Pointwise convergence of ergodic averages for polynomial sequences of translations on a nilmanifold

A. Leibman
Department of Mathematics
The Ohio State University
Columbus, OH 43210, USA
e-mail: leibman@math.ohio-state.edu

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Abstract

We show that the orbit of a point on a compact nilmanifold X under the action of a polynomial sequence of translations on X is well distributed on the union of several sub-nilmanifolds of X. This implies that the ergodic averages of a continuous function on X along a polynomial sequence of translations on X converge pointwise.

1. Formulations

1.1. Let G be a nilpotent Lie group and X be a compact homogeneous space of G, that is, $X = G/\Gamma$ where Γ is a closed uniform (=cocompact) subgroup of G; we will call X *a nilmanifold*. G acts on X by left translations: for $a \in G$ and $x = b\Gamma \in X$ one defines $ax = ab\Gamma$.

1.2. Let $x \in X$ and $a \in G$; it is proved in [Le] that the orbit $\{a^n x\}_{n\in\mathbb{Z}}$ of x under the action of a is uniformly distributed on a sub-nilmanifold of X. A much more general result of this sort was obtained in [R]: if X is a finite volume homogeneous space of a (not necessarily nilpotent) Lie group G and W is a connected subgroup of G generated by one-parameter subgroups whose Ad_G-actions are unipotent, then for any $x \in X$ there exists a closed subgroup $F \subseteq G$ such that $Fx = \overline{Wx}$ and Wx is uniformly distributed on Fx. In [Sh2] this theorem is extended to the case where W is not necessarily connected. In [Sh1] an analogous result was obtained for continuous polynomial trajectories $\{P(u)x\}_{u\in\mathbb{R}^k}$, with P being a polynomial mapping $\mathbb{R}^k \longrightarrow G$. We consider here discrete polynomial trajectories $\{g(n)x\}_{n\in\mathbb{Z}}$ on nilmanifolds only.

1.3. A sequence $\{g(n)\}_{n\in\mathbb{Z}}$ in G of the form $g(n) = a_1^{p_1(n)} \cdots a_m^{p_m(n)}$, where $a_1, \ldots, a_m \in G$ and $p_1, \ldots, p_m$ are polynomials taking on integer values on the integers, is called *polynomial*. Polynomial sequences in nilpotent groups arise very naturally, and many classical ergodic-theoretical results remain valid after replacing the sequence of powers T^n of a (unitary, continuous, measure preserving) transformation T by a polynomial sequence in a nilpotent group of transformations (see [L1], [L2], [BL]).

1.4. Our main goal is to establish the following fact:

Theorem A. *Let g be a polynomial sequence in G. For any $x \in X$, $f \in C(X)$ and Følner sequence $\{\Phi_N\}_{N=1}^\infty$ in $\mathbb{Z}$, $\lim_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} f\bigl(g(n)x\bigr)$ exists.*

1.5. We will denote by μ the G-invariant probability measure on X. A sequence $\{x_n\}_{n\in\mathbb{Z}}$ of points of X is said to be *well distributed* on X if for any open subset U of X the set $\bigl\{n \in \mathbb{Z} : x_n \in U\bigr\}$ has density $\mu(U)$ with respect to any Følner sequence in $\mathbb{Z}$. Equivalently, $\{x_n\}_{n\in\mathbb{Z}}$ is well distributed on X if for any continuous function f on X and any Følner sequence $\{\Phi_N\}_{N=1}^\infty$ in $\mathbb{Z}$ one has $\lim_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} f\bigl(x_n\bigr) = \int_X f\,d\mu$.

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1.6. A closed subset Y of X of the form $Y = Hx$, where $x \in X$ and H is a closed subgroup of G, will be called *a sub-nilmanifold* of X. We will show that the orbit of any point of X under the action of a polynomial sequence of translations on X is well distributed on the union of several sub-nilmanifolds of X:

Theorem B. *Let g be a polynomial sequence in G and let $x \in X$. There exist a connected closed subgroup H of G and points $x_1, x_2, \ldots, x_k \in X$, not necessarily distinct, such that the sets $Y_j = Hx_j$, $j = 1, \ldots, k$, are closed sub-nilmanifolds of X, $\overline{\mathrm{Orb}(x)} = \overline{\{g(n)x\}}_{n\in\mathbb{Z}} = \bigcup_{j=1}^{k} Y_j$, the sequence $g(n)x$, $n \in \mathbb{Z}$, cyclically visits the sets $Y_1, \ldots, Y_k$ and for each $j = 1, \ldots, k$ the sequence $\{g(j + nk)\}_{n\in\mathbb{Z}}$ is well distributed on Y_j.*

1.7. Example. The following simple example demonstrates that, unlike the linear case, in the polynomial case $\overline{\mathrm{Orb}(x)}$ need not be a sub-nilmanifold of X: for $G = \mathbb{R}$, $\Gamma = \mathbb{Z}$ and $g(n) = \frac{n^2}{3} \in \mathbb{R}$ one has $\mathrm{Orb}(0) = \{0, \frac{1}{3}\} \subset \mathbb{R}/\mathbb{Z}$.

