Nilsequences, null-sequences, and multiple correlation sequences

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Abstract

A (d-parameter) *basic nilsequence* is a sequence of the form $\psi(n) = f(a^n x)$, $n \in \mathbb{Z}^d$, where x is a point of a compact nilmanifold X, a is a translation on X, and $f \in C(X)$; a *nilsequence* is a uniform limit of basic nilsequences. If $X = G/\Gamma$ be a compact nilmanifold, Y is a subnilmanifold of X, $(g(n))_{n\in\mathbb{Z}^d}$ is a polynomial sequence in G, and $f \in C(X)$, we show that the sequence $\varphi(n) = \int_{g(n)Y} f$ is the sum of a basic nilsequence and a sequence that converges to zero in uniform density (a null-sequence). We also show that an integral of a family of nilsequences is a nilsequence plus a null-sequence. We deduce that for any invertible finite measure preserving system $(W, \mathcal{B}, \mu, T)$, polynomials $p_1, \ldots, p_k : \mathbb{Z}^d \longrightarrow \mathbb{Z}$, and sets $A_1, \ldots, A_k \in \mathcal{B}$, the sequence $\varphi(n) = \mu(T^{p_1(n)} A_1 \cap \ldots \cap T^{p_k(n)} A_k)$, $n \in \mathbb{Z}^d$, is the sum of a nilsequence and a null-sequence.

0. Introduction

Throughout the whole paper we will deal with "multiparameter sequences", – we fix $d \in \mathbb{N}$ and under "a sequence" will usually understand "a two-sided d-parameter sequence", that is, a mapping with domain $\mathbb{Z}^d$.

A (compact) r-step *nilmanifold* X is a factor space G/Γ, where G is an r-step nilpotent (not necessarily connected) Lie group and Γ is a discrete co-compact subgroup of G. Elements of G act on X by *translations*; an (r-step) *nilsystem* is an (r-step) nilmanifold $X = G/\Gamma$ with a translation $a \in G$ on it.

A basic r-step nilsequence is a sequence of the form $\psi(n) = f(\eta(n)x)$, $n \in \mathbb{Z}^d$, where x is a point of an r-step nilmanifold $X = G/\Gamma$, η is a homomorphism $\mathbb{Z}^d \longrightarrow G$, and $f \in C(X)$; *an r-step nilsequence* is a uniform limit of basic r-step nilsequences. The algebra of nilsequences is a natural generalization of Weyl's algebra of almost periodic sequences, which are just 1-step nilsequences. An "inner" characterization of nilsequences, in terms of their properties, is obtained in [HKM]; see also [HK2].

The term "nilsequence" was introduced in [BHK], where it was proved that for any ergodic finite measure preserving system $(W, \mathcal{B}, \mu, T)$, positive integer k, and sets

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$A_1,\dots,A_k\in\mathcal{B}$ the *multiple correlation sequence* $\varphi(n)=\mu\bigl(T^nA_1\cap\ldots\cap T^{kn}A_k\bigr)$, $n\in\mathbb{N}$, is a sum of a nilsequence and of a sequence that tends to zero in uniform density in $\mathbb{Z}^d$, *a null-sequence*. Our goal in this paper is to generalize this result to multiparameter polynomial multiple correlation sequences and general (non-ergodic) systems. We prove (in Section 5):

Theorem 0.1. *Let $(W,\mathcal{B},\mu,T)$ be an invertible measure preserving system with $\mu(W)<\infty$, let $p_1,\dots,p_k$ be polynomials $\mathbb{Z}^d\longrightarrow\mathbb{Z}$, and let $A_1,\dots,A_k\in\mathcal{B}$. Then the "multiple polynomial correlation sequence" $\varphi(n)=\mu\bigl(T^{p_1(n)}A_1\cap\ldots\cap T^{p_k(n)}A_k\bigr)$, $n\in\mathbb{Z}^d$, is a sum of a nilsequence and a null-sequence.*

(In [L6] this theorem was proved in the case $d=1$ and ergodic T.)

Based on the theory of nil-factors developed in [HK1] and, independently, in [Z], it is shown in [L3] that nilsystems are *characteristic* for multiple polynomial correlation sequences induced by ergodic systems, in the sense that, up to a null-sequence and an arbitrarily small sequence, any such correlation sequence comes from a nilsystem. This reduces the problem of studying "ergodic" multiple polynomial correlation sequences to nilsystems.

Let $X=G/\Gamma$ be a connected nilmanifold, let Y be a connected subnilmanifold of X, and let g be a polynomial sequence in G, that is, a mapping $\mathbb{Z}^d\longrightarrow G$ of the form $g(n)=a_1^{p_1(n)}\ldots a_r^{p_r(n)}$, $n\in\mathbb{Z}^d$, where $a_1,\dots,a_r\in G$ and $p_1,\dots,p_r$ are polynomials $\mathbb{Z}^d\longrightarrow\mathbb{Z}$. We investigate (in Section 3) the behavior of the sequence $g(n)Y$ of subnilmanifolds of X: we show that there is a subnilmanifold Z of X, containing Y, such that the sequence $g(n)$ only shifts Z along X, without distorting it, whereas, outside of a null-set of $n\in\mathbb{Z}^d$, $g(n)Y$ becomes more and more "dense" in $g(n)Z$:

Proposition 0.2. *Assume* (as we can) *that the orbit $g(n)Y$, $n\in\mathbb{Z}^d$, is dense in X, and let Z be the normal closure* (in the algebraic sense; see below) *of Y in X. Then for any $f\in C(X)$, the sequence $\lambda(n)=\int_{g(n)Y}f-\int_{g(n)Z}f$, $n\in\mathbb{Z}^d$, is a null-sequence.*

We have $\int_{g(n)Z}f=g(n)\hat{f}(g(n)e)$, $n\in\mathbb{Z}^d$, where $\hat{f}=E(f|X/Z)$ and $e=Z/Z\in X/Z$. (Here and below, $E(f|X')$ stands for the conditional expectation of a function $f\in L^1(X)$ with respect to a factor X' of X.) So, the sequence $\int_{g(n)Z}f$ is a basic nilseqeunce, and we obtain:

Theorem 0.3. *For any $f\in C(X)$ the sequence $\varphi(n)=\int_{g(n)Y}f$, $n\in\mathbb{Z}^d$, is the sum of a basic nilsequence and a null-sequence.*

Applying this result to the diagonal Y of the power X^k of the nilmanifold X, the polynomial sequence $g(n)=(a^{p_1(n)},\dots,a^{p_k(n)})$, $n\in\mathbb{Z}^d$, in G^k, and the function $f=1_{A_1}\otimes\ldots\otimes 1_{A_k}$, we obtain Theorem 0.1 in the ergodic case.

Our next step (Section 4) is to extend this result to the case of a non-ergodic T. Using the ergodic decomposition $W\longrightarrow\Omega$ of T we obtain a measurable mapping from Ω to the space of nilsequences–plus–null-sequences, which we then have to integrate over Ω. The integral of a family of null-sequences is a null-sequence, and creates no trouble. As for nilsequences, when we integrate them we arrive at the following problem: if $X=G/\Gamma$ is a nilmanifold, with $\pi\colon G\longrightarrow X$ being the factor mapping, and $\rho(a)$, $a\in G$, is a finite Borel measure on G, what is the limiting behavior of the measures $\pi_*(\rho(a^n))$ on X? (This is the question corresponding to the case $d = 1$; for $d \geq 2$ it is slightly more complicated.) We show that this sequence of measures tends to a linear combination of Haar measures on (countably many) subnilmanifolds of X, which are normal (and so travel, without distortion) in the closure of their orbits, and we again obtain:

Proposition 0.4. *For any $f \in C(X)$, the sequence $\varphi(n) = \int_G f(\pi(a^n))\,d\rho(a)$, $n \in \mathbb{Z}$, is a sum of a basic nilsequence and a null-sequence.*

(This proposition is a "nilpotent" extension of the following classical fact: if ρ is a finite Borel measure on the 1-dimensional torus $\mathbb{T}$, then its Fourier transform $\varphi(n) = \int_{\mathbb{T}} e^{-2\pi inx} d\rho(x)$ is the sum of an almost periodic sequence (a 1-step nilsequence; it corresponds to the atomic part of ρ) and a null-sequence (that corresponds to the non-atomic part of ρ).)

As a corollary, we obtain the remaining ingredient of the proof of Theorem 0.1:

Theorem 0.5. *Let Ω be a measure space and let φ_ω, $\omega \in \Omega$, be an integrable family of nilsequences; then the sequence $\varphi(n) = \int_\Omega \varphi_\omega(n)$ is a sum of a nilsequence and a null-sequence.*

Let us also mention *generalized* (or *bracket*) *polynomials*, – the functions constructed from ordinary polynomials using the operations of addition, multiplication, and taking the integer part, $[\cdot]$. (For example, $p_1[p_2[p_3] + p_4]$, where p_i are ordinary polynomials, is a generalized polynomial.) Generalized polynomials (gps) appear quite often (for example, the fractional part, and the distance to the nearest integer, of an ordinary polynomial are gps); they were systematically studied in [Hå1], [Hå2], [BL], and [L7]. Because of their simple definition, gps are nice objects to deal with. On the other hand, similarly to nilsequences, gps come from nilsystems: bounded gps (on $\mathbb{Z}^d$, in our case) are exactly the sequences of the form $h(g(n)x)$, $n \in \mathbb{Z}^d$, where h is a piecewise polynomial function on a nilmanifold $X = G/\Gamma$, $x \in X$, and g is a polynomial sequence in G (see [BL] or [L7]). Since any continuous function is uniformly approximable by piecewise polynomial functions (this follows by an application of the Weierstrass theorem in the fundamental domain of X), nilsequences are uniformly approximable by generalized polynomials. We obtain as a corollary that any multiple polynomial correlation sequence is, up to a null-sequence, uniformly approximable by generalized polynomials.

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1. Nilmanifolds

In this section we collect the facts about nilmanifolds that we will need below; details and proofs can be found in [M], [L1], [L2], [L4].

Throughout the paper, $X = G/\Gamma$ will be a compact nilmanifold, where G is a nilpotent Lie group and Γ is a discrete subgroup of G, and π will denote the factor mapping $G \longrightarrow X$. By 1_X we will denote the point $\pi(1_G)$ of X. By μ_X we will denote the normalized Haar measure on X.

