Orbits on a nilmanifold under the action of a polynomial sequence of translations

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Abstract

It is known that the closure $\overline{\mathrm{Orb}}_g(x)$ of the orbit $\mathrm{Orb}_g(x)$ of a point x of a compact nilmanifold X under a polynomial sequence g of translations of X is a disjoint finite union of subnilmanifolds of X. Assume that $g(0)=1_G$ and let A be the group generated by the elements of g; we show in this paper that for almost all points $x\in X$, $\overline{\mathrm{Orb}}_g(x)$ are congruent (that is, are translates of each other), with connected components of $\overline{\mathrm{Orb}}_g(x)$ equal to (some of) the connected components of $\overline{\mathrm{Orb}}_A(x)$.

1. Nilmanifolds, subnilmanifolds, polynomial sequences and orbits

Let X be a compact nilmanifold, that is, a compact homogeneous space of a (not necessarily connected) nilpotent Lie group G. Then X is isomorphic to (and will be identified with) G/Γ, where Γ is a closed uniform subgroup of G, with G acting on X by left translations. We will denote by π the factorization mapping $G\longrightarrow X$, and by 1_X the point $\pi(1_G)$, so that $\pi(a)=a1_X$, $a\in G$.

We will list, without proofs, some elementary facts about nilmanifolds; for more details see [M], [L1], [L2] and [L3].

1.1. If X is not connected, it consists of finitely many isomorphic components, which may be treated independently; throughout the paper we will assume for simplicity that X is connected. The connectedness of X does not imply that G is connected; let G^{o} be the identity component of G and let $\Gamma^{\mathrm{o}}=\Gamma\cap G^{\mathrm{o}}$. Then $X=G^{\mathrm{o}}/\Gamma^{\mathrm{o}}$, so that X is a homogeneous space of the connected group G^{o}. If X is interpreted this way, the elements of $G\setminus G^{\mathrm{o}}$ act on X not as translations but as unipotent affine transformations. (Example: the nilmanifold $X=\left(\begin{smallmatrix}1&\mathbb{Z}&\mathbb{R}\\0&1&\mathbb{R}\\0&0&1\end{smallmatrix}\right)/\left(\begin{smallmatrix}1&\mathbb{Z}&\mathbb{Z}\\0&1&\mathbb{Z}\\0&0&1\end{smallmatrix}\right)$ is isomorphic to the torus $\mathbb{R}^2/\mathbb{Z}^2$, on which the element $\left(\begin{smallmatrix}1&1&0\\0&1&\alpha\\0&0&1\end{smallmatrix}\right)$ of the group $G=\left(\begin{smallmatrix}1&\mathbb{Z}&\mathbb{R}\\0&1&\mathbb{R}\\0&0&1\end{smallmatrix}\right)$ acts as the transformation $(x,y)\mapsto(x+\alpha,y+x)$.)

Conversely, if X is a nilmanifold corresponding to a group G and A is a nilpotent Lie group of unipotent affine transformations of X, then the semidirect product $\widetilde{G}=G\rtimes A$

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is a nilpotent Lie group that contains both G and A, and X is a homogeneous space of $\widetilde{G}$ on which it acts by translations.

1.2. If the subgroup Γ is not discrete, then the connected component $\Gamma^{\rm o}$ of Γ is a normal subgroup of G, and we may pass from G to $G/\Gamma^{\rm o}$ without changing X (see [L1]). Thus, we may and will assume that Γ is a discrete subgroup of G.

1.3. After replacing the group $G^{\rm o}$ by its universal cover, we may and will assume that $G^{\rm o}$ is simply connected. One may then introduce Malcev coordinates on $G^{\rm o}$, that is, a system of one-parameter subgroups $e_i(t)$, $t \in \mathbb{R}$, $i = 1,\ldots,d$, such that the elements $e_1^1,\ldots,e_d^1$ generate Γ and any element a of $G^{\rm o}$ is uniquely representable in the form $a = e_1^{t_1}\ldots e_d^{t_d}$, $t_1,\ldots,t_d \in \mathbb{R}$. The "coordinate" mapping $\eta(t_1,\ldots,t_d) = a$ is a homeomorphism $\mathbb{R}^d \longrightarrow G$, with $\eta(\mathbb{Z}^d) = \Gamma$. In coordinates, the multiplication in $G^{\rm o}$ is given by a polynomial mapping $\mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$.

Let us say that a mapping $\varphi\colon \mathbb{R}^k \times \mathbb{Z}^l \longrightarrow G^{\rm o}$ *is polynomial* if it is polynomial in coordinates, that is, if $\eta^{-1}\circ\varphi\colon \mathbb{R}^k \times \mathbb{Z}^l \longrightarrow \mathbb{R}^d$ is a polynomial mapping. Since the change-of-Malcev-coordinates mapping is an invertible bi-polynomial transformation of $\mathbb{R}^d$, this definition does not depend on the choice of Malcev coordinates on $G^{\rm o}$.

1.4. *A subnilmanifold* Y of X is a closed subset of X of the form $Y = Hx$, where H is a closed subgroup of G and $x \in X$. Since $\pi(G^{\rm o}) = X$, after replacing H by $H \cap G^{\rm o}$ one may assume that $H \subseteq G^{\rm o}$. A subnilmanifold Y is a nilmanifold, since $Y \simeq H/\big((a\Gamma a^{-1}) \cap H\big)$ where a is any element of G with $\pi(a) = x$.

1.5. Given a closed subgroup H of $G^{\rm o}$ and a point $x \in X$, the set Hx may not be closed and so, may not be a subnilmanifold of X; Hx is closed iff $(a\Gamma a^{-1}) \cap H$ is a uniform subgroup of H, where a is any element of $\pi^{-1}(x)$. In particular, $H1_X = \pi(H)$ is closed iff $H \cap \Gamma$ is uniform in H; we will say that H is *rational* in this case. There are only countably many rational closed subgroups in G.

We say that an element a of G is *rational* if $a^n \in \Gamma$ for some $n \in \mathbb{N}$. A closed subgroup H of G is rational iff rational elements are dense in H ([L3]).

We say that a point $x = \pi(a) \in X$ is *rational* if $x = \pi(a)$ where a is rational in G. A subnilmanifold Y of X is rational if it contains at least one rational point of X, and in this case rational points are dense in Y. X has countably many rational subnilmanifolds. For any point $x \in X$ there are only countably many distinct subnilmanifolds in X that contain x. (See [L3].)