1.8. Regarding Theorem B the following question remains open to us: if $k \geq 2$, are the nilmanifolds $Y_1, \ldots, Y_k$ isomorphic to each other?

1.9. If Y is a sub-nilmanifold of X, $Y = Hx$, let μ_Y denote the H-invariant probability measure on Y. Using this notation, we get the following corollary of Theorem B:

Corollary. *For any $f \in C(X)$ and any Følner sequence $\{\Phi_N\}_{N=1}^{\infty}$ in $\mathbb{Z}$, $\lim_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} f\big(g(n)x\big) = \frac{1}{k}\sum_{j=1}^{k} \int_{Y_j} f\, d\mu_{Y_j}$.*

In particular, Theorem A follows.

1.10. Assume that X is connected, and let G^o be the identity component of G. Then X is a homogeneous space of G^o, $X = G^o/(\Gamma \cap G^o)$. Let $T = [G^o, G^o]\backslash X = G^o/\big((\Gamma \cap G^o)[G^o, G^o]\big)$. T is a compact connected abelian Lie group, that is, a torus; we will refer to it as to *the maximal factor-torus* of X. Let $p: X \longrightarrow T$ be the factorization mapping. In this situation we obtain a simple criterion of "ergodicity" of a polynomial sequence of translations of X (cf. [P1] and [P2]):

Theorem C. *Assume that X is connected, let $x \in X$ and let g be a polynomial sequence in G. The following are equivalent:*
(i) *the sequence $\{g(n)x\}_{n\in\mathbb{Z}}$ is dense in X;*
(ii) *$\{g(n)x\}_{n\in\mathbb{Z}}$ is well distributed on X;*
(iii) *the sequence $\{g(n)p(x)\}_{n\in\mathbb{Z}}$ is dense/well distributed on T.*

1.11. Let G be a nilpotent Lie group with a uniform subgroup Γ and let the discrete group G/G^o be finitely generated. Then one can show that G is a factor, $\eta: \widetilde{G} \longrightarrow G$, of a simply-connected nilpotent Lie group $\widetilde{G}$. Let $\widetilde{\Gamma} = \eta^{-1}(\Gamma)$. Further, there exists a connected simply-connected nilpotent Lie group $\widehat{G}$ with a uniform subgroup $\widehat{\Gamma}$ such that $\widetilde{G} \subseteq \widehat{G}$ and $\widetilde{\Gamma} = \widehat{\Gamma} \cap \widetilde{G}$. So, $X = G/\Gamma$ is isomorphic to a sub-nilmanifold of $\widehat{X} = \widehat{G}/\widehat{\Gamma}$, with all translations from G represented in $\widehat{G}$. It follows that when proving Theorem B, one may restrict himself to the case of a connected simply-connected G. We will not utilize this fact.

1.12. We first prove analogs of Theorems B and C in the "linear" case, where g is not a polynomial sequence but a group homomorphism from a finitely generated amenable group. These results are a very special case of general theorems of Ratner and Shah ([R], [Sh2]), but using a method of Parry ([P1] and [P2]) we can obtain a simple and independent proof thereof. Then we exploit Furstenberg's idea ([F], p. 31) to represent a "polynomial" orbit of a point on a nilmanifold as a projection of the "linear" orbit of a point on a "larger" nilmanifold.

2. Linear case

We suppose that G is a nilpotent Lie group, Γ is a closed uniform subgroup of G and $X = G/\Gamma$ is a compact nilmanifold.

2.1. We will denote by G^o the identity component of G. If X is connected, then $X = (G^o\Gamma)/\Gamma$ and $G = G^o\Gamma$. If X is disconnected then $X^o = (G^o\Gamma)/\Gamma \simeq G^o/(\Gamma \cap G^o)$ is a connected component of X and, since X is compact, X is a disjoint union of finitely many translates of X^o: $X = \bigcup_{j=1}^{l} b_j X^o$, $b_1, \dots, b_l \in G$. Thus, X is a homogeneous space of the group generated by G^o and $b_1, \dots, b_l$. When we study the action on X of a finitely generated subgroup A of G, we may replace G by the group generated by G^o, $b_1, \dots, b_l$ and the generators of A. Therefore, we may and will assume that the group G/G^o is finitely generated.