By G^c we will denote the identity component of G. Note that if G is disconnected,

X can be interpreted as a nilmanifold, $X=G'/\Gamma'$, in different ways: if, for example, X is connected, then also $X=G^c/(\Gamma\cap G^c)$. If X is connected and we study the action on X of a sequence $g(n)$, $g\colon\mathbb{Z}^d\longrightarrow G$, we may always assume that G is generated by G^c and the range $g(\mathbb{Z}^d)$ of g. Thus, we may (and will) assume that the group G/G^c is finitely generated.

Every nilpotent Lie group G is a factor of a torsion free nilpotent Lie group. (As such, a suitable "free nilpotent Lie group" F can be taken. If G^c has k_1 generators, G/G^c has k_2 generators, and G is r-step nilpotent, then $F=\mathcal{F}/\mathcal{F}_{r+1}$, where $\mathcal{F}$ is the free product of k_1 copies of $\mathbb{R}$ and k_2 copies of $\mathbb{Z}$, and $\mathcal{F}_{r+1}$ is the $(r+1)$st term of the lower central series of $\mathcal{F}$.) Thus, we may always assume that G is torsion-free. The identity component G^c of G is then an exponential Lie group, and for every element $a\in G^c$ there exists a (unique) one-parametric subgroup a^t such that $a^1=a$.

If G is torsion free, it possesses a *Malcev basis* compatible with Γ, which is a finite set $\{e_1,\ldots,e_k\}$ of elements of Γ, with $e_1,\ldots,e_{k_1}\in G^c$ and $e_{k_1+1},\ldots,e_k\notin G^c$, such that every element $a\in G$ can be uniquely written in the form $a=e_1^{u_1}\ldots e_k^{u_k}$ with $u_1,\ldots,u_{k_1}\in\mathbb{R}$ and $u_{k_1+1},\ldots,u_k\in\mathbb{Z}$, and with $a\in\Gamma$ iff $u_1,\ldots,u_k\in\mathbb{Z}$; we call $u_1,\ldots,u_k$ *the coordinates* of a. Thus, Malcev coordinates define a homeomorphism $G\simeq\mathbb{R}^{k_1}\times\mathbb{Z}^{k-k_1}$, $a\leftrightarrow(u_1,\ldots,u_k)$, which maps Γ onto $\mathbb{Z}^k$.

The multiplication in G is defined by the (finite) multiplication table for the Malcev basis of G, whose entries are integers; it follows that there are only countably many non-isomorphic nilpotent Lie groups with cocompact discrete subgroups, and countably many non-isomorphic compact nilmanifolds.

Let X be connected. Then, under the identification $G^c\leftrightarrow\mathbb{R}^{k_1}$, the cube $[0,1)^{k_1}$ is the fundamental domain of X. We will call the closed cube $Q=[0,1]^{k_1}$ *the fundamental cube of X in G^c* and simetimes identify X with Q. When X is identified with its fundamental cube Q, the measure μ_X corresponds to the standard Lebesgue measure μ_Q on Q.

In Malcev coordinates, multiplication in G is a polynomial operation: there are polynomials $q_1,\ldots,q_k$ in $2k$ variables with rational coefficients such that for $a=e_1^{u_1}\ldots e_k^{u_k}$ and $b=e_1^{v_1}\ldots e_k^{v_k}$ we have $ab=e_1^{q_1(u_1,v_1,\ldots,u_k,v_k)}\ldots e_k^{q_k(u_1,v_1,\ldots,u_k,v_k)}$. This implies that "life is polynomial" in nilpotent Lie groups: in coordinates, homomorphisms between these groups are polynomial mappings, and connected closed subgroups of such groups are images of polynomial mappings and are defined by systems of polynomial equations.

A subnilmanifold Y of X is a closed subset of the form $Y=Hx$, where H is a closed subgroup of G and $x\in X$. For a closed subgroup H of G, the set $\pi(H)=H1_X$ is closed (and so, is a subnilmanifold) iff the subgroup $\Gamma\cap H$ is co-compact in H; we will call the subgroup H with this property *rational*. Any subnilmanifold Y of X has the form $\pi(aH)=a\pi(H)$, where H is a closed rational subgroup of G.

If Y is a subnilmanifold of X with $1_X\in Y$, then $H=\pi^{-1}(Y)$ is a closed subgroup of G, and $Y=\pi(H)=H1_X$. H, however, does not have to be the minimal subgroup with this property: if Y is connected, then the identity component H^c of H also satisfies $\pi(H^c)=Y$.

The intersection of two subnilmanifolds is a subnilmanifold (if nonempty).

Given a subnilmanifold Y of X, by μ_Y we will denote the normalized Haar measure on Y. Translations of subnilmanifods are measure preserving: we have $a_*\mu_Y=\mu_{aY}$ for all

$a \in G$.

Let Z be a subnilmanifold of X, $Z = Lx$, where L is a closed subgroup of G. We say that Z is *normal* if L is normal. In this case the nilmanifold $\widehat{X} = X/Z = G/(L\Gamma)$ is defined, and X splits into a disjoint union of fibers of the factor mapping $X \longrightarrow \widehat{X}$. (Note that if L is normal in G^c only, then the factor $X/Z = G^c/(L\Gamma)$ is also defined, but the elements of $G \setminus G^c$ do not act on it.)

One can show that a subgroup L is normal iff $\gamma L \gamma^{-1} = L$ for all $\gamma \in \Gamma$; hence, $Z = \pi(L)$ is normal iff $\gamma Z = Z$ for all $\gamma \in \Gamma$.

If H is a closed rational subgroup of G then its normal closure L (the minimal normal subgroup of G containing H) is also closed and rational, thus $Z = \pi(L)$ is a subnilmanifold of X. We will call Z *the normal closure* of the subnilmanifold $Y = \pi(H)$. If L is normal then the identity component of L is also normal; this implies that the normal closure of a connected subnilmanifold is connected.

If X is connected, *the maximal factor-torus* of X is the torus $[G^c, G^c]\backslash X$, and *the nil-maximal factor-torus* is $[G, G]\backslash X$. The nil-maximal factor-torus is a factor of the maximal one.

If $\eta\colon \mathbb{Z}^d \longrightarrow G$ is a homomorphism, then for any point $x \in X$ the closure of the orbit $\eta(\mathbb{Z}^d)x$ of x in X is a subnilmanifold V of X (not necessarily connected), and the sequence $\eta(n)x$, $n \in \mathbb{Z}^d$, is well distributed in V. (This means that for any function $f \in C(V)$ and any Følner sequence (Φ_N) in $\mathbb{Z}^d$, $\lim_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} f(\eta(n)x) = \int_Y f\,d\mu_V$.) If X is connected, the sequence $\eta(n)x$, $n \in \mathbb{Z}^d$, is dense, and so, well distributed in X iff the image of this sequence is dense in the nil-maximal factor-torus of X. All the same is true for the orbit of any subnilmanifold Y of X: the closure of $\bigcup_{n\in\mathbb{Z}^d} \eta(n)Y$ is a subnilmanifold W of X; the sequence $\eta(n)Y$, $n \in \mathbb{Z}^d$, is well distributed in W (this means that for any function $f \in C(W)$ and any Følner sequence (Φ_N) in $\mathbb{Z}^d$, $\lim_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} \int_{\eta(n)Y} f(x)\,d\mu_{\eta(n)Y} = \int_Y f\,d\mu_V$); and, in the case X is connected, the sequence $\eta(n)Y$ is well distributed in X iff its image is dense in the nil-maximal factor-torus of X.

A polynomial sequence in G is a sequence of the form $g(n) = a_1^{p_1(n)} \cdots a_k^{p_k(n)}$, $n \in \mathbb{Z}^d$, where $a_1, \dots, a_k \in G$ and $p_1, \dots, p_k$ are polynomials $\mathbb{Z}^d \longrightarrow \mathbb{Z}$. Let g be a polynomial sequence in G and let $x \in X$. Then the closure V of the orbit $g(\mathbb{Z}^d)x$ is a finite disjoint union of connected subnilmanifolds of X, and $g(n)x$ visits these subnilmanifolds periodically: there exists $l \in \mathbb{N}$ such that for any $i \in \mathbb{Z}^d$, all the elements $g(lm+i)x$, $m \in \mathbb{Z}^d$, belong to the same connected component of V. If V is connected, then the sequence $g(n)x$, $n \in \mathbb{Z}^d$, is well distributed in V. In the case X is connected, the sequence $g(n)x$, $n \in \mathbb{Z}^d$, is dense, and so, well distributed in X iff the image of this sequence is dense in the maximal factor-torus of X. All the same is true for the orbit $g(Z^d)Y$ of any connected subnilmanifold Y of X under the action of g: its closure W is a finite disjoint union of connected subnilmanifolds of X, visited periodically; if W is connected, then the sequence $g(n)Y$, $n \in \mathbb{Z}^d$, is well distributed in W; and, if X is connected, the sequence $g(n)Y$ is well distributed in X iff its image is dense in the maximal factor-torus of X.

The following proposition, which is a corollary (of a special case) of the result obtained in [GT], says that "almost every" subnilmanifold of X is distributed in X "quite uniformly". (See Appendix in [L6] for details.)

Proposition 1.1. *Let X be connected. For any $C > 0$ and any $\varepsilon > 0$ there are finitely many subnilmanifolds $V_1, \ldots, V_r$ of X, connected and containing 1_X, such that for any connected subnilmanifold Y of X with $1_X \in Y$, if $Y \not\subseteq V_i$ for all $i \in \{1, \ldots, r\}$, then $\left|\int_Y f\,d\mu_Y - \int_X f\,d\mu_X\right| < \varepsilon$ for all functions f on X with $\sup_{x \neq y} |f(x) - f(y)|/\operatorname{dist}(x, y) \leq C$.*

(This is in complete analogy with the situation on tori: if X is a torus, for any $\varepsilon > 0$ there are only finitely many subtori $V_1, \ldots, V_r$ such that any subtorus Y of X that contains 0 and is not contained in $\bigcup_{i=1}^{r} V_i$ is ε-dense and "ε-uniformly distributed" in X.)