1.6. Let H be a closed connected subgroup of $G^{\rm o}$ and let $\tau\colon \mathbb{R}^r \longrightarrow H$ be Malcev coordinates on H. Then the mapping $\eta^{-1}\circ\tau\colon \mathbb{R}^r \longrightarrow \mathbb{R}^d$ is polynomial, and thus in coordinates H is the image of a polynomial mapping. Let us say that a subset S of $G^{\rm o}$ *is polynomial* if $\eta^{-1}(S)$ is an algebraic subset of $\mathbb{R}^d$, that is, is defined by one or several polynomial equations; this definition does not depend on the choice of Malcev coordinates on $G^{\rm o}$. Any closed connected subgroup H of $G^{\rm o}$ is a polynomial subset of $G^{\rm o}$; indeed, Malcev coordinates on $G^{\rm o}$ can be constructed so that they extend Malcev coordinates on H, and in these coordinates $\eta^{-1}(H)$ is even a linear (coordinate) subspace of $\mathbb{R}^d$. Since a translation by an element $a \in G^{\rm o}$ is an inveretible bi-polynomial transformation of $\mathbb{R}^d$, the set aH is polynomial in G^{o} as well.

Let us say that a set $P \subseteq X$ is *polynomial* in X if $P = \pi(S)$ where S is a polynomial subset in G^{o}. Note that a polynomial subset of X does not have to be closed in X. (It may even be dense in X, as a line with an irrational slope in the 2-dimensional torus $\mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^2$.)

Let us say that a subset of $\mathbb{R}^d$, G^{o} or X is *countably polynomial* if it is a countable (or finite) union of polynomial subsets. Note that any proper countably polynomial subset is of zero (Lebesgue) measure and of first category in the corresponding space.

1.7. Let A be a closed (possibly, discrete) subgroup of G. For $x \in X$, we will denote by $\mathrm{Orb}_A(x)$ the orbit of x under the action of A, $\mathrm{Orb}_A(x) = Ax$, and by $\overline{\mathrm{Orb}}_A(x)$ the closure of $\mathrm{Orb}_A(x)$. By abuse of language, we will also refer to $\overline{\mathrm{Orb}}_A(x)$ as *the orbit* of x under the action of A. It is shown in [L1] that for any $x \in X$, $\overline{\mathrm{Orb}}_A(x)$ is a (connected or disconnected) subnilmanifold of X. (See also [Le] and [Sh].) For any $x \in X$, the action of A on $\overline{\mathrm{Orb}}_A(x)$ is minimal, that is, $\overline{\mathrm{Orb}}_A(y) = \overline{\mathrm{Orb}}_A(x)$ for any $y \in \overline{\mathrm{Orb}}_A(x)$. It follows that $X = \bigcup_{x\in X} \overline{\mathrm{Orb}}_A(x)$ is a partition of X. In particular, if $\overline{\mathrm{Orb}}_A(x) = X$ for a point $x \in X$, then $\overline{\mathrm{Orb}}_A(y) = X$ for all $y \in X$.

The orbits of distinct points may not be translates of each other, and may even have different dimensions, as the following examples demonstrate:

1.8. Examples.
(1) Let $G = \left\{\left(\begin{smallmatrix}1 & x_1 & x_3\\ 0 & 1 & x_2\\ 0 & 0 & 1\end{smallmatrix}\right),\ x_1 \in \mathbb{Z},\ x_2, x_3 \in \mathbb{R}\right\}$ and $\Gamma = \left\{\left(\begin{smallmatrix}1 & x_1 & x_3\\ 0 & 1 & x_2\\ 0 & 0 & 1\end{smallmatrix}\right),\ x_1, x_2, x_3 \in \mathbb{Z}\right\}$; then $X = G/\Gamma$ is identified with the 2-dimensional torus $\mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^2$ with coordinates $x_2, x_3 \in \mathbb{T}$. Let $a = \left(\begin{smallmatrix}1 & 1 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{smallmatrix}\right) \in G$, then the action of a on X is given by $a(x_2, x_3) = (x_2, x_3 + x_2) \bmod 1$, $(x_2, x_3) \in X$. (Equivalently, without even mentioning nilpotent groups, $X = \mathbb{T}^2$ and a is the unipotent transformation of X defined by this formula.) Let $A = \{a^n\}_{n\in\mathbb{Z}}$. Then for $x = (x_2, x_3) \in X$, $\overline{\mathrm{Orb}}_A(x) = \{(x_2, u),\ u \in \mathbb{T}\} \simeq \mathbb{T}$ if x_1 is irrational, and is the finite set $\{(x_2, nx_1),\ n \in \mathbb{N}\}$ if x_1 is rational.

(2) Now let $G = \left\{\left(\begin{smallmatrix}1 & x_1 & x_3\\ 0 & 1 & x_2\\ 0 & 0 & 1\end{smallmatrix}\right),\ x_1, x_2, x_3 \in \mathbb{R}\right\}$ and $\Gamma = \left\{\left(\begin{smallmatrix}1 & x_1 & x_3\\ 0 & 1 & x_2\\ 0 & 0 & 1\end{smallmatrix}\right),\ x_1, x_2, x_3 \in \mathbb{Z}\right\}$; $X = G/\Gamma$ is then the 3-dimensional *Heisenberg manifold*. Let $a = \left(\begin{smallmatrix}1 & \alpha & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{smallmatrix}\right) \in G$ where α is an irrational number; then the action of a on X is given by $ax = \left(\begin{smallmatrix}1 & x_1+\alpha & x_3+\alpha x_2\\ 0 & 1 & x_2\\ 0 & 0 & 1\end{smallmatrix}\right) \bmod \Gamma$, $x = \left(\begin{smallmatrix}1 & x_1 & x_3\\ 0 & 1 & x_2\\ 0 & 0 & 1\end{smallmatrix}\right) \in X$. Let $A = \{a^n\}_{n\in\mathbb{Z}}$ and $x = \left(\begin{smallmatrix}1 & x_1 & x_3\\ 0 & 1 & x_2\\ 0 & 0 & 1\end{smallmatrix}\right) \in X$. If α and αx_2 are rationally independent, that is, if $x_2 \notin \mathbb{Q} + \frac{1}{\alpha}\mathbb{Q}$, the orbit of $x = \left(\begin{smallmatrix}1 & x_1 & x_3\\ 0 & 1 & x_2\\ 0 & 0 & 1\end{smallmatrix}\right)$ is the 2-dimensional torus $\overline{\mathrm{Orb}}_A(x) = \left\{\left(\begin{smallmatrix}1 & u & v\\ 0 & 1 & x_2\\ 0 & 0 & 1\end{smallmatrix}\right),\ u, v \in \mathbb{T}\right\}$. Otherwise $\overline{\mathrm{Orb}}_A(x)$ is a 1-dimensional torus or the union of several 1-dimensional tori; for example, if $x_2 = 0$ or $x_2 = \frac{1}{\alpha}$, then $\overline{\mathrm{Orb}}_A(x) = \left\{\left(\begin{smallmatrix}1 & u & x_3\\ 0 & 1 & x_2\\ 0 & 0 & 1\end{smallmatrix}\right),\ u \in \mathbb{T}\right\}$.