2.2. Let π be the factorization mapping $G \longrightarrow X = G/\Gamma$ and let $x = \pi(\mathbf{1}_G) \in X$. Let H be a closed subgroup of G. In general, the image of H in X, $\pi(H) = Hx = (H\Gamma)/\Gamma$, need not be a submanifold of X. H acts on Hx with $\mathrm{Stab}(x) = \Gamma \cap H$, so one has a continuous bijection $\xi\colon H/(\Gamma \cap H) \longrightarrow Hx$. If $\Gamma \cap H$ is uniform in H then $H/(\Gamma \cap H)$ is compact, so ξ is a homeomorphism and Hx is a homogeneous space of H. On the other hand, H is locally compact and separable, so when Hx is locally compact ξ is a homeomorphism ([MZ] Theorem 2.13). Thus, if Hx is closed, that is, if $H\Gamma$ is closed in G, then ξ is again a homeomorphism. It follows that the statements "Hx is a closed sub-nilmanifold of X", "$H\Gamma$ is closed in G" and "$\Gamma \cap H$ is uniform in H" are equivalent.

2.3. We will now list some properties of nilpotent Lie groups which we are going to use in the sequel. Most of this can be found in, or deduced from, [M].

2.4. Any connected nilpotent Lie group G is exponential, that is, the exponential mapping $\mathfrak{G} \longrightarrow G$ from the Lie algebra $\mathfrak{G}$ of G is surjective. It follows that for any $a \in G$ there exists a one-parameter subgroup $\{\alpha(t)\}_{t \in \mathbb{R}}$ in G such that $\alpha(1) = a$. We will write a^t for $\alpha(t)$, assuming that α is fixed for a.

2.5. Let G be a connected simply-connected nilpotent Lie group and Γ be a closed uniform subgroup of G. Then G possesses *a Malcev basis*, a finite set $\{a_1, \dots, a_l\} \subseteq \Gamma$ such that any $a \in G$ is uniquely representable in the form $a = a_1^{t_1} \cdots a_l^{t_l}$, $t_1, \dots, t_l \in \mathbb{R}$.

The correspondence $a \mapsto (t_1, \dots, t_l)$ produces a homeomorphism $G \longrightarrow \mathbb{R}^l$. Under this homeomorphism the multiplication in G is given by a polynomial mapping $\mathbb{R}^l \times \mathbb{R}^l \longrightarrow \mathbb{R}^l$. It follows that any polynomial sequence g in G can be written in the basis $\{a_1, \dots, a_l\}$: $g(n) = a_1^{p_1(n)} \cdots a_l^{p_l(n)}$, $p_1, \dots, p_l \in \mathbb{R}[n]$.

2.6. Any connected nilpotent Lie group G is a factor group of a connected simply-connected nilpotent Lie group $\widetilde{G}$. (One can take as $\widetilde{G}$ the universal cover of G.) Choose a Malcev basis in $\widetilde{G}$ and let $\{a_1, \dots, a_l\}$ be the projection of this basis to G. Then any $a \in G$ is representable (not necessarily uniquely) in the form $a = a_1^{t_1} \cdots a_l^{t_l}$, $t_1, \dots, t_l \in \mathbb{R}$.

If G is not connected, then the finitely generated group G/G^o also has *a basis*, that is, a subset $\{e_1, \dots, e_m\} \subseteq G$ such that every element of G/G^o is representable in the form $e_1^{n_1} \cdots e_m^{n_m} G^o$, $n_1, \dots, n_m \in \mathbb{Z}$. Every element of G is then representable in the form $a_1^{t_1} \cdots a_l^{t_l} e_1^{n_1} \cdots e_m^{n_m}$, $t_1, \dots, t_l \in \mathbb{R}$, $n_1, \dots, n_m \in \mathbb{Z}$. In the coordinates $(t_1, \dots, t_l, n_1, \dots, n_m)$ the multiplication in G is given by ordinary polynomials; it follows that any polynomial sequence in G can be written as $g(n) = a_1^{p_1(n)} \cdots a_l^{p_l(n)} e_1^{q_1(n)} \cdots e_m^{q_m(n)}$, where $p_1, \dots, p_l$ are polynomials $\mathbb{Z} \longrightarrow \mathbb{R}$ and $q_1, \dots, q_m$ are polynomials $\mathbb{Z} \longrightarrow \mathbb{Z}$.

2.7. If Γ is a uniform subgroup of G then, in the notation of 2.6, $a_1, \dots, a_l$ can be taken from Γ. If $G = G^o\Gamma$ then $e_1, \dots, e_m$ can also be chosen from Γ. Otherwise $G^o\Gamma$ has finite index in G and so, there exists $d \in \mathbb{N}$ such that $b^d \in G^o\Gamma$ for any $b \in G$.

