-1}\cdots a_{j,j+1}}, a_{i,l}^{a_{i,l-1}\cdots a_{i,j+1}}, a_{i,j})$

$\langle 15\rangle$ $(b_{i,l}^{b_{i,l-1}\cdots b_{i,j+1}}, b_{j,l}^{b_{j,l-1}\cdots b_{j,j+1}a_{j-1,j}\cdots a_{i+1,j}}, a_{i,j})$

$\langle 16\rangle$ $(a_{i,l}^{a_{i,l-1}\cdots a_{i,j}}, b_{j,l}^{b_{j,l-1}\cdots b_{j,j+1}a_{j-1,j}\cdots a_{1,j}b_{1,j}\cdots b_{i-1,j}}, b_{i,j})$

$\langle 17\rangle$ $(a_{j,l}^{a_{j,l-1}\cdots a_{j,j+1}}, b_{i,l}{}^{b_{i,l-1}\cdots b_{i,j+1}}, b_{i,j})$.

3.5. We can considerably simplify these relations (but we could not reduce them up to the ones in [Mar]). We are going to illustrate the method of the reduction on the relations $\langle 2\rangle, \langle 3\rangle, \langle 4\rangle, \langle 14\rangle$: the elements $a_{i,j}$ of $\mathscr{D}_n$ generate the subgroup $\mathscr{A}_n$ corresponding to the root system A_n; and these relations form a complete system of relations of this group. We use the evident implication $(a, c) \Rightarrow ((a, b) \Leftrightarrow (a, b^c))$ and the induction on l to prove that this set of relations is equivalent to the following one:

$$\langle 2'\rangle \quad (a_{i,j}, a_{k,l}),\ \langle 3'\rangle\ (a_{i,k}, a_{j,l}^{a_{j,k}}),\ \langle 4'\rangle\ (a_{j,k}, a_{i,l}),$$

$$\langle 14'\rangle\ (a_{i,j}, a_{j,l}, a_{i,l}).$$

$\langle 2'\rangle$: $(a_{i,j}, a_{k,p})$ for $k < p < l \Rightarrow (\langle 2\rangle \Leftrightarrow \langle 2'\rangle)$

$\langle 3'\rangle$: $(a_{i,k}, a_{j,p}^{a_{j,k}})$ for $k < p < l \Rightarrow (\langle 3\rangle \Leftrightarrow \langle 3'\rangle)$

$\langle 4'\rangle$: $(a_{i,j}, a_{k,p})$ for $k < p < l \Rightarrow (\langle 4\rangle \Leftrightarrow \langle 4'\rangle)$

$\langle 14'\rangle$: $a_{j,l}^{a_{j,l-1}\cdots a_{j,j+1}}a_{i,l}^{a_{i,l-1}\cdots a_{i,j+1}}$

$= (\text{by } \langle 3\rangle)\ (a_{j,l}^{a_{j,l-1}\cdots a_{j,j+1}}a_{i,l}^{a_{i,l-1}\cdots a_{i,j+2}})^{a_{i,j+1}}$

$= (\text{by } \langle 4\rangle)(a_{j,l}^{a_{j,l-1}\cdots a_{j,j+2}}a_{i,l}^{a_{i,l-1}\cdots a_{i,j+2}})^{a_{j,j+1}a_{i,j+1}} = \cdots$

$= (a_{j,l}a_{i,l})^{a_{j,l-1}a_{i,l-1}\cdots a_{j,j+1}a_{i,j+1}}$

and, as $(a_{i,j}, a_{j,p}a_{i,p})$ for $k < p < l$,

$$a_{i,j}a_{j,l}^{a_{j,l-1}\cdots a_{j,j+1}}a_{i,l}^{a_{i,l-1}\cdots a_{i,j+1}} = (a_{i,j}a_{j,l}a_{i,l})^{a_{j,l-1}a_{i,l-1}\cdots a_{j,j+1}a_{i,j+1}},$$

$$a_{j,l}^{a_{j,l-1}\cdots a_{j,j+1}}a_{i,l}^{a_{i,l-1}\cdots a_{i,j+1}}a_{i,j} = (a_{j,l}a_{i,l}a_{i,j})^{a_{j,l-1}a_{i,l-1}\cdots a_{j,j+1}a_{i,j+1}},$$