1.9. Let us say that two subsets Y_1, Y_2 of X *are congruent* if $V_2 = aV_1$ for some $a \in G^{\mathrm{o}}$. In the examples 1.8 we observe that (i) almost all points of X have congruent orbits (which we will call "generic" below); (ii) any "non-generic" orbit is a proper subnilmanifold of the "generic" one; and (iii) the points having a "non-generic" orbit are all contained in a countable union of proper subnilmanifolds of X. We will show that (i) and (ii) always hold; (iii) may fail (see example 2.4 below), and must be replaced by a weaker statement:

1.10. Theorem. *Let A be a closed subgroup of G. There exists a closed subnilmanifold Y_A of X such that*

(a) *for any $x \in X$ the orbit $\overline{\mathrm{Orb}}_A(x)$ is congruent to some subset of Y_A;*

(b) *there exists a proper countably polynomial subset $P \subset X$ such that for all $x \notin P$ the orbit $\overline{\mathrm{Orb}}_A(x)$ is congruent to Y_A.*

This theorem will be proven in Section 2. We will refer to the "standard" orbit Y_A in the formulation of the theorem as *the generic orbit for A*.

1.11. A (multiparameter) *polynomial sequence* in G is a sequence of the form $g(n) = a_1^{p_1(n)} \ldots a_r^{p_r(n)}$, $n \in \mathbb{Z}^l$, where $a_1, \ldots, a_n \in G$ and $p_1, \ldots, p_r$ are polynomials $\mathbb{Z}^l \longrightarrow \mathbb{Z}$. In the terminology introduced above, a polynomial sequence is just a polynomial mapping $\mathbb{Z}^l \longrightarrow G$. For $x \in X$ we will denote by $\mathrm{Orb}_g(x)$ the orbit of x under the action of g, $\mathrm{Orb}_g(x) = g(\mathbb{Z}^l)x = \{g(n)x,\ n \in \mathbb{Z}^l\}$, and by $\overline{\mathrm{Orb}}_g(x)$ the closure of $\mathrm{Orb}_g(x)$; by abuse of language, we will also refer to $\overline{\mathrm{Orb}}_g(x)$ as *the orbit* of x under the action of g. It is shown in [L2] that $\overline{\mathrm{Orb}}_g(x)$ is of the form $\bigcup_{i=1}^s Hx_i$, where H is a connected closed subgroup of G and $x_1, \ldots, x_s \in X$, and thus is either a connected subnilmanifold of X or the union of a finite collection of pairwise disjoint connected subnilmanifolds of same dimension. Let us call such a union *a FU-subnilmanifold*; in particular, any (connected or disconnected) subnilmanifold of X is a FU-subnilmanifold.

Let us say that a FU-subnilmanifold is *rational* if all its connected components are rational subnilmanifolds. It is shown in [L3] that for any rational point x of X, $\overline{\mathrm{Orb}}_g(x)$ is a rational FU-subnilmanifold.

1.12. The orbits under the action of a polynomial sequence do not have to partition X; in the following example, due to Frantzikinakis and Kra, the generic orbit is the whole space X, whereas nongeneric orbits are proper subnilmanifolds of X.

Example. Let X be the 3-dimensional torus $\mathbb{T}^3$ and let a and b be the transformations of X defined by $ax = (x_1+\alpha, x_2+2x_1+\alpha, x_3)$ and $bx = (x_1, x_2, x_3+\alpha)$, $x = (x_1, x_2, x_3) \in X$. (As mentioned in 1.1 above, since a and b are commuting unipotent transformations of X they can be viewed as elements of a nilpotent Lie group for which X is a homogeneous space.) Define $g(n) = a^n b^{n^2}$, $n \in \mathbb{Z}$. Then, for $x = (x_1, x_2, x_3)$, one has $g(n)x = (x_1 + n\alpha, x_2 + 2nx_1 + n^2\alpha, x_3 + n^2\alpha)$. If x_1 is irrational, $\overline{\mathrm{Orb}}_g(x) = X$. If x_1 is rational, $\overline{\mathrm{Orb}}_g(x)$ is a proper subtorus or a union of several 2-dimensional subtori of X. For example, if $x_1 = 0$, $\overline{\mathrm{Orb}}_g(x) = \{(u, v, v),\ u, v \in \mathbb{T}\}$.

1.13. We will show that, like in the case of a linear action, under the action of a polynomial sequence g almost all points of X have congruent orbits:

Theorem. *Let g be a polynomial sequence in G.*
I. *There exists a closed FU-subnilmanifold Y_g of X such that*
(a) *for any $x \in X$ the orbit $\overline{\mathrm{Orb}}_g(x)$ is congruent to some subset of Y_g;*
(b) *there exists a proper countably polynomial subset $P \subset X$ such that for all $x \notin P$ the orbit $\overline{\mathrm{Orb}}_g(x)$ is congruent to Y_g.*
II. *Assume that $g(0) = 1_G$, let A be the subgroup of G generated by the elements of g and let Y_A be the generic orbit for A. Then Y_g consists of one or several components of Y_A; in particular, if Y_A is connected, $Y_g = Y_A$.*

Part I of this theorem will be proved in Section 2, Part II will be proved in Section 4. In Section 3 we study the property of "normality" of generic orbits. In Section 5 we investigate the orbit of a subnilmanifold of X.