Lemma. *For any $b \in G$ there exists $c \in G^o$ such that $(bc)^d \in \Gamma$.*

Proof. Let $G = G_1 \supset G_2 \supset \dots \supset G_r \supset G_{r+1} = \{\mathbf{1}_G\}$ be the lower central series of G, and let G_i^o be the identity component of G_i, $i = 1, \dots, r$. Assume that $b^d = c\gamma$ with $c \in G_i^o$ and $\gamma \in \Gamma$. Then $(bc^{-1/d})^d = c'\gamma$ with $c' \in G_{i+1}^o$. By the (descending) induction on i, we are done. ∎

Now let $\{e_1, \dots, e_m\} \subseteq G$ be a basis of G/G^o. After replacing each e_j by $e_j c_j$ with an appropriate $c_j \in G^o$ we will have $e_j^d \in \Gamma$, $j = 1, \dots, m$.

2.8. Assume that Γ is not discrete and let Γ^o be the identity component of Γ. Then Γ^o is a normal subgroup of G. This fact is proved in [M] only in the case of connected G, but the argument works in the general case as well, and we will repeat it now. Let $\mathfrak{G}$ be the Lie algebra of G and let $\mathrm{Ad}\colon G \longrightarrow \mathrm{Aut}(\mathfrak{G})$, $a \mapsto \mathrm{Ad}_a$, be the adjoint representation of G. Let $a \in G$. Since Ad_a is unipotent, in a proper basis in $\mathfrak{G}$ the matrix representing Ad_a is upper triangular with unit diagonal. It follows that (in any basis) the entries of the matrix representing Ad_{a^t}, $t \in \mathbb{R}$, (or Ad_{a^n}, $n \in \mathbb{Z}$, if $a \notin G^o$) are polynomials in t (respectively, in n). Let $\mathfrak{L} \subseteq \mathfrak{G}$ be the tangent space of Γ^o. Then $a^{-t}\Gamma^o a^t = \Gamma^o$ iff $\mathrm{Ad}_{a^t}(\mathfrak{L}) = \mathfrak{L}$; this is a linear condition on the entries of Ad_{a^t} and so, a polynomial condition $P_a(t) = 0$ on t. Now, choose a basis $a_1, \ldots, a_l, e_1, \ldots, e_m$ in G with $a_1, \ldots, a_l, e_1^d, \ldots, e_m^d \in \Gamma$. Then for each a_i, $P_{a_i}(t) = 0$ for all $t \in \mathbb{Z}$, hence, $P_{a_i} \equiv 0$ and $a^{-t}\Gamma^o a^t = \Gamma^o$ for all $t \in \mathbb{R}$. Similarly, for each e_j, $P_{e_j}(n) = 0$ for all $n \in d\mathbb{Z}$, so $P_{e_j} \equiv 0$ and $e^{-n}\Gamma^o e^n = \Gamma^o$ for all $n \in \mathbb{Z}$. Since $a_1, \ldots, a_l, e_1, \ldots, e_m$ generate G, Γ^o is normal in G.

After replacing G by G/Γ^o and Γ by Γ/Γ^o we arrive at the situation where Γ is discrete. We thus may and will assume that Γ is a *discrete* uniform subgroup of G.

2.9. $[a,b]$ will stand for $a^{-1}b^{-1}ab$. We will denote by G_i the members of the lower central series of G, $G_1 = G$, $G_2 = [G,G]$ and $G_i = [G_{i-1}, G]$, $i = 3, \ldots, r$, where r is the nilpotency class of G. For each $i = 1, \ldots, r$, let G_i^o be the identity component of G_i. Note that $(G^o)_i \subseteq G_i^o$ and that this inclusion may be strict: for $G = \left\{ \begin{pmatrix} 1 & n & y \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} : n \in \mathbb{Z},\ x, y \in \mathbb{R} \right\}$ one has $G^o = \left\{ \begin{pmatrix} 1 & 0 & y \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} : x, y \in \mathbb{R} \right\}$ and $(G^o)_2 = \{\mathbf{1}_G\}$, whereas $G_2^o = G_2 = \left\{ \begin{pmatrix} 1 & 0 & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : y \in \mathbb{R} \right\}$.

2.10. Given $S \subseteq G$, by $\langle S \rangle$ we will denote the subgroup of G generated by S. Let $S \subseteq G$ be any set such that $G = \langle G^o, S \rangle$. Then G_i is generated by elements of the form $b = \big[\ldots[[b_1, b_2], b_3], \ldots, b_i\big]$ with $b_1, \ldots, b_i \in G^o \cup S$. If at least one of $b_1, \ldots, b_i$ belongs to G^o, then $b \in G_i^o$. If all $b_1, \ldots, b_i \in S$, then $b \in R_i := \langle S \rangle \cap G_i$. Hence,

Lemma. $G_i = \langle G_i^o, R_i \rangle$.