2. Nilsequences, null-sequences, and generalized polynomials

We will deal with the space $l^\infty = l^\infty(\mathbb{Z}^d)$ of bounded sequences $\varphi\colon \mathbb{Z}^d \longrightarrow \mathbb{C}$, with the norm $\|\varphi\| = \sup_{n \in \mathbb{Z}^d} |\varphi(n)|$.

A basic r-step nilsequence is an element of l^∞ of the form $\psi(n) = f(\eta(n)x)$, $n \in \mathbb{Z}^d$, where x is a point of an r-step nilmanifold $X = G/\Gamma$, η is a homomorphism $\mathbb{Z}^d \longrightarrow G$, and $f \in C(X)$. We will denote the algebra of basic r-step nilsequences by $\mathcal{N}_r^o$, and the algebra $\bigcup_{r \in \mathbb{N}} \mathcal{N}_r^o$ of all basic nilsequences by $\mathcal{N}^o$. We will denote the closure of $\mathcal{N}_r^o$, $r \in \mathbb{N}$, in l^∞ by $\mathcal{N}_r$, and the closure of $\mathcal{N}^o$ by $\mathcal{N}$; the elements of these algebras will be called *r-step nilsequences* and, respectively, *nilsequences*.

Given a polynomial sequence $g(n) = a_1^{p_1(n)} \cdots a_k^{p_k(n)}$, $n \in \mathbb{Z}^d$, in a nilpotent group with $\deg p_i \leq s$ for all i, we will say that g has *naive degree* $\leq s$. (The term "degree" was already reserved for another parameter of a polynomial sequence.) We will call a sequence of the form $\psi(n) = f(g(n)x)$, where x is a point of an r-step nilmanifold $X = G/\Gamma$, g is a polynomial sequence of naive degree $\leq s$ in G, and $f \in C(X)$, *a basic polynomial r-step nilsequence of degree* $\leq s$. We will denote the algebra of basic polynomial r-step nilsequences of degree $\leq s$ by $\mathcal{N}_{r,s}^o$ and the closure of this algebra in l^∞ by $\mathcal{N}_{r,s}$. It is shown in [L2] (see proof of Theorem B*) that any basic polynomial r-step nilsequence of degree $\leq s$ is a basic l-step nilsequence, where $l = 2rs$; we introduce this notion here only in order to keep trace of the parameters r, s of the "origination" of a nilsequence. So, for any r and s, $\mathcal{N}_{r,s}^o \subseteq \mathcal{N}_{2rs}^o$; since also $\mathcal{N}_r^o \subseteq \mathcal{N}_{r,1}^o$, we have $\bigcup_{r,s \in \mathbb{N}} \mathcal{N}_{r,s}^o = \mathcal{N}^o$.

We will also need the following lemma, saying, informally, that the operation of "alternation" of sequences preserves the algebras of nilsequences:

Lemma 2.1. *Let $k \in \mathbb{N}$ and let $\psi_i \in \mathcal{N}_{r,s}^o$ (respectively, $\mathcal{N}_r^o$), $i \in \{0, \ldots, k-1\}^d$. Then the sequence ψ defined by $\psi(n) = \psi_i(m)$ for $n = km + i$ with $m \in \mathbb{Z}^d$, $i \in \{0, \ldots, k-1\}^d$, is also in $\mathcal{N}_{r,s}^o$ (respectively, $\mathcal{N}_r^o$)).*

Proof. Put $I = \{0, \ldots, k-1\}^d$. For each $i \in I$, let $X_i = G_i/\Gamma_i$ be the r-step nilmanifold, g_i be the polynomial sequence in G_i of naive degree $\leq s$, $x_i \in X_i$ be the point, and $f_i \in C(X_i)$ be the function such that $\psi_i(n) = f(g_i(n)x_i)$, $n \in \mathbb{Z}^d$. If, for some i, G_i is not connected, it is a factor-group of a free r-step nilpotent group with both continuous and discrete generators, which, in its turn, is a subgroup of a free r-step nilpotent group with only continuous generators; thus after replacing, if needed, X_i by a larger nilmanifold and extending f_i to a continuous function on this nilmanifold we may assume that every G_i

is connected and simply-connected. In this case for any element $b \in G_i$ and any $l \in \mathbb{N}$ an l-th root $b^{1/l}$ exists in G_i, and thus the polynomial sequence $b^{p(n)}$ in G_i makes sense even if a polynomial p has non-integer rational coefficients. Thus, for each $i \in I$ we may construct a polynomial sequence g_i' in G_i, of the same naive degree as g_i, such that $g_i'(km+i) = g_i(m)$ for all $m \in \mathbb{Z}^d$. Put $G = \mathbb{Z}^d \times \prod_{i\in I} G_i$, $X = (\mathbb{Z}/(k\mathbb{Z}))^d \times \prod_{i\in I} X_i$, $g(n) = \big(n, (g_i'(n),\ i \in I)\big)$ for $n \in \mathbb{Z}^d$, $x = \big(0, (x_i,\ i \in I)\big) \in X$, and $f\big(i, (y_i,\ i \in I)\big) = f_i(y_i)$ for $(i, (y_i,\ i \in I)) \in X$. Then X is an r-step nilmanifold, $f \in C(X)$, and thus the sequence $\psi(n) = f(g(n)x) = f_i(g_i'(n)x_i) = f_i(g_i(m)x_i) = \psi_i(m)$, $n = km+i$, $m \in \mathbb{Z}^d$, $i \in I$, is in $\mathcal{N}_{r,s}$. ■

A set $S \subset \mathbb{Z}^d$ is said to be of *uniform* (or *Banach*) *density zero* if for any Følner sequence $(\Phi_N)_{N=1}^\infty$ in $\mathbb{Z}^d$, $\lim_{N\to\infty} |S \cap \Phi_N|/|\Phi_N| = 0$. A sequence $(\omega_n)_{n\in\mathbb{Z}^d}$ in a topological space Ω converges to $\omega \in \Omega$ *in uniform density* if for every neighborhood U of ω the set $\big(\{n \in \mathbb{Z}^d : \omega_n \notin U\}\big)$ is of uniform density zero.

We will say that a sequence $\lambda \in l^\infty$ is *a null-sequence* if it tends to zero in uniform density. This is equivalent to $\lim_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} |\lambda(n)| = 0$ for any Følner sequence $(\Phi_N)_{N=1}^\infty$ in $\mathbb{Z}^d$, and is also equivalent to $\lim_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} |\lambda(n)|^2 = 0$ for any Følner sequence $(\Phi_N)_{N=1}^\infty$ in $\mathbb{Z}^d$. We will denote the set of (bounded) null-sequences by $\mathcal{Z}$. $\mathcal{Z}$ is a closed ideal in l^∞.

The algebra $\mathcal{Z}$ is orthogonal to the algebra $\mathcal{N}$ in the following sense:

Lemma 2.2. *For any $\psi \in \mathcal{N}$ and $\lambda \in \mathcal{Z}$, $\|\psi + \lambda\| \geq \|\psi\|$.*

Proof. Let $c \geq \|\psi + \lambda\|$. Nilsystems are distal systems (see, for example, [L1]), which implies that every point returns to any its neighborhood regularly. It follows that if $|\psi(n)| > c$ for some n, then the set $\big\{n \in \mathbb{Z}^d : |\psi(n) > c|\big\}$ has positive lower density, and then $\psi(n) + \lambda(n) > c$ for some n, contradiction. Hence, $|\psi(n)| \leq c$ for all n. ■

It follows that $\mathcal{N} \cap \mathcal{Z} = 0$.

We will denote the algebras $\mathcal{N}_r^o + \mathcal{Z}$, $\mathcal{N}_r + \mathcal{Z}$, $\mathcal{N}_{r,s}^o + \mathcal{Z}$, $\mathcal{N}_{r,s} + \mathcal{Z}$, $\mathcal{N}^o + \mathcal{Z}$, and $\mathcal{N} + \mathcal{Z}$ by $\mathcal{M}_r^o$, $\mathcal{M}_r$, $\mathcal{M}_{r,s}^o$, $\mathcal{M}_{r,s}$, $\mathcal{M}^o$, and $\mathcal{M}$ respectively.

Lemma 2.2 implies:

Lemma 2.3. *The algebras $\mathcal{M}$, $\mathcal{M}_r$, and $\mathcal{M}_{r,s}$, $r, s \in \mathbb{N}$, are all closed, and the projections $\mathcal{M} \longrightarrow \mathcal{N}$, $\mathcal{M} \longrightarrow \mathcal{Z}$ are continuous.*

Proof. If a sequence $(\psi_n + \lambda_n)$ with $\psi_n \in \mathcal{N}$, $\lambda_n \in \mathcal{Z}$, converges to $\varphi \in l^\infty$, then since $\|\psi_n\| \leq \|\psi_n + \lambda_n\|$ for all n, the sequence (ψ_n) is Cauchy, and so converges to some $\psi \in \mathcal{N}$. Then (λ_n) also converges, to some $\lambda \in \mathcal{Z}$, and so $\varphi = \psi + \lambda \in \mathcal{M}$. All the same is true for $\mathcal{M}_r$ and $\mathcal{M}_{r,s}$, instead of $\mathcal{M}$, for all r and s.

For $\varphi_1 = \psi_1 + \lambda_1$ and $\varphi_2 = \psi_2 + \lambda_2$ with $\psi_1, \psi_2 \in \mathcal{N}$ and $\lambda_1, \lambda_2 \in \mathcal{Z}$ we have $\|\psi_1 - \psi_2\| \leq \|\varphi_1 - \varphi_2\|$, so the projection $\mathcal{M} \longrightarrow \mathcal{N}$, $\psi + \lambda \mapsto \psi$, is continuous, and so the projection $\mathcal{M} \longrightarrow \mathcal{Z}$, $\psi + \lambda \mapsto \lambda$, is also continuous. ■

Generalized polynomials on $\mathbb{Z}^d$ are the functions on $\mathbb{Z}^d$ constructed from ordinary polynomials using the operations of addition, multiplication, and the operation of taking the integer part. A function h on a nilmanifold X is said to be *piecewise polynomial* if it can be represented in the form $h(x)=q_i(x)$, $x\in Q_i$, $i=1,\dots,k$, where $X=\bigcup_{i=1}^k Q_i$ is a finite partition of X and, in Malcev coordinates on X, for every i the set Q_i is defined by a system of polynomial inequalities and q_i is a polynomial function. (Since multiplication in a nilpotent Lie group is polynomial, this definition does not depend on the choice of coordinates on X; see [BL].) It was shown in [BL] (and also, in a simpler way, in [L7]), that a sequence $v\in l^\infty$ is a generalized polynomial iff there is a nilmanifold $X=G/\Gamma$, a piecewise polynomial function h on X, a polynomial sequence g in G, and a point $x\in X$ such that $v(n)=h(g(n)x)$, $n\in\mathbb{Z}^d$.