so

$$a_{i,j}a_{j,l}^{a_{j,l-1}\cdots a_{j,j+1}}a_{i,l}^{a_{i,l-1}\cdots a_{i,j+1}}=a_{j,l}^{a_{j,l-1}\cdots a_{j,j+1}}a_{i,l}^{a_{i,l-1}\cdots a_{i,j+1}}a_{i,j}$$

$$\Leftrightarrow a_{i,j}a_{j,l}a_{i,l}=a_{j,l}a_{i,l}a_{i,j};$$

furthermore,

$$a_{j,l}^{a_{j,l-1}\cdots a_{i,j+1}}a_{i,j}a_{j,l}^{a_{j,l-1}\cdots a_{j,j+1}}$$

$$=(\text{by } \langle 4\rangle)\ a_{i,j+1}^{-1}a_{j,j+1}^{-1}a_{i,l}^{a_{i,l-1}\cdots a_{i,j+2}}a_{j,j+1}a_{i,j+1}a_{i,j}a_{j,l}^{a_{j,l-1}\cdots a_{j,j+1}}$$

(by induction)

$$=a_{i,j+1}^{-1}a_{j,j+1}^{-1}a_{i,l}^{a_{i,l-1}\cdots a_{i,j+2}}a_{i,j}a_{j,j+1}a_{i,j+1}a_{j,l}^{a_{j,l-1}\cdots a_{j,j+1}}$$

$$=(\text{by } \langle 3\rangle)\ (a_{i,l}^{a_{i,l-1}\cdots a_{j,j+2}}a_{i,j}a_{j,l}^{a_{j,l-1}\cdots a_{j,j+2}})^{a_{j,j+1}a_{i,j+1}}=\cdots$$

$$=(a_{i,l}a_{i,j}a_{j,l})^{a_{j,l-1}a_{i,l-1}\cdots a_{j,j+1}a_{i,j+1}},$$

and hence,

$$a_{i,l}^{a_{i,l-1}\cdots a_{i,j+1}}a_{i,j}a_{j,l}^{a_{j,l-1}\cdots a_{j,j+1}}=a_{i,j}a_{j,l}^{a_{j,l-1}\cdots a_{j,j+1}}a_{i,l}^{a_{i,l-1}\cdots a_{i,j+1}}$$

$$\Leftrightarrow a_{i,l}a_{i,j}a_{j,l}=a_{i,j}a_{j,l}a_{i,l}.$$

Therefore, we have the following presentation of $\mathscr{A}_n$:

$$\{a_{i,j},\ 1\leqslant i<j\leqslant n\,|\,(a_{i,j},\ a_{k,l}),\ (a_{i,k},\ a_{j,p}^{a_{j,k}}),\ (a_{j,k},\ a_{i,l}),\ (a_{i,j},\ a_{j,l},\ a_{i,l})$$
$$\text{for } 1\leqslant i<j<k<l\leqslant n\}.$$

REMARK. The Burau presentation of $\mathscr{A}_n$ can be obtained by choosing the reference point of the fundamental group on the line $t=0$.

By similar computations, we obtain the following complete set of relations in $\mathscr{D}_n$:

$$(a_{k,l}^{a_{k-1,l}\cdots a_{1,l}},\ b_{k,l}),$$
$$(a_{i,j},\ a_{k,l}),\ (a_{i,k},\ a_{j,l}^{a_{i,l}}),\ (a_{j,k},\ a_{i,l}),$$
$$(b_{i,j},\ a_{k,l}),\ (b_{i,k},\ a_{j,l}^{a_{j,k}}),\ (b_{j,k},\ a_{i,l}^{a_{i,k}}),$$
$$(a_{i,j},\ b_{k,l}^{b_{k,l-1}\cdots b_{k,k+1}a_{k-1,k}\cdots a_{1,k}b_{1,k}\cdots b_{k-1,k}}),\ (a_{i,k},\ b_{j,l}),\ (a_{j,k},\ b_{i,l}),$$
$$(b_{i,j},\ b_{k,l}^{b_{k,l-1}\cdots b_{k,k+1}a_{k-1,k}\cdots a_{1,k}b_{1,k}\cdots b_{k-1,k}}),\ (b_{i,k},\ b_{j,l}),\ (b_{j,k},\ b_{i,l}^{b_{i,k}}),$$
$$(a_{i,j},\ a_{j,l},\ a_{i,l}),\ (a_{i,j}^{a_{i,j-1}\cdots a_{i,i+1}},\ b_{i,l},\ b_{j,l}),$$
$$(b_{i,j},\ a_{i,l}^{a_{i,l-1}\cdots a_{i,j}},\ b_{j,l}^{b_{j,l-1}\cdots b_{j,j+1}a_{j-1,j}\cdots a_{1,j}b_{1,j}\cdots b_{i-1,j}}),\ (b_{i,j},\ a_{j,l},\ b_{i,l}).$$