2.11. For each $i = 1, \ldots, r$, G_i and $G_i\Gamma$ are closed subgroups of G and $(G_i\Gamma)/\Gamma$ is a closed submanifold of X. This fact is well known in the case where G is connected and simply-connected ([M]); here is the sketch of the proof in the general case.

Define $\Gamma_i = \Gamma \cap G_i$, $i = 1, \ldots, r$. Fix i. We have a continuous mapping $G_i/\Gamma_i \longrightarrow (G_i\Gamma)/\Gamma$. If G_i/Γ_i is compact, then $(G_i\Gamma)/\Gamma \simeq G_i/\Gamma_i$ is a closed submanifold of X, and so, $G_i\Gamma$ is a closed subgroup of G. In this case $G_i\Gamma$ is locally compact and since Γ is countable, G_i is closed in $G_i\Gamma$ and therefore in G. Hence, we are done if we show that there exists a compact subset K_i in G_i such that $G_i = K_i\Gamma_i$. Following 2.6 and 2.7 above, choose a basis $B = \{a_1, \ldots, a_l, e_1, \ldots, e_m\}$ in G with $a_1, \ldots, a_l, e_1^d, \ldots, e_m^d \in \Gamma$. G_i/G_{i+1} is an abelian group generated by finitely many continuous and/or discrete generators of the form $b = \big[\ldots[[b_1, b_2], b_3], \ldots, b_i\big]$ with $b_1, \ldots, b_i \in B$. For any such b, $b^{d^i} \in \Gamma_i G_{i+1}$, thus $G_i = K_i'\Gamma_i G_{i+1} = K_i' G_{i+1}\Gamma_i$, where K_i' is the image of a "cube" $[0, d^i]^n \times \{0, \ldots, d^i\}^k$ in G_i/G_{i+1}. By (the descending) induction on i, $G_{i+1} = K_{i+1}\Gamma_{i+1}$ with compact K_{i+1}, so $G_i = K_i' K_{i+1}\Gamma_i$.

2.12. We define $X_i = G_i \backslash X = G/(G_i\Gamma)$. Then X decomposes into a tower $X = X_{r+1} \longrightarrow X_r \longrightarrow \ldots \longrightarrow X_2 \longrightarrow X_1 = \{\cdot\}$ of compact nilmanifolds. In particular, X_2 is a compact abelian Lie group, that is, a finite dimensional torus or a union of several tori. For each i, the fibers of the projection $X_{i+1} \longrightarrow X_i$ are isomorphic to the compact abelian Lie group $G_i/(G_{i+1}\Gamma_i)$.

2.13. Example. Let $G = \left\{ \begin{pmatrix} 1 & n & y \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} : n \in \mathbb{Z},\ x, y \in \mathbb{R} \right\}$ and $\Gamma = \left\{ \begin{pmatrix} 1 & n & k \\ 0 & 1 & m \\ 0 & 0 & 1 \end{pmatrix} : n, m, k \in \mathbb{Z} \right\}$. Then $r = 2$, X is the 2-dimensional torus $\big\{(x, y),\ x, y \in \mathbb{R}/\mathbb{Z}\big\}$, $G_2 = \left\{ \begin{pmatrix} 1 & 0 & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : y \in \mathbb{R} \right\}$ and X_2 is the 1-dimensional torus $\big\{(x),\ x \in \mathbb{R}/\mathbb{Z}\big\}$.

2.14. Theorem. (Cf. [AGH], Ch.4, Theorem 3.) *The action of G on X is distal.*

Proof. $X \longrightarrow X_r \longrightarrow \ldots \longrightarrow X_2$ is a tower of isometric extensions, which implies the result. In more detail, let $x, y \in X$, $x \neq y$, and let $i \leq r$ be such that the images of x and y in X_{i+1} are distinct whereas their images in X_i coincide. Let us factor G by G_{i+1} and replace X by X_{i+1}, then G_i is in the center of G. We have $y = cx$ with $c \in G_i$. Let $\mathrm{dist}(\cdot,\cdot)$ be a distance on X. Since X is compact, $\mathrm{dist}(z, cz) > \delta > 0$ for all $z \in X$. So, $\mathrm{dist}(ax, ay) = \mathrm{dist}(ax, acx) = \mathrm{dist}(ax, cax) > \delta$ for all $a \in G$. ■

2.15. We now fix a finitely generated amenable group A and a homomorphism $\varphi: A \longrightarrow G$. A acts on X by translations: $(\varphi(u))(x) = \varphi(u)x$, $u \in A$, $x \in X$.