Let $\mathcal{P}^o$ be the algebra of bounded generalized polynomials on $\mathbb{Z}^d$ and $\mathcal{P}$ be the closure of $\mathcal{P}^o$ in l^∞. Since (by the Weierstrass approximation theorem) any continuous function on a compact nilmanifold X is uniformly approximable by piecewise polynomial functions, any basic nilsequence is uniformly approximable by bounded generalized polynomials, and so, is contained in $\mathcal{P}$. Hence, $\mathcal{N}\subset\mathcal{P}$, and $\mathcal{M}\subset\mathcal{P}+\mathcal{Z}$. The inverse inclusion does not hold, since not all piecewise polynomial functions are uniformly approximable by continuous functions; however, they are – on the complement of a set of arbitrarily small measure, which implies that generalized polynomials are also approximable by nilsequences, – in a certain weaker topology in l^∞.

3. Distribution of a polynomial sequence of subnilmanifolds

Let Y be a connected subnilmanifold of the (connected) nilmanifold X, and let $g(n)$, $n\in\mathbb{Z}^d$, be a polynomials sequence in G. We will investigate how the sequence $g(n)Y$ of sunilmanifolds of X is distributed in X.

Proposition 3.1. *Let $X=G/\Gamma$ be a connected nilmanifold, let Y be a connected subnilmanifold of X, and let $g\colon\mathbb{Z}^d\longrightarrow G$ be a polynomial sequence in G with $g(0)=1_G$. Assume that $g(\mathbb{Z}^d)Y$ is dense in X, and that G is generated by G^c and the range $g(\mathbb{Z}^d)$ of g. Let Z be the normal closure of Y in X; then for any $f\in C(X)$, $\lambda(n)=\int_{g(n)Y}f\,d\mu_{g(n)Y}-\int_{g(n)Z}f\,d\mu_{g(n)Z}$, $n\in\mathbb{Z}^d$, is a null-sequence.*

Proof. Let $f\in C(X)$ and let $\varepsilon>0$; we have to show that the set $\bigl\{n\in\mathbb{Z}^d:\bigl|\int_{g(n)Y}f\,d\mu_{g(n)Y}-\int_{g(n)Z}f\,d\mu_{g(n)Z}\bigr|\geq\varepsilon\bigr\}$ has zero uniform density in $\mathbb{Z}^d$. After replacing f by a close function we may assume that f is Lipschitz, so that $C=\sup_{x\neq y}|f(x)-f(y)|/\operatorname{dist}(x,y)$ is finite. Choose Malcev's coordinates in G^c, and let $Q\subset G^c$ be the corresponding fundamental cube. Since Z is normal in X, $aZ=bZ$ whenever $a=b\bmod\Gamma$, and $\bigcup_{a\in Q}aZ$ is a partition of X.

We first want to determine for which $a\in G$ one has $\bigl|\int_{aY}f_{aY}\,d\mu_{aY}-\int_{aZ}f\,d\mu_{aZ}\bigr|\geq\varepsilon$. For every $b\in Q$, by Proposition 1.1, applied to the nilmanifold bZ, there exist proper subnilmanifolds $V_{b,1},\dots,V_{b,r_b}$ of Z such that $\bigl|\int_W f\,d\mu_W-\int_{bZ}f_{bZ}\,d\mu_{bZ}\bigr|<\varepsilon/2$ whenever W is a subnilmanifold of bZ with $b\in W\not\subseteq bV_{b,i}$, $i=1,\dots,r_b$. By continuity, for each $b\in Q$ there exists a neighborhood U_b of b such that for all $a\in U_b$, $\bigl|\int_W f\,d\mu_W-\int_{aZ}f\,d\mu_{aZ}\bigr|<\varepsilon$ whenever $a\in W\subseteq aZ$ and $W\not\subseteq aV_{b,i}$, $i=1,\dots,r_b$. Using the compactness of the closure of Q, we can choose $b_1,\dots,b_l\in Q$ such that $Q\subseteq\bigcup_{j=1}^l U_{b_j}$; let $V=\bigcup_{\substack{j=1,\dots,l\\ i=1,\dots,r_j}}V_{b_j,i}$. Then for any $b\in Q$, for any subnilmanifold W of bZ with $b\in W\not\subseteq V$ one has $\bigl|\int_W f\,d\mu_W-$

$\int_{bZ} f\,d\mu_{bZ}\big| < \varepsilon$. Now let $a \in G$, and let $b \in Q$ be such that $a = b \bmod \Gamma$. Then $aY \subseteq aZ = bZ$, thus, if $aY \not\subseteq bV$, then $\big|\int_{aY} f\,d\mu_{aY} - \int_{aZ} f\,d\mu_{aZ}\big| < \varepsilon$. Hence, $\big|\int_{aY} f\,d\mu_{aY} - \int_{aZ} f\,d\mu_{aZ}\big| \geq \varepsilon$ only if $(a1_X, aY) \subseteq (b1_X, bV)$ for some $b \in Q$.

Let $N = \big\{(b1_X, bV),\ b \in Q\big\}$; we have to prove that the the set $\big\{n \in \mathbb{Z}^d : (g(n)1_X, g(n)Y) \subseteq N\big\}$ has zero uniform density in $\mathbb{Z}^d$. For this purpose we are going to find the closure of the sequence $\widetilde{Y}_n = (g(n)1_X, g(n)Y)$, $n \in \mathbb{Z}^d$, (more exactly, of the union $\bigcup_{n\in\mathbb{Z}^d} \widetilde{Y}_n$), the orbit of the subnilmanifold $\widetilde{Y} = (1_X, Y)$ of $X \times X$ under the polynomial action $(g(n), g(n))$, $n \in \mathbb{Z}^d$. Assume for simplicity that the closure R of the orbit $\big\{g(n)1_X,\ n \in \mathbb{Z}^d\big\}$ is connected, and let P be the closed connected subgroup of G such that $\pi(P) = R$. (If R is disconnected we pass to a sublattice of $\mathbb{Z}^d$ and its cosets to deal with individual connected components of R.) We will also assume that $Y \ni 1_X$.

Lemma 3.2. *The closure of the sequence $(\widetilde{Y}_n)$ is the subnilmanifold $D = \big\{(a1_X, aZ),\ a \in P\big\} = \big\{(a1_X, aZ),\ a \in P \cap Q\big\}$ of $X \times X$.*

Proof. Let L be the closed connected subgroup of G such that $\pi(L) = Z$, and let $K = \big\{(a, au),\ a \in P,\ u \in L\big\}$; since L is normal in G, K is a (closed rational) subgroup of $G \times G$, and we have $D = K/((\Gamma \times \Gamma) \cap K)$.

For any $n \in \mathbb{Z}^d$ we have $\widetilde{Y}_n \subseteq D$ (since $g(n)1_X \in R$, so $g(n) \in P\Gamma$, so $g(n)L \subseteq P\Gamma L = PL\Gamma$), and we have to show that the sequence $(\widetilde{Y}_n)$ is dense in D. For this it suffices to prove that the image of this sequence is dense in the maximal torus $T = [K, K]\backslash D$ of D. Since L is normal, we have $[K, K] = \big\{(a, au),\ a \in [P, P],\ u \in [P, L][L, L]\big\}$, and the torus T is the factor of the commutative group $K/[K, K]$ by the image Λ in this group of the lattice $\Gamma \times \Gamma$. Let H be the closed connected subgroup of G such that $\pi(H) = Y$; then $L = H[H, G]$, so

$$K/[K, K] = \big\{(a, avw),\ a \in P,\ v \in H,\ w \in [H, G]\big\}/\big\{(a, au),\ a \in [P, P],\ u \in [P, L][L, L]\big\}.$$

By assumption, G is generated by G^c and g. Since the orbit $\{g(n)Z,\ n \in \mathbb{Z}^d\}$ is dense in X and Z is normal, the orbit $\{g(n)1_{X/Z},\ n \in \mathbb{Z}^d\}$ is dense in X/Z, so $P/(P \cap L) = G^c/L$, so $G^c = PL$. Hence, $[H, G] = [H, g][H, P][H, L]$. For any n, $g(n) = u_n\gamma_n$ for some $u_n \in P$ and $\gamma_n \in \Gamma$, thus, modulo $[P, L][L, L]$, the group $[H, G]$ is generated by $\{[H, \gamma_n],\ n \in \mathbb{Z}^d\}$.

The closure B of the image of the sequence $(\widetilde{Y}_n)$ in T is a subtorus of T. Since the sequence $g(n)1_X$ is dense in $R = \pi(P)$, the subtorus $T_1 = \{(a, a),\ a \in P\}/([K, K]\Lambda)$ of T is the closure of the image of the sequence $(g(n)1_X, g(n)1_X)$ and so, is contained in B. Also, the subtorus $T_2 = \{(1_X, u),\ u \in H\}/([K, K]\Lambda)$ is contained in B. Finally, for $n \in \mathbb{Z}^d$ and $c \in H$ we have

$$(g(n), g(n)c) = (u_n, u_n\gamma_n c) = (u_n 1_X, u_n c[c, \gamma_n^{-1}]\gamma_n).$$

Taken modulo $[K, K]\Lambda$, these elements of B generate T modulo $T_1 + T_2$, so $B = T$. ∎

It follows that the sequence $\widetilde{Y}_n$, $n \in \mathbb{Z}^d$, is well distributed in D. The set $N = \big\{(b1_X, bV),\ b \in Q\big\}$ is a compact subset of D of zero measure, thus, the set $\big\{n \in \mathbb{Z}^d : (g(n)1_X, g(n)Y) \subseteq N\big\}$ has zero uniform density in $\mathbb{Z}^d$. ∎

Theorem 3.3. *Let $X = G/\Gamma$ be an r-step nilmanifold, let Y be a subnilmanifold of X, let g be a polynomial sequence in G of naive degree $\leq s$, let $f \in C(X)$. Then the sequence $\varphi(n) = \int_{g(n)Y} f \, d\mu_{g(n)Y}$, $n \in \mathbb{Z}^d$, is contained in $\mathcal{M}^o_{r,s}$.*

Proof. We may assume that $Y \ni 1_X$. After replacing f by $f(g(0)x)$, $x \in X$, we may assume that $g(0) = 1_X$. We may also replace X by the closure of the orbit $g(\mathbb{Z}^d)Y$, and we may assume that G is generated by G^c and the range of g.