3.6. The induction used in our derivation of relations for $\mathscr{D}_n$ shows that $\mathscr{D}_{n-1}$ is in a natural way a subgroup of $\mathscr{D}_n$. In fact, this inclusion is a section of split sequence (3)

$$1 \to \mathscr{G} \to \mathscr{D}_n \rightleftarrows \mathscr{D}_{n-1} \to 1$$

corresponding to the locally non-trivial bundle $\mathbf{C}^n \backslash \mathscr{D}_n \to \mathbf{C}^{n-1} \backslash \mathscr{D}_{n-1}$ induced by the linear projection $(x_1, \ldots, x_{n-1}, x_n) \mapsto (x_1, \ldots, x_{n-1})$, so that $\mathscr{D}_n = \mathscr{G} \rtimes \mathscr{D}_{n-1}$ is the semidirect product. We are going to produce an infinite presentation of $\mathscr{G}$ in terms of generators and relations on the case $n = 3$. By the general formulas,

$$\begin{aligned}\mathscr{D}_3 \simeq \{b_{1,2}, a_{1,2}, b_{2,3}, b_{1,3}, a_{1,3}, a_{2,3} |\ & (a_{1,2}, b_{1,2}), (a_{1,2}^{-1} a_{1,3} a_{1,2}, b_{1,2}^{-1} b_{1,3} b_{1,2}), (a_{1,2}^{-1} b_{2,3} a_{1,2}, b_{1,2}, a_{1,2}^{-1} a_{1,3} a_{1,2}),\\ & (b_{1,3}, b_{1,2}, a_{2,3}), (a_{2,3}, a_{1,2}^{-1} b_{2,3} a_{1,2}), (a_{1,3}, a_{1,2}, a_{2,3}), (b_{2,3}, a_{1,2}, b_{1,3})\}.\end{aligned}$$

For an appropriate choice of generators, we obtain the relations

$$[a_{1,3}, b_{1,3}] = [a_{2,3}, b_{2,3}] = 1 \quad \text{in } \mathscr{G} = \ker(\mathscr{D}_3 \to \mathscr{D}_2 = (a_{1,2}, b_{1,2})).$$

But this is not all. We will obtain a complete description of $\mathscr{G}$ in considering the projection $p \colon \mathbf{C}^3 \backslash D_3 \to \mathbf{C}^2 \backslash D_2$, $p(x_1, x_2, x_3) = (x_1, x_2)$. We have the following picture (Figure 12) in an appropriate real plane section. The group $\mathscr{D}_2$ is commutative with the generators $a_{1,2}, b_{1,2}$. So, all the relators of $\mathscr{G}$ are of the form

$$[a_{1,3}^{a_{1,2}^n b_{1,2}^m}, b_{1,3}^{a_{1,2}^n b_{1,2}^m}], \ [a_{2,3}^{a_{1,2}^n b_{1,2}^m}, b_{2,3}^{a_{1,2}^n b_{1,2}^m}], \quad n, m \in \mathbf{Z}.$$