2.16. Since the action of A on X is distal, we have:

Corollary. (See, for example, [F], Corollary on page 160.) *X decomposes into the union of disjoint closed subsets, $X = \bigcup Y_\theta$, which are invariant and minimal with respect to the action of A, that is, for any θ and any $x \in Y_\theta$, $\overline{\varphi(A)x} = Y_\theta$.*

2.17. By μ we denote the G-invariant probability measure on X.

Theorem. *Let $N = \langle G^o, \varphi(A)\rangle$. The action of A is ergodic on X (with respect to μ) iff it is ergodic on $Z = [N,N]\backslash X$.*

Proof. We may assume that $G = N$, then $Z = X_2$. We follow the line of the proof of Theorem 3 in [P2]. We use induction on r, the nilpotency class of G; for $r = 1$ the statement is trivial. Assume that the action of A is ergodic on X_2 and assume that $f \in L^2(X)$ is A-invariant, $\varphi(u)f = f$ for any $u \in A$. The compact abelian group $D = G_r/\Gamma_r$ acts on X and this action commutes with the action of G. Therefore $L^2(X)$ decomposes into a direct sum of A-invariant eigenspaces of D. We may assume that f belongs to one of these eigenspaces, that is, that $cf = \lambda(c)f$, $\lambda(c) \in \mathbb{C}$, $|\lambda(c)| = 1$, for all $c \in G_r$. Also, we may assume that $|f| \equiv 1$.

We have $c(af) = \lambda(c)(af)$ for any $a \in G$ and $c \in G_r$, and $\varphi(u)(bf) = \lambda\big([\varphi(u), b]\big)(bf)$ for any $b \in G_{r-1}$ and $u \in A$. Hence, for any $b \in G_{r-1}$ the function $(bf)f^{-1}$ factors through $X_r = G_r\backslash X$ and is an eigenfunction for A. Let E be the group of eigenfunctions of A on X_r under multiplication, and let C be the subgroup of constants in E. By induction on r, the action of A is ergodic on $X_r = G_r\backslash X$. Hence, the eigenspaces of A in $L^2(X_r)$ are one-dimensional, and so, E/C is discrete. We have a continuous mapping $\lambda: G_{r-1} \longrightarrow E$, $b \mapsto (bf)f^{-1}$. By the connectedness argument, $\lambda(G^o_{r-1}) \subseteq C$. Put $\lambda(a) = 1$ for all $a \in \varphi(A)$. Since $G = \big\langle G^o, \varphi(A)\big\rangle$, Lemma 2.10 implies that $G_{r-1} \subseteq \big\langle G^o_{r-1}, \varphi(A)\big\rangle$, and hence $\lambda(G_{r-1}) \subseteq C$.

It follows that f is G_r-invariant. Indeed, G_r is generated by $[G_{r-1}, G^o]$ and $[G_{r-1}, \varphi(A)]$. On $[G_{r-1}, \varphi(A)]$, λ is identically 1. Extend λ to a mapping $G \longrightarrow \mathbb{C}$ by $\lambda(a) = \int_X (af)f^{-1}d\mu$, $a \in G$. For any $b \in G_{r-1}$ and $a \in G^o$ we have $\lambda(ab) = \int_X (abf)f^{-1}d\mu = \lambda(b)\int_X (af)f^{-1}d\mu = \lambda(b)\lambda(a)$ and $\lambda(ba) = \int_X (baf)f^{-1}d\mu = \int_X (af)(b^{-1}f^{-1})d\mu = \lambda(b)\int_X (af)f^{-1}d\mu = \lambda(b)\lambda(a) = \lambda(ab)$. On the other hand, $\lambda(ba) = \lambda(ab)\lambda([b,a])$. Since λ is continuous, there exists a neighborhood V of $\mathbf{1}_G \in G^o$ such that for any $a \in V$ one has $\lambda(a) \neq 0$ and so, $\lambda([b,a]) = 1$. Since G^o is exponential, for any $d \in G^o$ there exist $m \in \mathbb{N}$ and $a \in V$ such that $a^m = d$, and so, $\lambda([b,d]) = \lambda([b,a^m]) = \lambda([b,a]^m) = \lambda([b,a])^m = 1$. We obtain that $\lambda|_{G_r} \equiv 1$. Hence, f factors through X_r and by induction on r, $f = \mathrm{const}$. ■

2.18. Assume that X is connected and consider $T = [G^o, G^o]\backslash X$, *the maximal factor-torus of* X. Since Z is a factor of T, we have

Corollary. *If X is connected, then the action of A is ergodic on X iff it is ergodic on T.*