First, let X and Y be both connected. Let Z be the normal closure of Y in X; then by Proposition 3.1, $\varphi(n) = \int_{g(n)Z} f \, d\mu_{g(n)Z} + \lambda_n$, $n \in \mathbb{Z}^d$, with $\lambda \in \mathcal{Z}$. Define $\widehat{X} = X/Z$, $\hat{x} = \{Z\} \in \widehat{X}$, and $\hat{f} = E(f|\widehat{X}) \in C(\widehat{X})$; then $\int_{g(n)Z} f \, d\mu_{g(n)Z} = \hat{f}(g(n)\hat{x})$, $n \in \mathbb{Z}^d$, and the sequence $\hat{f}(g(n)\hat{x})$, $n \in \mathbb{Z}^d$, is in $\mathcal{N}^o_{r,s}$, so $\varphi \in \mathcal{M}^o_{r,s}$.

Now assume that Y is connected but X is not. Then, by [L2], there exists $k \in \mathbb{N}$ such that $\overline{g(k\mathbb{Z}^d + i)Y}$ is connected for every $i \in \{0, \dots, k-1\}^d$. Thus, for every $i \in \{0, \dots, k-1\}^d$, $\varphi(kn+i) \in \mathcal{M}^o_{r,s}$, and the assertion follows from Lemma 2.1.

Finally, if Y is disconnected and $Y_1, \dots, Y_r$ are the connected components of Y, then $\int_{g(n)Y} f \, d\mu_{g(n)Y} = \sum_{j=1}^r \int_{g(n)Y_j} f \, d\mu_{g(n)Y_j}$, $n \in \mathbb{Z}^d$, and the result holds since it holds for $Y_1, \dots, Y_r$. ■

4. Integrals of null- and of nil-sequences

On l^∞ and, thus, on $\mathcal{N}$, $\mathcal{Z}$ and $\mathcal{M}$ we will assume the Borel σ-algebra induced by the weak topology.

We start with integration of null-sequences:

Lemma 4.1. *Let (Ω, ν) be a measurable space and let $\Omega \longrightarrow \mathcal{Z}$, $\omega \mapsto \lambda_\omega$, be an integrable mapping. Then the sequence $\lambda(n) = \int_\Omega \lambda_\omega(n) \, d\nu$ is in $\mathcal{Z}$ as well.*

(We say that a mapping $\Psi \colon \Omega \longrightarrow l^\infty$ is *integrable* if it is measurable and $\int_\Omega \|\Psi\| \, d\nu < \infty$.)

Proof. For each $\omega \in \Omega$, $\lim_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} |\lambda_\omega(n)| = 0$ for any Følner sequence $(\Phi_N)_{N=1}^\infty$ in $\mathbb{Z}^d$. By the dominated convergence theorem,

$$\limsup_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} |\lambda(n)| = \limsup_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} \Bigl| \int_\Omega \lambda_\omega(n) \, d\nu \Bigr|$$
$$\leq \lim_{N\to\infty} \int_\Omega \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} |\lambda_\omega(n)| \, d\nu = \int_\Omega \lim_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} |\lambda_\omega(n)| \, d\nu = 0.$$

So, $\lambda \in \mathcal{Z}$. ■

For nilsequences we have:

Proposition 4.2. *Let (Ω, ν) be a measure space and let $\Omega \longrightarrow \mathcal{N}$, $\omega \mapsto \varphi_\omega$, be an integrable mapping. Then the sequence $\varphi(n) = \int_\Omega \varphi_\omega(n) \, d\nu$ belongs to $\mathcal{M}$. (If, for some r, $\varphi_\omega \in \mathcal{N}_r$ for all ω, then $\varphi \in \mathcal{M}_r$.)*

To simplify notation, let us start with the case $d=1$. We are going to reduce Proposition 4.2 to a statement concerning a sequence of measures on a nilmanifold.[1] Since $\mathcal{M}$ is closed in l^∞, we are allowed to replace the mapping φ_ω from Ω to $\mathcal{N}$ by a close mapping φ'_ω: we are done if for any $\varepsilon>0$ we can find a mapping $\Omega\longrightarrow\mathcal{N}$, $\omega\mapsto\varphi'_\omega$, with $\|\int_\Omega\|\varphi_\omega-\varphi'_\omega\|_{l^\infty}\,d\nu<\varepsilon$ and such that the assertion of Proposition 4.2 holds for φ'_ω. Fix $\varepsilon>0$. First, after replacing Ω by $\Omega'\subseteq\Omega$ with $\nu(\Omega')<\infty$ such that $\int_{\Omega\setminus\Omega'}\|\varphi_\omega\|_{l^\infty}\,d\nu<\varepsilon$, we may assume that $\nu(\Omega)<\infty$. Next, since the set $\mathcal{N}^o$ of basic nilsequences is dense in $\mathcal{N}$, we may replace the nilsequences φ_ω by close basic nilsequences, if we manage to do this in a measurable way. We will, as we may, deal with $\mathbb{R}$-valued nilsequences. Let $X=G/\Gamma$ be a nilmanifold where G is a simply connected nilpotent Lie group and Γ is a lattice in G, and let $\pi\colon G\longrightarrow X$ be the projection. We may assume that G has the same number of connected components as X, then G is homeomorphic to $\mathbb{R}^{\dim G}\times F$, where F is a finite set, with Γ corresponding to $\mathbb{Z}^{\dim G}$; this homeomorphism induces a natural metric on G and on X. For $k\in\mathbb{N}$ let Q_k be the set of elements of G at the distance $\leq k$ from 1_G and let L_k be the set of Lipschitz functions on X with Lipshitz constant k and of modulus $\leq k$. The subset $Q_k\times L_k$ of $G\times C(X)$ is compact; the "nilsequence reading" mapping $\Psi\colon G_k\times C(X_k)\longrightarrow\mathcal{N}$, $\Psi(a,h)(n)=h(\pi(a^n))$, is continuous with respect to the weak topology on $\mathcal{N}$; thus the set $\mathcal{L}_{X,k}=\Psi(Q_k\times L_k)\subset\mathcal{N}^o$ is compact in this topology. Fix a countable set S dense in $\mathcal{L}_{X,k}$ in the weak topology and enumerate it. Let $\varphi\in\mathcal{N}$. For each $j\in\mathbb{N}$ let ψ_j be the element of S for which

(i) the sum $\sum_{n=-j}^{j}|\varphi(n)-\psi_j(n)|$ is minimal;

(ii) among the elements of S for which (i) holds, the vector $\bigl(\psi(0),\psi(-1),\psi(1),\ldots,\psi(-j),\psi(j)\bigr)$ is minimal for $\psi=\psi_j$ with respect to the lexicographic order;

(iii) and among the elements of S for which (i) and (ii) hold, ψ_j has the minimal number under the ordering of S.

Put $\zeta_{X,k,j}(\varphi)=\psi_j$; then $\zeta_{X,k,j}$ is a measurable mapping $\mathcal{N}\longrightarrow\mathcal{L}_{X,k}$. For any $\varphi\in\mathcal{N}$ the sequence $\psi_j=\zeta_{X,k,j}(\varphi)$ converges in $\mathcal{L}_k$: indeed, $\mathcal{L}_{X,k}$ is compact, and any convergent subsequence of this sequence converges to the same element of $\mathcal{L}_{X,k}$, namely, to $\psi\in\mathcal{Q}_k$ which is closest to φ in the l^∞-norm, and among such, which is minimal with respect to the lexicographic order of its entries. Put $\zeta_{X,k}(\varphi)=\lim_{j\to\infty}\zeta_{X,k,j}(\varphi)$, $\varphi\in\mathcal{N}$; then $\zeta_{X,k}$ is a measurable mapping $\mathcal{N}\longrightarrow\mathcal{L}_{X,k}$ that maps each nilsequence to a closest in l^∞-norm element of $\mathcal{L}_{X,k}$. It also follows that the function $\partial_{X,k}(\varphi)=\min_{\psi\in\mathcal{L}_{X,k}}\|\varphi-\psi\|_{l^\infty}$ is measurable on $\mathcal{N}$.

In each class of isomorphic nilmanifolds choose a representative X (along with G, Γ, a homeomorphism $G\longrightarrow\mathbb{R}^{\dim G}\times F$, and a metric on G and X); let $\mathcal{X}$ be the set of these representatives. Since there exists only countably many nonisomorphic nilmanifolds, $\mathcal{X}$ is countable. Introduce a well ordering of $\mathcal{X}$ satisfying $X'<X$ when $\dim X'<\dim X$. For every $X\in\mathcal{X}$ put $\Omega_{X,k}=\bigl\{\omega\in\Omega:\partial_{X,k}(\varphi_\omega)<\varepsilon/\nu(\Omega)\bigr\}$ and $\Omega_X=\bigcup_{k=1}^\infty\Omega_{X,k}$; these are measurable subsets of Ω. The union $\bigcup_{X\in\mathcal{X}}\bigcup_{k=1}^\infty\mathcal{L}_{X,k}$ is dense in $\mathcal{N}$, thus $\bigcup_{X\in\mathcal{X}}\bigcup_{k=1}^\infty\Omega_{X,k}=\Omega$. Next define $\Omega'_X=\Omega_X\setminus\bigcup_{X'<X}\Omega_{X'}$, $X\in\mathcal{X}$; these are disjoint sets that partition Ω. Finally, for each $X\in\mathcal{X}$ and $k\in\mathbb{N}$ put $\Omega'_{X,k}=\Omega'_X\setminus\bigcup_{k'<k}\Omega'_{X,k'}$.