Using prescription 3.3, we can rewrite them in terms of $\mathscr{G}$. Denote

$$\begin{aligned}
\tilde{a}_{1,3}^{n,m} = \langle a_{1,3}, & ((a_{2,3}b_{1,3})^n a_{2,3} (a_{2,3}b_{1,3})^{-n} \\
& (a_{1,3}b_{2,3})^n a_{1,3} (a_{1,3}b_{2,3})^{-n})^m (a_{1,3}b_{2,3})^n \rangle, \\
\tilde{a}_{2,3}^{n,m} = \langle a_{2,3}, & ((a_{2,3}b_{1,3})^n a_{2,3} (a_{2,3}b_{1,3})^{-n} \\
& (a_{1,3}b_{2,3})^n a_{1,3} (a_{1,3}b_{2,3})^{-n})^m (a_{2,3}b_{1,3})^n \rangle, \\
\tilde{b}_{1,3}^{n,m} = \langle b_{1,3}, & ((a_{2,3}b_{1,3})^{n+1} a_{2,3} (a_{2,3}b_{1,3})^{-n-1} \\
& (a_{1,3}b_{2,3})^{n+1} a_{1,3} (a_{1,3}b_{2,3})^{-n-1})^m (a_{2,3}b_{1,3})^{n+1} \rangle, \\
\tilde{b}_{2,3}^{n,m} = \langle b_{2,3}, & ((a_{2,3}b_{1,3})^{n+1} a_{2,3} (a_{2,3}b_{1,3})^{-n-1} \\
& (a_{1,3}b_{2,3})^{n+1} a_{1,3} (a_{1,3}b_{2,3})^{-n-1}) (a_{1,3}b_{2,3})^{n+1} \rangle,
\end{aligned}$$

$n, m \in \mathbf{Z}$.

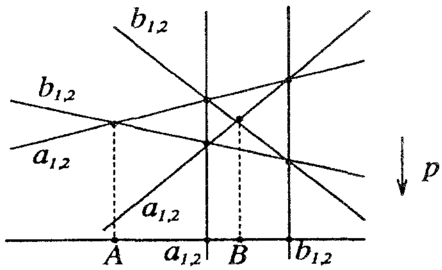

Fig. 12.

Then

$$\mathscr{G}=\{a_{1,3}, b_{1,3}, a_{2,3}, b_{2,3} \mid [\tilde{a}_{1,3}^{n,m}, (\tilde{a}_{2,3}^{n,m})^{-1}\tilde{b}_{1,3}^{n,m}\tilde{a}_{2,3}^{n,m}], [\tilde{a}_{2,3}^{n,m}, \tilde{b}_{2,3}^{n,m}]\}.$$

The three simplest relators are

$$[a_{1,3}, b_{1,3}], [a_{2,3}, a_{1,3}b_{2,3}a_{1,3}^{-1}], [a_{1,3}b_{2,3}a_{1,3}b_{2,3}^{-1}a_{1,3}^{-1}, a_{2,3}b_{1,3}a_{2,3}^{-1}].$$

After the substitution $b'_{2,3}=a_{1,3}b_{2,3}a_{1,3}^{-1}$, we have

$$[a_{1,3}, b_{1,3}], [a_{2,3}, b'_{2,3}], [b'_{2,3}a_{1,3}b'^{-1}_{2,3}, a_{2,3}b_{1,3}a_{2,3}^{-1}]$$

and we see that the third relation is not a consequence of the first two.

3.7. We can view the method used for the computation of the relations of $\mathscr{D}_n$ as a general one in the real case of hyperplane arrangement: practically, we have computed $\mathscr{D}_n$ as an extension of the group generated by the set $\{a_{i,n}, b_{i,n}\, i<n\}$ by $\mathscr{D}_{n-1}$, which is represented by the other generators $a_{i,j}$, $b_{i,j}$, $1 \leqslant i < j \leqslant n-1$, that is by the elements such that the equations of the corresponding hyperplanes do not contain x_n.

In the general case, we can do the same thing: divide the set of the hyperplanes S into the subgroups S_k, where S_k consists of the hyperplanes whose equations contain x_k and do not contain x_l, $l \geqslant k+1$. Rewrite the equations of the hyperplanes from S_k into the form $x_k=\Sigma_{i=1}^{k-1}\alpha_i x_i$ and define an order in S:

$$a<b, \text{ if } a \in S_j,\ b \in S_i,\ j>i \quad \text{or}$$

$$a,\ b \in S_k,\ a=\left\{x_k+\sum_{i=1}^{k-1}\alpha_i x_i=0\right\},\ b=\left\{x_k+\sum_{i=1}^{k-1}\beta_i x_i=0\right\},$$

$\alpha_p<\beta_p$ for some $p<k$, and $\alpha_i=\beta_i \forall i: p<i \leqslant k-1$,

that is $S_j<S_i$ for $j>i$, and in one S_k the order is lexigraphical.