2. The generic orbits

2.1. Theorem 1.10 and (the first part of) Theorem 1.13 are corollaries of the following simple general fact:

Theorem. *Let M be a set and let $\varphi: \mathbb{R}^k \times M \longrightarrow G$ be a mapping; assume that for each fixed $m \in M$, φ is polynomial with respect to $\mathbb{R}^k$, and for each $t \in \mathbb{R}^k$ the set $Y_t = \pi(\varphi(t, M))$ is a rational FU-subnilmanifold of X. Then there exist a FU-subnilmanifold Y_φ of X and a proper countably polynomial subset $S \subset \mathbb{R}^k$ such that*
(a) *$Y_t \subseteq Y_\varphi$ for all $t \in \mathbb{R}^k$;*
(b) *$Y_t = Y_\varphi$ for all $t \notin S$.*

Proof. Let Y_φ be the minimal FU-subnilmanifold of X such that $Y_t \subseteq Y_\varphi$ for all $t \in \mathbb{R}^k$. Assume that Z is a rational FU-subnilmanifold of X such that $Z \not\supseteq Y$; then there exists $t_0 \in \mathbb{R}^k$ such that $Y_{t_0} \not\subseteq Z$. Let $Z_1, \ldots, Z_s$ be connected components of Z and let $H_1, \ldots, H_s$ be connected closed subgroups of G^{o} such that $Z_i = \pi(H_i)$, $i = 1, \ldots, s$. There exists $m_0 \in M$ such that $\varphi(t_0, m_0) \notin \bigcup_{i=1}^s H_i$. Each H_i is a polynomial subset of G^{o}, and the mapping $t \mapsto \varphi_t(m_0)$, $t \in \mathbb{R}^k$, is polynomial, thus the set $S_Z = \left\{t \in \mathbb{R}^k : \varphi(t, m_0) \in \bigcup_{i=1}^s H_i\right\}$ is a proper polynomial subset of $\mathbb{R}^k$. For any $t \notin S_Z$ we have $\varphi(t, m_0) \notin Z$ and so, $Y_t \neq Z$. We now put $S = \bigcup S_Z$, where Z runs over the set of rational FU-subnilmanifolds of X (which is countable by 1.5). ■

2.2. We will now deduce a generalization of Theorem 1.10:

Theorem. *Let V be a connected subnilmanifold of X, let K be a connected component of $\pi^{-1}(V)$ and A be a closed subgroup of G. There exists a closed subnilmanifold $Y_{V,A}$ of X such that*
(a) *for any $x \in V$ one has $\overline{\mathrm{Orb}}_A(x) \subseteq aY_{V,A}$ whenever $a \in K$, $\pi(a) = x$;*
(b) *there exists a proper countably polynomial subset $P \subset V$ such that for any $x \in V \setminus P$ one has $\overline{\mathrm{Orb}}_A(x) = aY_{V,A}$ whenever $a \in K$, $\pi(a) = x$.*

We call the subnilmanifold $Y_{V,A}$ *the generic orbit for A on V*; in the case $V = X$, $Y_{V,A}$ is just *the generic orbit for A* and will be denoted by Y_A.

Proof. We may assume that $\dim V \geq 1$, $V \ni 1_X$ and K is a connected closed subgroup of G^{o}. Let $\tau: \mathbb{R}^r \longrightarrow V$ be a (Malcev) coordinate system on K. Define a mapping $\varphi: \mathbb{R}^r \times A \longrightarrow G$ by $\varphi(t,b) = \tau(t)^{-1} b \tau(t)$. Then φ is a polynomial mapping, and for each $t \in \mathbb{R}^k$ and $a = \tau(t)$ the set

$$Y_t = \overline{\pi(\varphi(t,A))} = \overline{\pi(a^{-1}Aa)} = \overline{a^{-1}Aa1_X} = \overline{\mathrm{Orb}}_{a^{-1}Aa}(1_X)$$

is a rational subnilmanifold of X. By Theorem 2.1, there exist a FU-subnilmanifold $Y_{V,A} \subseteq X$ and a proper countably polynomial subset $S \subset \mathbb{R}^k$ such that $Y_t \subseteq Y_{V,A}$ for all $t \in \mathbb{R}^k$ and $Y_t = Y_{V,A}$ for all $t \in \mathbb{R}^k \setminus S$. Since all Y_t are subnilmanifolds, $Y_{V,A}$ is also a subnilmanifold. Finally, for any $x \in V$, $x = \pi(a)$ with $a = \tau(t) \in K$, we have

$$\overline{\mathrm{Orb}}_A(x) = \overline{Ax} = \overline{Aa1_X} = a\overline{(a^{-1}Aa1_X)} = a\overline{(\tau(t)^{-1}A\tau(t))1_X} = a\overline{\pi(\varphi(t,A))} = aY_t,$$

so $\overline{\mathrm{Orb}}_A(x) \subseteq aY_{V,A}$, and $\overline{\mathrm{Orb}}_A(x) = aY_{V,A}$ whenever $x \notin P = \pi(\tau(S))$. ■

2.3. We generalize Theorem 1.13.I in a similar manner:

Theorem. *Let V be a connected subnilmanifold of X, let K be a connected component of $\pi^{-1}(V)$ and let $g: \mathbb{Z}^l \longrightarrow G$ be a polynomial sequence in G. There exists a closed FU-subnilmanifold $Y_{V,g}$ of X such that*
(a) *for any $x \in V$ one has $\overline{\mathrm{Orb}}_g(x) \subseteq aY_{V,g}$ whenever $a \in K$, $\pi(a) = x$;*
(b) *there exists a proper countably polynomial subset $P \subset V$ such that for any $x \in V \setminus P$ one has $\overline{\mathrm{Orb}}_g(x) = aY_{V,g}$ whenever $a \in K$, $\pi(a) = x$.*

We call the FU-subnilmanifold $Y_{V,g}$ *the generic orbit for g on V*; in the case $V = X$, $Y_{V,g}$ is just *the generic orbit for g* and will be denoted by Y_g.

Proof. We may assume that $g(0) = 1_G$, $\dim V \geq 1$, $V \ni 1_X$ and K is a connected closed subgroup of G^{o}. Let $\tau: \mathbb{R}^r \longrightarrow V$ be a (Malcev) coordinate system on K. Define a mapping $\varphi: \mathbb{R}^r \times \mathbb{Z}^l \longrightarrow G$ by $\varphi(t,n) = \tau(t)^{-1} g(n) \tau(t)$, then φ is a polynomial mapping. For $t \in \mathbb{R}^k$ let $Y_t = \overline{\pi(\varphi(t,\mathbb{Z}^l))}$. Putting $a = \tau(t)$, we get

$$Y_t = \overline{\pi\big(\varphi(t,\mathbb{Z}^l)\big)} = \overline{\pi\big(a^{-1}g(\mathbb{Z}^l)a\big)} = \overline{a^{-1}g(\mathbb{Z}^l)a1_X} = \overline{\mathrm{Orb}}_{g^a}(1_X),$$