[1] The argument that follows has been changed; I thank B. Host for pointing to me out a mistake in the previos version of the paper.

Now, for $\omega \in \Omega$ define $\psi_\omega = \zeta_{X,k}(\varphi_\omega)$ when $\omega \in \Omega'_{X,k}$, $X \in \mathcal{X}$, $k \in \mathbb{N}$; then $\omega \mapsto \psi_\omega$ is a measurable mapping $\Omega \longrightarrow \mathcal{N}^o$ with $\|\psi_\omega - \varphi_\omega\|_{l^\infty} < \varepsilon/\nu(\Omega)$ for all $\omega \in \Omega$. We may now replace φ_ω by ψ_ω, $\omega \in \Omega$; moreover, we may also deal with the sets Ω'_X separately, and therefore assume that φ_ω, $\omega \in \Omega$, are all read off the same nilmanifod $X = G/\Gamma$: $\varphi_\omega = \Psi(a,h)$ with $a \in G$ and h being a Lipschitz function on X. (And, in addition, by our construction, φ_ω is not readable off any nilmanifold X' with $X' < X$.)

We now claim that for each $\omega \in \Omega$, φ_ω has only countably many preimages under this mapping; by Lusin's theorem about the existence of a measurable section, this will imply that we can choose elements $a_\omega \in Q_k$, $h_\omega \in L_k$ with $\varphi_\omega = \Psi(a_\omega, h_\omega)$, $\omega \in \Omega$, such that the mapping $\omega \mapsto (a_\omega, h_\omega)$ is measurable. We get use of the following fact:

Lemma 4.2a. *Let nilmanifolds $X_i = G_i/\Gamma_i$, elements $a_i \in G_i$, and functions $h_i \in C(X_i)$, $i = 1, 2$, be such that the orbits $\{\pi_i(a_i^n)\}_{n\in\mathbb{Z}}$, where π_i are the projections $G_i \longrightarrow X_i$, are dense in X_i, $i = 1, 2$, and the triples (X_1, a_1, h_1) and (X_2, a_2, h_2) produce the same nilsequence: $\varphi(n) = h_1(\pi_1(a_1^n)) = h_2(\pi_2(a_2^n))$, $n \in \mathbb{Z}$. Then there exists a common factor $(\widehat{X}, \hat{a}, \hat{h})$ of (X_1, a_1, h_1) and (X_2, a_2, h_2) such that $\varphi(n) = \hat{h}(\hat{\pi}(\hat{a}^n))$ (where $\hat{\pi}$ is the projection $\widehat{G} \longrightarrow \widehat{X}$).*

Proof. Let $\widetilde{G} = G_1 \times G_2$, $\widetilde{X} = X_1 \times X_2$, $\tilde{\pi} = \pi_1 \times \pi_2 \colon \widetilde{G} \longrightarrow \widetilde{X}$. Let $\tilde{a} = (a_1, a_2) \in \widetilde{G}$, and let Y be the closure of the orbit of $\tilde{a}$ in $\widetilde{X}$, $Y = \overline{\{\tilde{\pi}(\tilde{a}^n),\ n \in \mathbb{Z}\}}$. Let p_1 and p_2 be the projections of Y to X_1 and to X_2 respectively. Let $Y_1 \subseteq X_1$ and $Y_2 \subseteq X_2$ be "the fibers" of the projections p_2 and p_1 respectively: $p_2^{-1}(1_{X_2}) = Y_1 \times \{1_{X_2}\}$ and $p_1^{-1}(1_{X_1}) = Y_2 \times \{1_{X_1}\}$. For any $n \in \mathbb{Z}$ we have $\varphi(n) = h_1(p_1(\tilde{\pi}(\tilde{a}^n))) = h_2(p_2(\tilde{\pi}(\tilde{a}^n)))$; since the orbit $\{\tilde{\pi}(\tilde{a}^n)\}_{n\in\mathbb{Z}}$ is dense in Y, this implies that $h_1 \circ p_1 = h_2 \circ p_2$, so h_1 is constant on the fibers $b_1 Y_1$, $b_1 \in G_1$, of p_2 and h_2 is constant on the fibers $b_2 Y_2$, $b_2 \in G_2$, of p_1, and therefore we may factorize X_1 by Y_1 and X_2 by Y_2 (and Y by $Y_1 \times Y_2$) to get the same factor $(\widehat{X}, \hat{a}, \hat{h})$. ∎

Now, assume that for some $\omega \in \Omega$, $\varphi_\omega = \Psi(a_1, h_1) = \Psi(a_2, h_2)$, $a_1, a_2 \in G$ and h_1, h_2 are Lipschitz functions on X. Let, by Lemma 4.2a, $(\widehat{X}, \hat{a}, \hat{h})$ be a common factor of (X, a_1, h_1) and (X, a_2, h_2) such that $\varphi_\omega = \Psi(\hat{a}, \hat{h})$. Since φ_ω cannot be read off a nilmanifold $\widehat{X}$ with $\dim \widehat{X} < \dim X$, there must be $\dim \widehat{X} = \dim X$. However, for any pair $(\widehat{X}, \hat{a})$, "nilmanifold with a translation", there are only countably many (up to isomorphism) pairs (X, a) extending $(\widehat{X}, \hat{a})$ and with $\dim X = \dim \widehat{X}$.

Thus, we arrive at the following situation: we have a nilmanifold $X = G/\Gamma$ and a measurable function $\Omega \longrightarrow G \times C(X)$, $\omega \mapsto (a_\omega, h_\omega)$, such that for every $\omega \in \Omega$ one has $\varphi_\omega(n) = h_\omega(\pi(a_\omega^n))$, $n \in \mathbb{Z}$. Let $H(\omega, x) = h_\omega(x)$, $\omega \in \Omega$, $x \in X$; then $\varphi(n) = \int_\Omega H(\omega, \pi(a_\omega^n)) d\nu(\omega)$, $n \in \mathbb{Z}$. Choose a basis $f_1, f_2, \ldots$ in $C(X)$; the function H is representable in the form $H(\omega, x) = \sum_{i=1}^\infty \theta_i(\omega) f_i(x)$, where convergence is uniform with respect to x for any ω; we are done if we prove the assertion for the functions $\theta_i(\omega) f_i(x)$ instead of H. So, let $\theta \in L^1(\Omega)$ and $f \in C(X)$; we have to show that the sequence $\varphi(n) = \int_\Omega \theta(\omega) f(\pi(a_\omega^n))\, d\nu(\omega)$ is in $\mathcal{M}$. We may also assume that $\theta \geq 0$. Let $\tau \colon \Omega \longrightarrow G$ be the mapping defined by $\tau(\omega) = a_\omega$, and let $\rho = \tau_*(\theta\nu)$; then ρ is a finite measure on G and $\varphi(n) = \int_G f(\pi(a^n))\, d\rho(a)$. Thus, Proposition 4.2 will follow from the following:

Proposition 4.3. *Let $X = G/\Gamma$ be a nilmanifold, let ρ be a finite Borel measure on G, and let $f \in C(X)$. Then the sequence $\varphi(n) = \int_G f(\pi(a^n))\, d\rho(a)$, $n \in \mathbb{Z}$, is in $\mathcal{M}^o$. (If X*

is an r-step nilmanifold, then $\varphi \in \mathcal{M}_r^o$.)

Proof. We may and will assume that X is connected. Let $\tilde{\rho} = \pi_*(\rho)$; we decompose $\tilde{\rho}$ in the following way:

Lemma 4.4. *There exists an at most countable collection $\mathcal{V}$ of connected subnilmaniolds of X (which may include X itself and singletons) and finite Borel measures ρ_V, $V \in \mathcal{V}$, on G such that $\rho = \sum_{V \in \mathcal{V}} \rho_V$ and for every $V \in \mathcal{V}$, $\mathrm{supp}(\tilde{\rho}_V) \subseteq V$ and $\tilde{\rho}_V(W) = 0$ for any proper subnilmanifold W of V, where $\tilde{\rho}_V = \pi_*(\rho_V)$.*

Proof. Let $\mathcal{V}_0$ be the (at most countable) set of the singletons $V = \{x\}$ in X (connected 0-dimensional subnilmanifolds of X) for which $\tilde{\rho}(V) > 0$. For each $V \in \mathcal{V}_0$ let ρ_V be the restriction of ρ to $\pi^{-1}(V)$ (that is, $\tilde{\rho}_V(A) = \rho(A \cap \pi^{-1}(V))$ for measurable subsets A of G), and let $\rho_1 = \rho - \sum_{V \in \mathcal{V}_0} \rho_V$ and $\tilde{\rho}_1 = \pi_*(\rho_1)$. Now let $\mathcal{V}_1$ be the (at most countable) set of connected 1-dimensional subnilmanifolds of X for which $\tilde{\rho}_1(V) > 0$, for each $V \in \mathcal{V}_1$ let ρ_V be the restriction of ρ_1 to $\pi^{-1}(V)$, and $\rho_2 = \rho - \sum_{V \in \mathcal{V}_1} \rho_V$, $\tilde{\rho}_2 = \pi_*(\rho_2)$. (Note that for $V_1, V_2 \in \mathcal{V}_1$, the subnilmanifold $V_1 \cap V_2$, if nonempty, has dimension 0, so $\tilde{\rho}_1(V_1 \cap V_2) = 0$.) And so on, by induction on the dimension of the subnilmanifolds; at the end, we put $\mathcal{V} = \bigcup_{i=0}^{\dim X} \mathcal{V}_i$. ■

By Lemma 2.3, it suffices to prove the assertion for each of ρ_V instead fo ρ. So, we will assume that the measure ρ is supported by a connected subnilmanifold V of X and $\rho(W) = 0$ for any proper subnilmanifold W of V.