If we define now a plane (t, s) in $\mathbf{C}^n$ in the same way as for $\mathscr{D}_n$ (see (7)), this order will be the original one, that is it will coincide with the order of $s_a(-\infty)$, where $s = s_a(t)$ is the equation of $a \in S$. This order will also coincide with the order of the intersections of every element $a \in S_k$ with the elements of the previous groups S_i, $i \leqslant k-1$.

The order of the intersections of elements of the same group is easily defined: the intersection (a, b), where

$$a = \{x_k + \Sigma_{i=1}^{k-1} \alpha_i x_i = 0\},$$

$$b = \{x_k + \Sigma_{i=1}^{k-1} \beta_i x_i = 0\},$$

occurs with the junior

$$c = \{\Sigma_{i=1}^{k-1} (\alpha_i - \beta_i) x_i = 0\};$$

we have only to define the place of this equation in the original hierarchy.

This method is especially useful in the case in which all the intersections of the elements of the same group are double, as in the $\mathscr{D}_n$ case. Then we can make all the conjugations by the elements of the junior groups only. But, if the deviation from this condition is not big, the method works as well.

3.8. As an illustration, we will compute the fundamental group of the B_n arrangement, formed by the following hyperplanes in $\mathbf{C}^n$:

$$a_{i,j} = \{x_j - x_i = 0\},\ b_{i,j} = \{x_j + x_i = 0\},\ c_i = \{x_i = 0\},\ 1 \leqslant i, j \leqslant n.$$

There are fourfold intersections $(a_{k,l}, c_l, b_{k,l}, c_k)$ here, and to conjugate c_l we have to use the elements $a_{i,l}$, which have already undergone a conjugation by the moment of the intersection.

We provide the resulting list of relations below ($1 \leqslant i < j < k < l \leqslant n$):

⟨1⟩ $(a_{k,l}^{a_{k,l-1} \cdots a_{k,k+1}}, c_l^{\tilde{a}_{l-1,l}^{-1} \cdots \tilde{a}_{k+1,l}^{-1}}, b_{k,l}^{b_{k,l-1} \cdots b_{k,k+1} a_{k-1,k} \cdots a_{1,k}}, c_k)$,

where $\tilde{a}_{j,l} = a_{j,l}^{a_{j,l-1} \cdots a_{j,j+1}}$

⟨2⟩ $(a_{k,l}^{a_{k,l-1} \cdots a_{k,k+1}}, a_{i,j})$

⟨3⟩ $(a_{j,l}^{a_{j,l-1} \cdots a_{j,k}}, a_{i,k})$

⟨4⟩ $(a_{i,l}^{a_{i,l-1} \cdots a_{i,k+1}}, a_{j,k})$

⟨5⟩ $(a_{k,l}^{a_{k,l-1} \cdots a_{k,k+1}}, b_{i,j})$

⟨6⟩ $(a_{j,l}^{a_{j,l-1} \cdots a_{j,k}}, b_{i,k})$

⟨7⟩ $(a_{i,l}^{a_{i,l-1} \cdots a_{i,k}}, b_{j,k})$

⟨8⟩ $(b_{k,l}^{b_{k,l-1} \cdots b_{k,k+1} a_{k-1,k} \cdots a_{1,k} c_k b_{1,k} \cdots b_{k-1,k}}, a_{i,j})$

$\langle 9\rangle$ $(b_{j,l}^{b_{j,l-1}\cdots b_{j,k+1}}, a_{i,k})$

$\langle 10\rangle$ $(b_{i,l}^{b_{i,l-1}\cdots b_{i,k+1}}, a_{j,k})$

$\langle 11\rangle$ $(b_{k,l}^{b_{k,l-1}\cdots b_{k,k+1}a_{k-1,k}\cdots a_{1,k}c_k b_{1,k}\cdots b_{k-1,k}}, b_{i,j})$

$\langle 12\rangle$ $(b_{j,l}^{b_{j,l-1}\cdots b_{j,k+1}}, b_{i,k})$

$\langle 13\rangle$ $(b_{i,l}^{b_{i,l-1}\cdots b_{i,k}}, b_{j,k})$

$\langle 14\rangle$ $(a_{j,l}^{a_{j,1}\cdots a_{j,j+1}}, a_{i,l}^{a_{i,l-1}\cdots a_{i,j+1}}, a_{i,j})$