where g^a is the polynomial mapping $g^a(n) = a^{-1}g(n)a$, $n \in \mathbb{Z}^l$. Hence, Y_t is a rational FU-subnilmanifold of X. By Theorem 2.1, there exist a FU-subnilmanifold $Y_{V,g} \subseteq X$ and a proper countably polynomial subset $S \subset \mathbb{R}^k$ such that $Y_t \subseteq Y_{V,g}$ for all $t \in \mathbb{R}^k$ and $Y_t = Y_{V,g}$ for all $t \in \mathbb{R}^k \setminus S$. For any $x \in V$, $x = \pi(a)$ with $a = \tau(t) \in K$, we have

$$\begin{aligned}\overline{\mathrm{Orb}}_g(x) = \overline{g(\mathbb{Z}^l)x} = \overline{g(\mathbb{Z}^l)a1_X} = a\big(\overline{a^{-1}g(\mathbb{Z}^l)a1_X}\big) &= a\big(\overline{\tau(t)^{-1}g(\mathbb{Z}^l)\tau(t)1_X}\big) \\ &= a\big(\overline{\pi(\varphi(t,\mathbb{Z}^l))}\big) = aY_t,\end{aligned}$$

so $\overline{\mathrm{Orb}}_g(x) \subseteq aY_{V,g}$, and $\overline{\mathrm{Orb}}_g(x) = aY_{V,g}$ whenever $x \notin P = \pi(\tau(S))$. ■

2.4. The following example shows that the set of points having non-generic orbits may not be a union of subnilmanifolds of X.

Example. Let $G = \left\{ \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & x_{1,4} \\ 0 & 1 & x_{2,3} & x_{2,4} \\ 0 & 0 & 1 & x_{3,4} \\ 0 & 0 & 0 & 1 \end{pmatrix},\ x_{i,j} \in \mathbb{R} \right\}$, $\Gamma = \left\{ \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & x_{1,4} \\ 0 & 1 & x_{2,3} & x_{2,4} \\ 0 & 0 & 1 & x_{3,4} \\ 0 & 0 & 0 & 1 \end{pmatrix},\ x_{i,j} \in \mathbb{Z} \right\}$ and $X = G/\Gamma$. Let $b = \begin{pmatrix} 1 & \alpha & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, where α is an irrational number, and $A = \{b^n\}_{n \in \mathbb{Z}}$. For $a = \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & x_{1,4} \\ 0 & 1 & x_{2,3} & x_{2,4} \\ 0 & 0 & 1 & x_{3,4} \\ 0 & 0 & 0 & 1 \end{pmatrix} \in G$ one finds that $a^{-1}b^n a = \begin{pmatrix} 1 & n\alpha & n\alpha x_{2,3} & n\alpha x_{2,4} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $n \in \mathbb{Z}$. So, the generic orbit for A is the 3-dimensional torus $Y_A = \begin{pmatrix} 1 & u & v & w \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$; when the numbers $\alpha, \alpha x_{2,3}, \alpha x_{2,4}$ are rationally dependent, the point $\pi(a)$ has a nongeneric orbit, which is a 1- or a 2-dimensional subtorus of Y_A. Let $Q = \left\{ \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & x_{1,4} \\ 0 & 1 & x_{2,3} & 0 \\ 0 & 0 & 1 & x_{3,4} \\ 0 & 0 & 0 & 1 \end{pmatrix},\ x_{i,j} \in \mathbb{R} \right\} \subset G$, then every $x \in \pi(Q)$ has a 2-dimensional nongeneric orbit. Q is a connected polynomial subset of G with $\pi(Q)$ dense in X.

3. The normality of generic orbits

If g is a polynomial sequence in G with generic orbit Y_g on X and x is a point of X having generic orbit under the action of g, then $\overline{\mathrm{Orb}_g(x)} = aY_g$ *for all* $a \in G^{\mathrm{o}}$ with $\pi(a) = x$. This gives us some additional information about generic orbits.

3.1. Let us say that a subnilmanifold Y of X is *normal* if $Y = Hx$ where $x \in X$ and H is a normal subgroup of G^{o}.

3.2. The importance of the notion of normality is manifested by the following fact:

Proposition. *Let Y be a connected subnilmanifold of X, $Y = Hx$ where $x \in X$ and H is a connected closed subgroup of G^{o}. The following are equivalent:*
(i) *Y is normal;*
(ii) *the sets Hy are closed in X for all $y \in X$;*
(iii) *the subnilmanifolds aY, $a \in G^{\mathrm{o}}$, partition X.*

Proof. If Y is normal then H is normal in G^{o}, so $aH = Ha$ and thus, $aY = Hax$ for all $a \in G^{\mathrm{o}}$. The sets aH, $a \in G^{\mathrm{o}}$, are closed and the sets Hax, $a \in G^{\mathrm{o}}$, partition X, so we have (ii) and (iii).

Assume that the sets Hy are all closed. This means that the sets $\pi(Ha)$, $a \in G^{\mathrm{o}}$, are closed, and so, the sets $\pi(a^{-1}Ha)$, $a \in G^{\mathrm{o}}$, are closed. Thus, for any $a \in X$, $a^{-1}Ha$ is a rational subgroup of G^{o}; since there are only countably many of such and $a^{-1}Ha$ continuously depends on a, $a^{-1}Ha = H$ for all $a \in G^{\mathrm{o}}$.

Let now the sets aY, $a \in G^{\mathrm{o}}$, partition X. We may assume that $x = 1_X$, so that $Y = \pi(H)$ and $1_X \in Y$. Then for any $\gamma \in \Gamma^{\mathrm{o}}$, γY contains 1_X, thus $\gamma Y = Y$. Let $\gamma \in \Gamma^{\mathrm{o}}$; then $\gamma H\Gamma^{\mathrm{o}} = H\Gamma^{\mathrm{o}}$ and, since H is connected, $\gamma H = H\gamma'$ for some $\gamma' \in \Gamma^{\mathrm{o}}$. So, $\gamma H\gamma^{-1} = H\gamma'\gamma^{-1}$, and since $\gamma H\gamma^{-1}$ is a subgroup of G^{o}, $\gamma H\gamma^{-1} = H$. It remains to apply the following lemma:

3.3. Lemma. *If a subgroup H of G^{o} is normalized by Γ^{o} then H is normal in G^{o}.*

Sketch of the proof. G^{o} is an exponential group, which means that for any $a \in G^{\mathrm{o}}$ there exists a one-parametric flow $t \to a^t$, $t \in \mathbb{R}$, passing through a. Let $a \in G^{\mathrm{o}}$ be such that $a^{t_0} \in \Gamma$ for some nonzero $t_0 \in \mathbb{R}$. The condition "a^t normalizes H" is polynomial with respect to t, so, since a^{nt_0} normalizes H for all $n \in \mathbb{Z}$, a^t normalizes H for all $t \in \mathbb{R}$. G^{o} is generated by elements $a \in G^{\mathrm{o}}$ with $a^{t_0} \in \Gamma$ for some nonzero $t_0 \in \mathbb{R}$, thus G^{o} normalizes H. ■

3.4. If Y is a normal subnilmanifold of X, the *factor-nilmanifold* X/Y is defined. Indeed, assume that $1_X \in Y$ and let H be the closed normal subgroup of G^{o} such that $Y = \pi(H)$. Then $\pi^{-1}(Y) = H\Gamma^{\mathrm{o}}$ is a closed uniform subgroup of G^{o}; define $Z = G^{\mathrm{o}}/(H\Gamma^{\mathrm{o}})$. Z is a nilmanifold, and the fibers of the natural mapping $X \longrightarrow Z$ are translates of Y.

3.5. We will now show:

Theorem. *Let A be a subgroup of G and Y_A be the generic orbit for A. The connected components of Y_A are normal subnilmanifolds of X.*

Proof. Let P be the set, introduced in Theorem 2.2, of points whose orbits under the action of A are nongeneric on X. Let $x \notin P$; we may assume that $x = 1_X$. Then, by Theorem 2.2, for any $\gamma \in \Gamma^{\mathrm{o}}$, $\overline{\mathrm{Orb}}_A(1_X) = \overline{\mathrm{Orb}}_A(\pi(\gamma)) = \gamma Y_A$. So, $\gamma Y_A = Y_A$ for all $\gamma \in \Gamma^{\mathrm{o}}$. Let H be the closed subgroup of G^{o} such that $Y_A = \pi(H)$ and let H^{o} be the identity component of H. Let $\gamma \in \Gamma^{\mathrm{o}}$, then $\gamma H\Gamma^{\mathrm{o}} = H\Gamma^{\mathrm{o}}$, and $\gamma H^{\mathrm{o}} = H^{\mathrm{o}}c\gamma'$ for some $c \in H$ and $\gamma' \in \Gamma^{\mathrm{o}}$. So $\gamma H^{\mathrm{o}}\gamma^{-1} = H^{\mathrm{o}}c\gamma'\gamma^{-1}$, and since $\gamma H^{\mathrm{o}}\gamma^{-1}$ is a subgroup of G^{o}, $\gamma H^{\mathrm{o}}\gamma^{-1} = H^{\mathrm{o}}$. Hence, H^{o} is normalized by Γ^{o}; by Lemma 3.3, H^{o} is normal in G^{o}. ■

3.6. Similarly, we have

Theorem. *If Y_g is the generic orbit for a polynomial sequence g in G then the connected components of Y_g are normal subnilmanifolds of X.*

Proof. Let P be the set, introduced in Theorem 2.3, of points whose orbits under the action of g are nongeneric on X. Let $x \notin P$; we may assume that $x = 1_X$. Then, by Theorem 2.3, for any $\gamma \in \Gamma^{\mathrm{o}}$, $\overline{\mathrm{Orb}}_g(1_X) = \overline{\mathrm{Orb}}_g(\pi(\gamma)) = \gamma Y_g$. So, $\gamma Y_g = Y_g$ for all $\gamma \in \Gamma^{\mathrm{o}}$. Let H be the connected closed subgroup of G^{o} such that $Y_g = \bigcup_{i=1}^{s} Hx_i$. Let $\gamma \in \Gamma^{\mathrm{o}}$, then $\gamma H = Hc\gamma'$ for some $c \in G^{\mathrm{o}}$ and $\gamma' \in \Gamma^{\mathrm{o}}$. So $\gamma H\gamma^{-1} = Hc\gamma'\gamma^{-1}$, and since $\gamma H\gamma^{-1}$ is a subgroup of G^{o}, $\gamma H\gamma^{-1} = H$. Hence, H is normalized by Γ^{o}; by Lemma 3.3, H is normal in G^{o}. ■

3.7. Let us informally describe the picture we have got. Let A be a subgroup of G. If $\overline{\mathrm{Orb}}_A(x) = X$ for some point $x \in X$ then $\overline{\mathrm{Orb}}_A(y) = X$ for all $y \in X$. Otherwise, the generic orbit Y_A for A is a proper subnilmanifold of X. Let Y be a connected component of Y_A, then Y is normal in X and thus the factor-nilmanifold $Z = X/Y$ is defined; the fibers of the projection mapping $\eta\colon X \longrightarrow Z$ are translates of Y. A acts on Z in a finite way; after passing to a subgroup of finite index in A we may assume that the action of A on Z is trivial, and A acts only on the fibers of η. For almost every $z \in Z$ the action of A on $\eta^{-1}(z)$ is minimal, that is, the subnilmanifold $\eta^{-1}(z)$ is the orbit of all its points. If a fiber $V=\eta^{-1}(z)$ is not the orbit of its points then the generic orbit $Y_{V,A}$ of points of V is a proper subnilmanifold of V, V is partitioned by translates of its connected component, etc.

For the action on X of a polynomial sequence g the picture is similar. A difference is that orbits of distinct points do not partition X; they may contain one another, or have a nontrivial intersection. That is, assuming $g(0)=1_G$, even if a translate $V=aY_g$ of the generic orbit Y_g for g is the orbit of some its point, $V=\overline{\mathrm{Orb}}_g(x)$, it does not have to be true for all other points of V; however, in this case $V=\overline{\mathrm{Orb}}_g(y)$ for almost all $y\in V$.

4. Relation between linear and polynomial generic orbits

Let $g\colon\mathbb{Z}^l\longrightarrow G$ be a polynomial sequence in G with $g(0)=1_G$ and let A be the subgroup of G generated by the elements of g. Let $Y_g\subseteq X$ be the generic orbit for g and $Y_A\subseteq X$ be the generic orbit for A. We will investigate the relation between Y_g and Y_A. Clearly, $Y_g\subseteq Y_A$.