First, let $V = X$:

Lemma 4.5. *Let ρ be a finite Borel measure on G such that for $\tilde{\rho} = \pi_*(\rho)$ one has $\tilde{\rho}(W) = 0$ for any proper subnilmanifold W of X. Then for any $f \in C(X)$ the sequence $\varphi(n) = \int_G f(\pi(a^n))\,d\rho(a)$, $n \in \mathbb{Z}$, converges to $\int_X f\,d\mu_X$ in uniform density.*

Proof. We may assume that $\int_X f\,d\mu_X = 0$; we then have to show that φ is a null-sequence. Let (Φ_N) be a Følner sequence in $\mathbb{Z}$. By the dominated convergence theorem we have

$$\begin{aligned}
\lim_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} |\varphi(n)|^2 &= \lim_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} \int_G f(\pi(a^n))\,d\rho(a) \int_G \bar{f}(\pi(b^n))\,d\rho(b) \\
&= \lim_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} \int_{G\times G} f(\pi(a^n))\bar{f}(\pi(b^n))\,d(\rho\times\rho)(a,b) \\
&= \int_{G\times G} \lim_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} f\otimes\bar{f}(\pi^{\times 2}(a^n,b^n))\,d\rho^{\times 2}(a,b) \\
&= \int_{G\times G} F(a,b)\,d\rho^{\times 2}(a,b),
\end{aligned}$$

where $\pi^{\times 2} = \pi\times\pi$, $\rho^{\times 2} = \rho\times\rho$, and $F(a,b) = \lim_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} f\otimes\bar{f}(\pi^{\times 2}(a^n,b^n))$, $a, b \in G$. For $a, b \in G$, if the sequence $u_n = \pi^{\times 2}(a^n, b^n)$, $n \in \mathbb{Z}$, is well distributed in $X \times X$ then $F(a,b) = \int_{X\times X} f\otimes\bar{f}\,d\mu_{X\times X} = \int_X f\,d\mu \int_X \bar{f}\,d\mu = 0$. So, $F(a,b) \neq 0$ only if the sequence (u_n) is not well distributed in $X \times X$, which only happens if the point

$\pi^{\times 2}(a,b)$ is contained in a proper subnilmanifold D of $X \times X$ with $1_{X\times X} \in D$. So,

$$\lim_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} |\varphi(n)|^2 \le \sum_{D\in\mathcal{D}} \int_{(\pi^{\times 2})^{-1}(D)} |F(a,b)|\, d\rho^{\times 2}(a,b),$$

where $\mathcal{D}$ is the (countable) set of proper subnilmanifolds of $X \times X$ containing $1_{X\times X}$. Let $D \in \mathcal{D}$; then either for any $x \in X$ the fiber $W'_x = \{y \in X : (x,y) \in D\}$ of D over x is a proper subnilmanifold of X, or for any $y \in X$ the fiber $W''_y = \{x \in X : (x,y) \in D\}$ of D over y is a proper subnilmanifold of X, (or both). Since, by our assumption, $\tilde{\rho}(W) = 0$ for any proper subnilmanifold W of X, in either case $\tilde{\rho}^{\times 2}(D) = 0$, so $\rho^{\times 2}((\pi^{\times 2})^{-1}(D)) = 0$. Hence, $\lim_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} |\varphi(n)|^2 = 0$, which means that $\varphi \in \mathcal{Z}$. ■

Thus, in this case, φ is a constant plus a null-sequence, that is, $\varphi \in \mathcal{M}^o$.

Let now V be of the form $V = cY$, where Y is a (proper) connected subnilmanifold of X with $1_X \in Y$ and $c \in G^c$. We may and will assume that the orbit $\{c^n Y,\ n \in \mathbb{Z}\}$ of Y is dense in X. Let Z be the normal closure of Y in X. In this situation the following generalization of Lemma 4.5 does the job:

Lemma 4.6. *Let Z be a normal subnilmanifold of X and let $c \in G$ be such that $\{c^n Z,\ n \in \mathbb{Z}\}$ is dense in X. Let ρ be a finite Borel measure on G such that for $\tilde{\rho} = \pi_*(\rho)$ one has $\operatorname{supp}(\tilde{\rho}) \subseteq cZ$ and $\tilde{\rho}(cW) = 0$ for any proper normal subnilmanifold W of Z. Let $\varphi(n) = \int_G f(\pi(a^n))\, d\rho(a)$, $n \in \mathbb{Z}$, let $\widehat{X} = X/Z$, and let $\hat{f} = E(f|\widehat{X})$. Then $\varphi - \hat{f}(\pi(c^n)) \in \mathcal{Z}$.*

Proof. After replacing f by $f - \hat{f}$ we will assume that $E(f|\widehat{X}) = 0$; we then have to prove that φ is a null-sequence. Let $L = \pi^{-1}(Z)$. Let (Φ_N) be a Følner sequence in $\mathbb{Z}$; as in Lemma 4.5, we obtain

$$\lim_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} |\varphi(n)|^2 = \int_{G\times G} F(a,b)\, d\rho^{\times 2}(a,b) = \int_{(cL)\times(cL)} F(a,b)\, d\rho^{\times 2}(a,b),$$

where $F(a,b) = \lim_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} f \otimes \bar{f}(\pi^{\times 2}(a^n, b^n))$, $a, b \in L$. Let us "shift" ρ to the origin, by replacing it by $c_*^{-1}\rho(a)$, $a \in G$, so that now $\operatorname{supp}(\rho) \subseteq L$, $\operatorname{supp}(\tilde{\rho}) \subseteq Z$, and

$$\lim_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} |\varphi(n)|^2 = \int_{L\times L} F(a,b)\, d\rho^{\times 2}(a,b),$$

where $F(a,b) = \lim_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} f \otimes \bar{f}(\pi^{\times 2}((ca)^n, (cb)^n))$, $a, b \in L$.

For $a, b \in L$, the sequence $u_n = \pi^{\times 2}((ca)^n, (cb)^n)$, $n \in \mathbb{Z}$, is contained in $X \times_{\widehat{X}} X = \big\{(x,y) : \delta(x) = \delta(y)\big\}$, where δ is the factor mapping $X \longrightarrow \widehat{X}$. If this sequence is well distributed in $X \times_{\widehat{X}} X$, then $F(a,b) = \int_{X\times_{\widehat{X}} X} f \otimes \bar{f}\, d\mu_{X\times_{\widehat{X}} X} = \int_{\widehat{X}} E(f|\widehat{X}) E(\bar{f}|\widehat{X})\, d\mu_{\widehat{X}} = 0$. So, $F(a,b) \ne 0$ only if the sequence (u_n) is not well distributed in $X \times_{\widehat{X}} X$, which only happens if the image $(\tilde{u}_n)$ of (u_n) is not well distributed in the nil-maximal factor-torus T of $X \times_{\widehat{X}} X$. Using additive notation on T we have $\tilde{u}_n = n\tilde{c} + n\tilde{a} + n\tilde{b}$, $n \in \mathbb{Z}$, where T contains the direct sum $S \oplus S$ of two copies of a torus S, $\tilde{a} \in S \oplus \{0\}$, $\tilde{b} \in \{0\} \oplus S$, and the sequence $(n\tilde{c})$ is dense in the factor-torus $T/(S\oplus S)$. The sequence $(\tilde{u}_n)$ is not dense in T only if the point $(\tilde{a},\tilde{b})$ is contained in a proper subtorus R of $S\oplus S$, and either for each $\tilde{x}\in S$ the fiber $\{\tilde{y}\in S:(\tilde{x},\tilde{y})\in R\}$ is a proper subtorus of S, or for each $\tilde{y}\in S$ the fiber $\{\tilde{x}\in S:(\tilde{x},\tilde{y})\in R\}$ is a proper subtorus of S (or both). Without loss of generality, assume that the first possibility holds. Then, returning back to $X\times_{\widehat{X}}X$, we obtain that the sequence (u_n) is not well distributed in this space only if the point $\pi^{\times 2}(a,b)$ is contained in a subnilmanifold D (the preimage of the torus R) in $Z\times Z$ with $D\ni 1_{X\times_{\widehat{X}}X}$ such that for every $x\in Z$ the fiber $W_x=\{y\in Z:(x,y)\in D\}$ is a proper normal subnilmanifold of Z. Since, by our assumption, $\tilde{\rho}(W_x)=0$ for all x, we have $\tilde{\rho}^{\times 2}(D)=0$, so $\rho^{\times 2}((\pi^{\times 2})^{-1}(D))=0$. The function $F(a,b)$ may only be nonzero on the union of a countable collection of the subnilmanifolds D like this, so $\int_{L\times L}F(a,b)\,d\rho^{\times 2}(a,b)=0$. Hence, $\lim_{N\to\infty}\frac{1}{|\Phi_N|}\sum_{n\in\Phi_N}|\varphi(n)|^2=0$, which means that $\varphi\in\mathcal{Z}$. ■

Since, in the notation of Lemma 4.6, the sequence $\hat{f}(\pi(c^n))=\hat{f}(c^n1_X)$, $n\in\mathbb{Z}$, is a basic nilsequence, φ is a sum of a nilsequence and a null-sequence, so $\varphi\in\mathcal{M}^o$ in this case as well. ■

The proof of Proposition 4.2 in the case $d\geq 2$ is not much harder than in the case $d=1$, and we will only sketch it. For each $\omega\in\Omega$, instead of a single element $a_\omega\in G_\omega$ we now have d commuting elements $a_{\omega,1},\ldots,a_{\omega,d}\in G_\omega$. After passing to a single nilmanifold $X=G/\Gamma$, we obtain d mappings $\tau_i\colon\Omega\longrightarrow G$, $\omega\mapsto a_{\omega,i}$, $i=1,\ldots,d$, and so, the mapping $\tau=(\tau_1,\ldots,\tau_d)\colon\Omega\longrightarrow G^d$. We define a measure ρ on G^d by $\rho=\tau_*(\theta\nu)$; then Proposition 4.2 follows from the following modification of Proposition 4.3:

Proposition 4.7. *Let $X=G/\Gamma$ be a nilmanifold, let ρ be a finite Borel measure on G^d, and let $f\in C(X)$. Then the sequence $\varphi(n_1,\ldots,n_d)=\int_{G^d}f(\pi(a_1^{n_1}\ldots a_d^{n_d}))\,d\rho(a_1,\ldots,a_d)$, $(n_1,\ldots,n_d)\in\mathbb{Z}^d$, is in $\mathcal{M}^o$. (If X is an r-step nilmanifold, then $\varphi\in\mathcal{M}^o_r$.)*

The proof of this proposition is the same as of Proposition 4.3, with a^n replaced by $a_1^{n_1}\ldots a_d^{n_d}$, and the mapping $G\longrightarrow X$, $a\mapsto\pi(a)$, replaced by the mapping $G^d\longrightarrow X$, $(a_1,\ldots,a_d)\mapsto\pi(a_1\ldots a_d)$.