$\langle 15\rangle$ $(b_{i,l}^{b_{i,l-1}\cdots b_{i,j+1}}, b_{j,l}^{b_{j,l-1}\cdots b_{j,j+1}a_{j-1,j}\cdots a_{i+1,j}}, a_{i,j})$

$\langle 16\rangle$ $(a_{i,l}^{a_{i,l-1}\cdots a_{i,j}}, b_{j,l}^{b_{j,l-1}\cdots b_{j,j+1}a_{j-1,j}\cdots a_{1,j}c_j b_{1,j}\cdots b_{i-1,j}}, b_{ij})$

$\langle 17\rangle$ $(a_{j,l}^{a_{j,l-1}\cdots a_{j,j+1}}, b_{i,l}^{b_{i,l-1}\cdots b_{i,j+1}}, b_{i,j})$

$\langle 18\rangle$ $(a_{j,l}^{a_{j,l-1}\cdots a_{j,j+1}}, c_i)$

$\langle 19\rangle$ $(a_{i,l}^{a_{i,l-1}\cdots a_{i,j}}, c_j)$

$\langle 20\rangle$ $(b_{j,l}^{b_{j,l-1}\cdots b_{j,j+1}a_{j-1,j}\cdots a_{1,j}c_j b_{1,j}\cdots b_{j-1,j}}, c_i)$

$\langle 21\rangle$ $(b_{i,l}^{b_{i,l-1}\cdots b_{i,j+1}}, c_j)$

$\langle 22\rangle$ $(c_l^{\tilde{a}_{l-1,l}^{-1}\cdots \tilde{a}_{j+1,l}^{-1}}, a_{i,j})$

$\langle 23\rangle$ $(c_l^{\tilde{a}_{l-1,l}^{-1}\cdots \tilde{a}_{j,l}^{-1}}, b_{i,j})$.

3.9. We worked with the generators of the fundamental group of the complement of a hyperplane arrangement which were chosen in the following way: these were the loops lying in a generic complex straight line and passing along straight segments to the hyperplanes of the arrangement: one loop for each hyperplane. One can ask, whether the condition that all of the segments be in the same complex straight line is necessary for the loops to form a set of generators. We investigate the case of a real arrangement in this section.

PROPOSITION. *Let $L_1, \ldots, L_d$ be a set of hyperplanes defined by real equations in $\mathbf{C}^n$ and let the coordinates of the reference point O be real. Let a_i, $i = 1, \ldots, d$ be the loop which passes along the segment of a real line from O to L_i bypassing L_j for $j \neq i$ by small arcs in the counterclockwise direction in the complexification of the real line, then passes around L_i and returns back following the same path. Then the corresponding elements of the fundamental group $G = \pi_1(\mathbf{C}^n \setminus \bigcup_{i=1}^d L_i)$ generate it.*

Proof. Draw an arbitrary straight line $q \in \mathbf{R}^n$ intersecting all of the L_i and containing O. Reorder the hyperplanes according to their distance from O measured along q. Firstly, if all of the segments joining O with the hyperplanes of the arrangement are contained in q, we have a system of generators of G; it is obtained from the standard one $a_1, \ldots, a_d$ corresponding to the complexification of q by an invertible transformation (see Figure 13).

When we rotate q around O in a two-dimensional plane in $\mathbf{R}^n$, the corresponding elements $a_{1,q}, \ldots, a_{d,q}$ of G are being replaced by certain conjugates only at the moments at which q passes through the intersections of certain hyperplanes of the arrangement; but the element a_1 corresponding to the hyperplane nearest to O along q stays unchanged (Figure 14). This means that the loops corresponding to arbitrary segments joining O with L_1 represent the same element a_1 of the group.

The element a_2 corresponding to the segment joining O with L_2 can get a conjugation only by a_1 at the moment of passing through an intersection of hyperplanes; continuing this reasoning, we see that the elements $a'_1, \ldots, a'_d$ corresponding to an arbitrary set of segments can be expressed in terms of $a_1, \ldots, a_d$ by the rule $a'_j = a_j^{\pm A_j}$, where A_j is a word in the alphabet $\{a_i^{\pm 1}, i < j\}$. Hence, conversely, the generators $a_1, \ldots, a_d$ can be expressed in terms of $a'_1, \ldots, a'_d$, and the latter elements generate G.