4.1. Let us first assume that Y_g is connected. Let $x\in X$ be a point of X that has generic orbit under the action of g; let $x=\pi(a)$, $a\in G^{\mathrm{o}}$, so that $\overline{\mathrm{Orb}}_g(x)=aY_g$. For any $y\in aY_g$ by Theorem 2.3(a) and Theorem 3.6 we have $\overline{\mathrm{Orb}}_g(y)\subseteq aY_g$, so $g(n)y\in aY_g$ for all $n\in\mathbb{Z}^l$. It follows that A preserves $\overline{\mathrm{Orb}}_g(x)$ and hence, $\overline{\mathrm{Orb}}_A(x)\subseteq\overline{\mathrm{Orb}}_g(x)$. Since this is true for almost all points of X, we have $Y_A\subseteq Y_g$.

4.2. We obtain the following result:

Proposition. *Let $g\colon\mathbb{Z}^l\longrightarrow G$ be a polynomial sequence in G with $g(0)=1_G$, let A be the subgroup of G generated by the elements of g, let $Y_g\subseteq X$ be the generic orbit for g and $Y_A\subseteq X$ be the generic orbit for A. If Y_g is connected, then $Y_g=Y_A$.*

4.3. The case where Y_g is not connected can be reduced to the previous one. It is proven in [L2] that, given a point $x\in X$, there exists a subgroup ω of finite index in $\mathbb{Z}^l$ such that, for the restriction g_ω of g on ω the orbit $\overline{\mathrm{Orb}}_{g_\omega}(x)$ is connected. It follows that for some subgroup $\omega\subseteq\mathbb{Z}^l$ of finite index the generic orbit Y_{g_ω} for g_ω is connected. (Indeed, since $\mathbb{Z}^l$ has only countably many subgroups, there exists a subgroup ω of finite index for which the set of x with connected orbits under the action of g_ω has positive measure.) Y_{g_ω} is then a connected component of Y_g.

Let A_ω be the group generated by the elements of g_ω; by Proposition 4.2, the generic orbit for A_ω is Y_{g_ω}. It is easy to see that A_ω has finite index in A, thus Y_{g_ω} coincides with one of the connected components of Y_ω. Hence, the connected components of Y_g have the same dimension as components of Y_A, and so, coincide with them. This proves Theorem 1.13.II:

4.4. Theorem. *Let $g\colon\mathbb{Z}^l\longrightarrow G$ be a polynomial sequence in G with $g(0)=1_G$, let A be the subgroup of G generated by the elements of g, let $Y_g\subseteq X$ be the generic orbit for g and $Y_A\subseteq X$ be the generic orbit for A. Then Y_g is a union of connected components of Y_A.*

4.5. Remark. If V is a connected subnilmanifold of X, the generic orbit $Y_{V,g}$ for g on V may not be a union of connected components of the generic orbit $Y_{V,A}$ for A on V. This can already be seen on the trivial example where V is a single point.

4.6. Corollary. *The connected components of Y_g are all congruent.*

4.7. An open question. I cannot answer the following question: are the connected components of any nongeneric orbit for g also congruent to each other?

4.8. Let $L = [G^{\mathrm{o}}, G^{\mathrm{o}}]\backslash X$; we will call L *the maximal factor-torus* of X.

Let g be a polynomial sequence in G. It is proven in [L2] that if $\overline{\mathrm{Orb}}_g(u) = L$ for a point $u \in L$ then $\overline{\mathrm{Orb}}_g(x) = X$ for any $x \in X$.

For "linear" actions on X a stronger statement holds. Now let $N = [G, G]\backslash X$. N is a factor-torus of L, and dealing with N is easier than with L since G acts on the torus N by conventional, "abelian" shifts, whereas on L it may act by "sqew-shifts", that is, by unipotent affine transformations (see Example 1.81 above). Let A be a subgroup of G; then, assuming that G is generated by G^{o} and A, one has $\overline{\mathrm{Orb}}_A(x) = X$ for all $x \in X$ whenever $\overline{\mathrm{Orb}}_A(v) = N$ for some $v \in N$. Example 1.12 shows that an analogous statement does not hold for a polynomial action; we, however, get the following:

4.9. Corollary. *Let g be a polynomial sequence in G and assume that G is generated by G^{o} and elements of g. Let $N = [G, G]\backslash X$, and assume that $\overline{\mathrm{Orb}}_g(v) = N$ for some $v \in N$. Then the generic orbit for g is equal to X.*

Proof. We may assume that $g(0) = 1_G$. Let A be the group generated by the elements of g. Then $\overline{\mathrm{Orb}}_A(v) = N$, so the generic orbit for A is X, and by Proposition 4.2, X is the generic orbit for g. ∎

5. Orbits of a subnilmanifold

Let V be a connected subnilmanifold of X; we will assume for simplicity that $V \ni 1_X$ and so, $V = \pi(K)$ where K is a connected closed subgroup of G^{o}. For a subgroup A of G or a polynomial sequence $g\colon \mathbb{Z}^l \longrightarrow G$ we may now investigate (the closures of) the orbits $\overline{\mathrm{Orb}}_A(V) = \overline{AV}$ and $\overline{\mathrm{Orb}}_g(V) = \overline{g(\mathbb{Z}^l)V}$ of V under the action of A and g respectively. It is shown in [L3] that $\overline{\mathrm{Orb}}_A(V)$ is a subnilmanifold and $\overline{\mathrm{Orb}}_g(V)$ is a FU-subnilmanifold of X; in this section we will study a relation between these orbits of V and the generic orbits for A and g on V.

5.1. We first extend the notion of normality of a subnilmanifold introduced in 3.1. Let Y be a subnilmanifold of X, $Y = Hx$ where $x \in X$ and H is a closed subgroup of G^{o}. Let us say that Y is *normal with respect to V* if K normalizes H.

5.2. Proposition. *Let H be a closed subgroup of G^{o} and let $x \in V$. If the subnilmanifold $Y = Hx$ of X is normal with respect to V, then*
(i) *the sets Hy, $y \in V$, are all closed;*
(ii) *the set $W = HV$ is a subnilmanifold of X with* $\dim W = \dim V + \dim Y - \dim(V \cap Y)$,

and the sets aY, $a \in K$, partition W;
(iii) *the subnilmanifolds $aY \cap V$, $a \in K$, of V are all congruent and partition V.*

Proof. We may assume that $x = 1_X$ and so, $Y = \pi(H)$. Since K normalizes H, the set $HK = KH$ is a closed subgroup of G^o. $\Gamma \cap K$ is uniform in K and $\Gamma \cap H$ is uniform in H, thus $\Gamma \cap (KH)$ is uniform in KH. Thus, $W = \pi(HK) = HV$ is a subnilmanifold of X.