Uniting Proposition 4.2 with Lemma 4.1, we obtain:

Theorem 4.8. *Let (Ω,ν) be a measure space and let $\Omega\longrightarrow\mathcal{M}$, $\omega\mapsto\varphi_\omega$, be an integrable mapping. Then the sequence $\varphi(n)=\int_\Omega\varphi_\omega(n)\,d\nu$ is in $\mathcal{M}$ as well. If, for some r, $\varphi_\omega\in\mathcal{M}_r$ for all ω, then $\varphi\in\mathcal{M}_r$.*

5. Multiple polynomial correlation sequences and nilsequences

Now let $(W,\mathcal{B},\mu)$ be a probability measure space and let T be an ergodic invertible measure preserving transformation of W. Let $p_1,\ldots,p_k$ be polynomials $\mathbb{Z}^d\longrightarrow\mathbb{Z}$. Let $A_1,\ldots,A_k\in\mathcal{B}$ and let $\varphi(n)=\mu\big(T^{p_1(n)}A_1\cap\ldots\cap T^{p_k(n)}A_k\big)$, $n\in\mathbb{Z}^d$; or, more generally, let $f_1,\ldots,f_k\in L^\infty(W)$ and $\varphi(n)=\int_W T^{p_1(n)}f_1\cdot\ldots\cdot T^{p_k(n)}f_k d\mu$, $n\in\mathbb{Z}^d$. Then, given $\varepsilon>0$, there exist an r-step nilsystem (X,a), $X=G/\Gamma$, $a\in G$, and functions $\tilde{f}_1,\ldots,\tilde{f}_k\in L^\infty(X)$ such that, for $\phi(n)=\int_X a^{p_1(n)}\tilde{f}_1\cdot\ldots\cdot a^{p_k(n)}\tilde{f}_k d\mu_X$, $n\in\mathbb{Z}^d$, the set $\big\{n\in\mathbb{Z}^d:|\phi(n)-\varphi(n)|>$

$\varepsilon\}$ has zero uniform density. Moreover, there is a universal integer r that works for all systems $(W,\mathcal{B},\mu,T)$, functions h_i, and ε, and depends only on the polynomials p_i; for the minimal such r, the integer $c=r-1$ is called *the complexity* of the system $\{p_1,\dots,p_k\}$ (see [L5]).

(Here is a sketch of the proof, for completeness; for more details see [HK1] and [BHK]. By [L3], there exists $c\in\mathbb{N}$, which only depends on the polynomials p_i, such that, if (V,ν,S) is an ergodic probability measure preserving system and $Z_c(V)$ is the c-th Host-Kra-Ziegler factor of V and $h_1,\dots,h_k\in L^\infty(V)$ are such that $E(h_i|Z_c(V))=0$ for some i, then for any Følner sequence (Φ_N) in $\mathbb{Z}^d$ one has $\lim_{N\to\infty}\frac{1}{|\Phi_N|}\sum_{n\in\Phi_N}\int_V S^{p_1(n)}h_1\cdot\ldots\cdot S^{p_k(n)}h_k d\nu=0$. Applying this to the ergodic components of the system $(W\times W,\mu\times\mu,T\times T)$ and the functions $h_i=f_i\otimes\bar{f}_i$, $i=1,\dots,k$, we obtain that for any Følner sequence (Φ_N) in $\mathbb{Z}^d$,

$$\lim_{N\to\infty}\frac{1}{|\Phi_N|}\sum_{n\in\Phi_N}\Big|\int_{W\times W}T^{p_1(n)}f_1\cdot\ldots\cdot T^{p_k(n)}f_k d\mu\Big|^2$$
$$=\lim_{N\to\infty}\frac{1}{|\Phi_N|}\sum_{n\in\Phi_N}\int_{W\times W}T^{p_1(n)}f_1(x)\cdot T^{p_1(n)}\bar{f}_1(y)\cdot\ldots$$
$$\cdot T^{p_k(n)}f_k(x)\cdot T^{p_k(n)}\bar{f}_k(y)d(\mu(x)\times\mu(y))=0$$

whenever, for some i, the function $f_i\otimes\bar{f}_i$ has zero conditional expectation with respect to almost all ergodic components of $Z_c(W\times W)$. This is so if $E(f_i|Z_{c+1}(W))=0$, and we obtain that the sequence $\int_W T^{p_1(n)}f_1\cdot\ldots\cdot T^{p_k(n)}f_k d\mu$ tends to zero in uniform density whenever $E(f_i|Z_r(W))=0$ for some i, where $r=c+1$. It follows that for any $f_1,\dots,f_k\in L^\infty(W)$ the sequence

$$\int_W T^{p_1(n)}f_1\cdot\ldots\cdot T^{p_k(n)}f_k d\mu-\int_{Z_r(W)}T^{p_1(n)}E(f_1|Z_r(W))\cdot\ldots\cdot T^{p_k(n)}E(f_k|Z_r(W))d\mu_{Z_r(W)}$$

tends to zero in uniform density. Now, $Z_r(W)$ has the structure of the inverse limit of a sequence of r-step nilmanifolds on which T acts as a translation; given $\varepsilon>0$, we can therefore find an r-step nilmanifold factor X of W such that $\|E(f_i|Z_r(W))-E(f_i|X)\|_{L^\infty(W)}<\varepsilon/\prod_{j=1}^k\|f_j\|_{L^\infty(W)}$ for all i. Putting $\tilde{f}_i=E(f_i|X)$, $i=1,\dots,k$, and denoting the translation induced by T on X by a, we then have

$$\Big|\int_{Z_r(W)}T^{p_1(n)}E(f_1|Z_r(W))\cdot\ldots\cdot T^{p_k(n)}E(f_k|Z_r(W))d\mu_{Z_r(W)}$$
$$-\int_X a^{p_1(n)}\tilde{f}_1\cdot\ldots\cdot a^{p_k(n)}\tilde{f}_k d\mu_X\Big|<\varepsilon$$

for all n, which implies the assertion.)

So, there exists $\lambda\in\mathcal{Z}$ such that $\|\varphi-(\psi+\lambda)\|<\varepsilon$. After replacing $\tilde{f}_i$ by L^1-close continuous functions, we may assume that $\tilde{f}_1,\dots,\tilde{f}_k\in C(X)$, and still $\|\varphi-(\psi+\lambda)\|<\varepsilon$. Applying Theorem 3.3 to the nilmanifold $X^k=G^k/\Gamma^k$, the diagonal subnilmanifold $Y=\big\{(x,\dots,x),\ x\in X\big\}\subseteq X^k$, the polynomial sequence $g(n)=(a^{p_1(n)},\dots,a^{p_k(n)})$, $n\in\mathbb{Z}^d$, in G^k, and the function $f(x_1,\dots,x_k)=\tilde{f}_1(x_1)\cdot\ldots\cdot\tilde{f}_k(x_k)\in C(X^k)$, we obtain that $\psi\in\mathcal{M}^o(r,s)$, so also $\psi+\lambda\in\mathcal{M}^o(r,s)$. Since ε is arbitrary and, by Lemma 2.3, $\mathcal{M}_{r,s}$ is the closure of $\mathcal{M}^o_{r,s}$, we obtain:

Proposition 5.1. *Let $(W, \mathcal{B}, \mu, T)$ be an ergodic invertible probability measure preserving system, let $f_1, \ldots, f_k \in L^\infty(W)$, and let $p_1, \ldots, p_k$ be polynomials $\mathbb{Z}^d \longrightarrow \mathbb{Z}$. Then the sequence $\varphi(n) = \int_W T^{p_1(n)} f_1 \cdot \ldots \cdot T^{p_k(n)} f_k d\mu$, $n \in \mathbb{Z}^d$, is in $\mathcal{M}$. If the complexity of the system $\{p_1, \ldots, p_k\}$ is c and $\deg p_i \leq s$ for all i, then $\varphi_n \in \mathcal{M}_{c+1,s}$.*

Let now $(W, \mathcal{B}, \mu, T)$ be a non-ergodic (or, rather, not necessarily ergodic) system. Let $\mu = \int_\Omega \mu_\omega d\nu(\omega)$ be the ergodic decomposition of μ. For each $\omega \in \Omega$, let $\varphi_\omega(n) = \int_W T^{p_1(n)} f_1 \cdot \ldots \cdot T^{p_k(n)} f_k d\mu_\omega$, $n \in \mathbb{Z}^d$; then $\omega \mapsto \varphi_\omega$ is a measurable mapping $\Omega \longrightarrow l^\infty$, and $\varphi(n) = \int_\Omega \varphi_\omega(n) \, d\nu(\omega)$, $n \in \mathbb{Z}^d$. By Proposition 5.1, for each $\omega \in \Omega$ we have $\varphi_\omega \in \mathcal{M}_{c+1,s} \subseteq \mathcal{M}_l$, where $l = 2(c+1)s$. By Theorem 4.8 we obtain:

Theorem 5.2. *Let $(W, \mathcal{B}, \mu, T)$ be an invertible probability measure preserving system, let $f_1, \ldots, f_k \in L^\infty(W)$, and let $p_1, \ldots, p_k$ be polynomials $\mathbb{Z}^d \longrightarrow \mathbb{Z}$. Then the sequence $\varphi(n) = \int_W T^{p_1(n)} f_1 \cdot \ldots \cdot T^{p_k(n)} f_k d\mu$, $n \in \mathbb{Z}^d$, is in $\mathcal{M}$. If the complexity of the system $\{p_1, \ldots, p_k\}$ is c and $\deg p_i \leq s$ for all i, then $\varphi_n \in \mathcal{M}_l$, where $l = 2(c+1)s$.*

Since $\mathcal{M} \subseteq \mathcal{P} + \mathcal{Z}$, where $\mathcal{P}$ is the closure in l^∞ of the algebra of bounded generalized polynomials (see the last paragraph of Section 2), we get as a corollary:

Corollary 5.3. *Up to a null-sequence, the sequence φ is uniformly approximable by generalized polynomials.*

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