2.19. Theorem. (Cf. [P1].) *If the action of A is ergodic on X then the action of A is uniquely ergodic on X. Hence, $\{\varphi_u x\}_{u\in A}$ is well distributed on X for any $x \in X$.*

Proof. We argue as in [F], proof of Proposition 3.10. A point $x \in X$ is said to be *generic for* μ (with respect to φ) if $\{\varphi(u)x\}_{u\in\mathcal{A}}$ is well distributed on X with respect to μ, that is, for any $f \in C(X)$ and any Følner sequence $\{\Phi_N\}_{N=1}^\infty$, $\lim\limits_{N\to\infty} \frac{1}{|\Phi_N|} \sum\limits_{u\in\Phi_N} f\big(\varphi(u)x\big) = \int_X f\,d\mu$. Let $P \subseteq X$ be the set of points generic for μ; since the action of A is ergodic, $\mu(P) = 1$. Let π_r be the projection $X \longrightarrow X/G_r = X_r$ and let $Q = \pi_r(P)$. Since the elements of G_r commute with $\varphi(A)$ and preserve μ, if $x \in X$ is generic for μ, then cx is also generic for μ for any $c \in G_r$. So, $G_rP = P$ and so, $P = \pi_r^{-1}(Q)$.

Let μ' be another measure on X ergodic with respect to the action of A. By induction on r, the action of A is uniquely ergodic on X_r, and so, the projections of μ and μ' onto X_r coincide. It follows that $\mu'(P)=\mu'(Q)=\mu(Q)=1$. That is, almost all (with respect to μ') points of X are not generic for μ'. This contradicts the ergodicity of μ'. ■

2.20. Corollary. (Cf. [P1].) *Let $N=\langle G^o,\varphi(A)\rangle$ and $Z=[N,N]\backslash X$. The action of A is ergodic on X iff X is minimal with respect to the action of A, and iff Z is minimal with respect to the action of A.*

Proof. If X is minimal with respect to the action of A, then Z is also minimal. If Z is minimal, then, since Z is a compact abelian group, the action of A is ergodic on Z, and by Theorem 2.17 the action of A is ergodic on X.

Now assume that X is not minimal; then by Theorem 2.14 we have a nontrivial decomposition of X into closed A-invariant subsets. It follows that the action of A is not uniquely ergodic on X and by Theorem 2.19, is not ergodic. ■

2.21. Theorem. (Cf. [Sh2], Theorem 1.3.) *For any $x\in X$ there exists a closed subgroup $E\subseteq G$ such that $\overline{\varphi(A)x}=Ex$. Consequently, $Y=\overline{\varphi(A)x}$ is a nilmanifold, and $\{\varphi(u)x\}_{u\in A}$ is well distributed on Y.*

Proof. Let $\pi\colon G\longrightarrow X$ be the factorization mapping; we may assume that $x=\pi(\mathbf{1}_G)$. After passing to a subgroup of finite index in A we may assume that $\varphi(A)$ preserves the connected component X^o of x in X. We may therefore assume that X is connected.

Let $N=\langle G^o,\varphi(A)\rangle$, $Z=[N,N]\backslash X$ and $p\colon X\longrightarrow Z$ be the factorization mapping. If $\overline{\varphi(A)x}\neq X$, then by Corollary 2.20, $\varphi(A)0=p\big(\varphi(A)x\big)$ is not dense in the torus Z. Hence it is contained in a proper closed subtorus $Z'\subset Z$. The projection $p\circ\pi\colon G\longrightarrow Z$ is a homomorphism, thus $G'=(\pi_2\circ\pi)^{-1}(Z')$ is a closed subgroup of G with $\dim G'<\dim G$ and $\varphi(A)\subseteq G'$. Induction on $\dim G$ proves the first statement.

We obtain that $Y\simeq E/(\Gamma\cap E)$ is a nilmanifold; the last statement of the theorem now follows from Corollary 2.20 and Theorem 2.19. ■

2.22. Remark. The group E in Theorem 2.21 is not uniquely determined, and does not have to contain $\varphi(A)$. However, among the groups E satisfying the conclusion of the theorem there is a maximal one, $E=\big\{a\in G: a(\overline{\varphi(A)x})=\overline{\varphi(A)x}\big\}$, and for this E one has $\varphi(A)\subseteq E$.

3. Reduction of the polynomial case to the linear case

We start with some group theoretical preliminaries.