3.10. COROLLARY OF THE PROOF. If a hyperplane L_i can be joined with O by a segment which does not intersect other hyperplanes, the same element of G corresponds to any segment joining O with L_i.

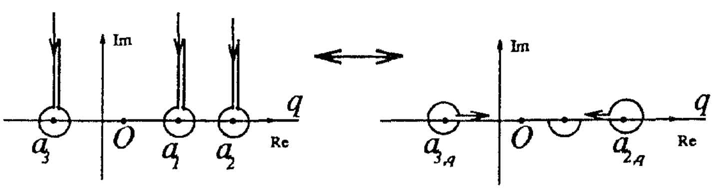

Fig. 13. $a_{1,q} \equiv a_1$, $a_{2,q} \equiv a_2^{a_1^{-1}}$, $a_{3,q} \equiv a_3^{-1}$.

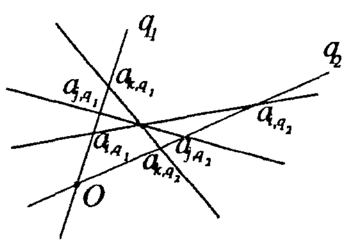

Fig. 14. $a_{i,q_2} = a_{i,q_1}$, $a_{k,q_2} = a_{k,q_1}$, $a_{j,q_2} = a_{j,q_1}^{a_{i,q_1}^{-1}} = a_{j,q_1}^{a_{k,q_1}}$ for $i > j > k$.

3.11. PROOF OF PROPOSITION 0.8. It is enough to prove that the monotone broken line consisting of two links can be straightened by the help of a homotopy in $\mathbf{C}^n \backslash S$. Fix the end points P_1, P_3 of the line and move its middle point P_2 along a segment of a straight line, which does not meet any intersections of hyperplanes of S, to the segment $[P_1, P_3]$ (Figure 15). Until P_2 arrives at a hyperplane of the arrangement, this movement does not cause any problems: it is equivalent to a movement along the real axis of the punctures; all the punctures should be passed around in the counterclockwise direction in the complexifications of the segments $[P_1, P_2]$, $[P_2, P_3]$ (Figure 16). Passing of P_2 through a hyperplane is shown on Figure 17.

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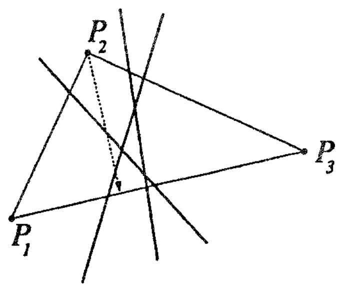

Fig. 15.

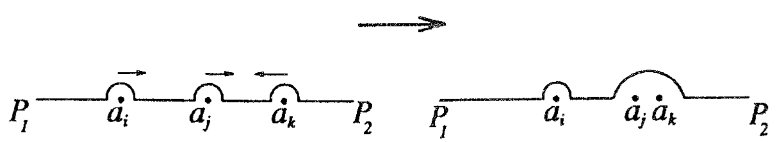

Fig. 16.

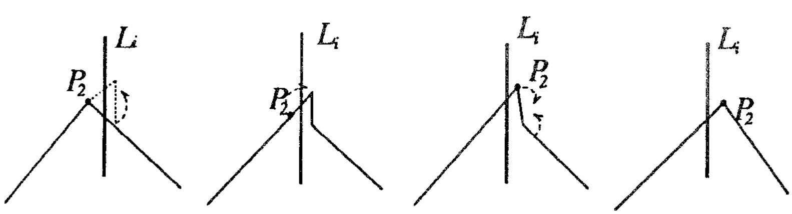

Fig. 17.

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Note added in proof: After the completion of our work on the typescript of the paper, we found out that our procedure for calculating a presentation of the fundamental group of the complement of an arrangement of hyperplanes had been obtained earlier by W. Arvola, as described in Section 5.3 of the monograph: P. Orlik, and M. Terao, *Arrangements of Hyperplanes*, Springer-Verlag, Berlin, Heidelberg, New York, 1992.