H is a normal subgroup of KH, thus Y is a normal subnilmanifold of W. Hence, the sets Hy, $y \in V$, are equal to the sets aY, $a \in K$, are closed and partition W. $H \cap K$ is a normal subgroup of K, thus $Y \cap V$ is a normal subnilmanifold of V, so the sets $aY \cap V = aY \cap aV = a(Y \cap V)$ partition V. The factor-nilmanifold W/Y is isomorphic to the factor-nilmanifold $V/(V \cap Y)$, so $\dim W = \dim Y + \dim(V/(V \cap Y)) = \dim Y + \dim V - \dim(V \cap Y)$. ■

5.3. Let us denote the subnilmanifold $W = HV$, appearing in Proposition 5.2, by YV.

5.4. We now have:

Theorem. *Let A be a subgroup of G. The connected components of the generic orbit $Y_{V,A}$ of A on V are normal with respect to V.*

5.5. We need an extension of Lemma 3.3:

Lemma. *Let H and K be subgroups of G^o, let Λ be a uniform subgroup of K and assume that Λ normalizes H. Then K normalizes H.*

The proof of this lemma is similar to the proof of Lemma 3.3.

5.6. Proof of Theorem 5.4. Let $\Lambda = \Gamma \cap K$, this is a uniform subgroup of K. Let P be the set, introduced in Theorem 2.2, of points of V whose orbits under the action of A are nongeneric on V. Let $x \in V \setminus P$; we may assume that $x = 1_X$. Then, by Theorem 2.2, for any $\lambda \in \Lambda$, $\overline{\mathrm{Orb}}_A(1_X) = \overline{\mathrm{Orb}}_A(\pi(\lambda)) = \lambda Y_{V,A}$. So, $\lambda Y_{V,A} = Y_{V,A}$ for all $\lambda \in \Lambda$. Let H be the closed subgroup of G^o such that $Y_{V,A} = \pi(H)$ and let H^o be the identity component of H. For $\lambda \in \Lambda$ we have $\lambda H\Gamma^o = H\Gamma^o$, and $\lambda H^o = H^o c\gamma$ for some $c \in H$ and $\gamma \in \Gamma^o$. So $\lambda H^o \lambda^{-1} = H^o c\gamma\lambda^{-1}$, and since $\lambda H^o \lambda^{-1}$ is a subgroup of G^o, $\lambda H^o \lambda^{-1} = H^o$. Hence, H^o is normalized by Λ; by Lemma 5.5, H^o is normalized by K. ■

5.7. As a corollary, we get

Theorem. *Let A be a subgroup of G and Y be a connected component of the generic orbit $Y_{V,A}$ of A on V. The connected components of the orbit $\overline{\mathrm{Orb}}_A(V)$ of V under the action of A are translates of YV.*

Proof. If $Y_{V,A} = Y$ is connected, it is normal with respect to V, thus YV is defined and is a closed subnilmanifold of X. For every point $x \in V$, $x = \pi(a)$ with $a \in K$, we have $\overline{\mathrm{Orb}}_A(x) \subseteq aY \subseteq YV$, thus $\overline{\mathrm{Orb}}_A(V) \subseteq YV$. For almost every point $x \in V$, $x = \pi(a)$ with $a \in K$, we have $\overline{\mathrm{Orb}}_A(x) = aY$, thus $\bigcup_{a \in K} \overline{\mathrm{Orb}}_A(x)$ is dense in YV, and so, $\overline{\mathrm{Orb}}_A(V) = YV$.

If $Y_{V,A}$ is not connected and Y is its connected component, we can find in A a subgroup B of finite index such that $Y_{V,B} = Y$. Thus, $\overline{\mathrm{Orb}}_B(V) = YV$. Now, $A = \bigcup_{i=1}^{s} b_i B$ for some $b_1,\dots,b_s\in A$, and thus $\overline{\mathrm{Orb}}_A(V)=\bigcup_{i=1}^s b_iYV$. ∎

5.8. Similarly, we have

Theorem. *Let g be a polynomials sequence in G. The connected components of the generic orbit $Y_{V,g}$ of g on V are normal with respcet to V.*

Proof. Let $g\colon\mathbb{Z}^l\longrightarrow G$; after passing to a subgroup of finite index in $\mathbb{Z}^l$ we may assume that $Y_{V,g}$ is connected. Next, we may assume that $g(0)=1_G$. Let P be the set, introduced in Theorem 2.3, of points whose orbits under the action of g are nongeneric on X. Let $x\notin P$; we may assume that $x=1_X$. Let $\Lambda=\Gamma\cap K$, this is a uniform subgroup of K. By Theorem 2.3, for any $\lambda\in\Lambda$, $\overline{\mathrm{Orb}}_g(1_X)=\overline{\mathrm{Orb}}_g(\pi(\lambda))=\lambda Y_{V,g}$. So, $\lambda Y_{V,g}=Y_{V,g}$ for all $\lambda\in\Lambda$. Let H be the connected closed subgroup of G^{o} such that $Y_{V,g}=\pi(H)$. Let $\lambda\in\Lambda$, then $\lambda H=Hc\gamma$ for some $\gamma\in\Gamma^{\mathrm{o}}$. So $\gamma H\lambda^{-1}=H\gamma\lambda^{-1}$, and since $\lambda H\lambda^{-1}$ is a subgroup of G^{o}, $\lambda H\lambda^{-1}=H$. Hence, H is normalized by Λ; by Lemma 5.5, H is normalized by K. ∎

5.9. And as a corollary we obtain

Theorem. *Let g be a polynomial sequence in G. Every connected component of the orbit $\overline{\mathrm{Orb}}_g(V)$ of V under the action of g is a translate of YV, where Y is a connected component of the generic orbit $Y_{V,g}$ of g on V.*

Proof. Again, by passing to a subgroup of finite index in $\mathbb{Z}^l$ the problem is reduced to the case $Y_{V,g}=Y$ is connected. Y is normal with respect to V, thus YV is a closed subnilmanifold of X. For every point $x\in V$, $x=\pi(a)$ with $a\in K$, we have $\overline{\mathrm{Orb}}_g(x)\subseteq aY\subseteq YV$, thus $\overline{\mathrm{Orb}}_g(V)\subseteq YV$. For almost every point $x\in V$, $x=\pi(a)$ with $a\in K$, we have $\overline{\mathrm{Orb}}_g(x)=aY$, thus $\bigcup_{a\in K}\overline{\mathrm{Orb}}_g(x)$ is dense in YV, and so, $\overline{\mathrm{Orb}}_g(V)=YV$. ∎

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