3.1. Let $\mathcal{F}$ be the free group generated by continuous generators $a_1,\dots,a_l$ and discrete generators $e_1,\dots,e_m$, that is, the group of words in the alphabet $\big\{a_1^{t_1},\dots,a_l^{t_l},e_1^{n_1},\dots,e_m^{n_m}\big\}_{\substack{t_i\in\mathbb{R}\\ n_j\in\mathbb{Z}}}$. Let $\mathcal{F}=\mathcal{F}_1\supset\mathcal{F}_2\supset\dots$ be the lower central series of $\mathcal{F}$: $\mathcal{F}_{i+1}=[\mathcal{F}_i,\mathcal{F}]$, $i\in\mathbb{N}$. Let $r\in\mathbb{N}$; we will call the nilpotent Lie group $F=\mathcal{F}/\mathcal{F}_{r+1}$ *the free nilpotent Lie group (of class r, with l continuous and m discrete generators)*. The discrete subgroup of F generated by the set $\{a_1,\dots,a_l,e_1,\dots,e_m\}$ is uniform in F; we will denote it by $\Gamma(F)$.

3.2. Proposition. *Let G be a nilpotent Lie group of class $\le r$, let G^o be the identity component of G, and let F be a free nilpotent Lie group of class r with continuous generators $a_1,\dots,a_l$ and discrete generators $e_1,\dots,e_m$. Any mapping $\eta\colon\{a_1,\dots,a_l,e_1,\dots,e_m\}\longrightarrow G$ with $\eta\big(\{a_1,\dots,a_l\}\big)\subseteq G^o$ extends to a homomorphism $F\longrightarrow G$.*

Proof. The connected nilpotent Lie group G^o is exponential, and so, for any $i=1,\dots,l$ there exists a one-parameter subgroup $\{\alpha_i(t)\}_{t\in\mathbb{R}}$ in G such that $\eta(a_i)=\alpha_i(1)$. Thus, η extends to a homomorphism $\eta\colon\mathcal{F}\longrightarrow G$ from the free group $\mathcal{F}$ generated by $\{a_1^{t_1},\dots,a_l^{t_l},e_1,\dots,e_m\}_{t_i\in\mathbb{R}}$ so that $\eta(a_i^t)=\alpha_i(t)$, $t\in\mathbb{R}$, $i=1,\dots,l$. Since $\eta(\mathcal{F}_{r+1})\subseteq G_{r+1}=\{\mathbf{1}_G\}$, η factors to a homomorphism $F\longrightarrow G$. ■

3.3. Let us say that a Lie group G is *finitely generated* if G is generated by a set of the form $\big\{a_1^{t_1},\dots,a_l^{t_l},e_1,\dots,e_m\big\}_{t_i\in\mathbb{R}}$. (If G^o is the identity component of G, then G is finitely generated iff the discrete group G/G^o is finitely generated in the conventional sense.)

Proposition. *Let G be a finitely generated nilpotent Lie group. Then G is a factor of a finitely generated free nilpotent Lie group.*

Proof. Let G have nilpotency class r, $a_1,\ldots,a_l \in G^o$ be the continuous and $e_1,\ldots,e_m \in G$ the discrete generators of G. Let F be the free nilpotent Lie group of class r with continuous generators $a_1,\ldots,a_l$ and discrete generators $e_1,\ldots,e_m$. By Proposition 3.2, there exists a homomorphism $\eta\colon F \longrightarrow G$ which is identical on $a_1,\ldots,a_l,e_1,\ldots,e_m$. Clearly, η is surjective. ■

3.4. Lemma. *Let G be a nilpotent group, $G_2=[G,G]$, and let H be a subgroup of G such that $HG_2=G$. Then $H=G$.*

Proof. Let $G=G_1\supset G_2\supset\ldots\supset G_r\supset G_{r+1}=\{\mathbf{1}_G\}$ be the lower central series of G. By induction on r, $HG_r=G$, and it is only to be checked that $G_r\subseteq H$. G_r is generated by elements of the form $[b,a]$ with $a\in G$ and $b\in G_{r-1}$. Let $c\in H$ be such that $cG_2=aG_2$ and $d\in H\cap G_{r-1}$ be such that $dG_r=bG_r$. Then $[d,c]\in H$ and $[d,c]=[b,a]$. ■

3.5. Proposition. *Let F be a free nilpotent Lie group, let $F_2=[F,F]$ and let a self-homomorphism τ of F be such that the induced self-homomorphism of F/F_2 is invertible. Then τ is also invertible.*

Proof. Since $\tau(F)F_2=F$, $\tau(F)=F$ by Lemma 3.4. It follows from Proposition 3.2 that there exists a homomorphism $\sigma\colon F\longrightarrow F$ such that $\tau\circ\sigma=\mathrm{Id}_F$. Since σ induces an automorphism of F/F_2, σ is also surjective. Hence, $\sigma=\tau^{-1}$. ■

3.6. Remark. Actually, Proposition 3.5 holds for any simply-connected finitely generated nilpotent Lie group; we do not need this in such generality.

3.7. We say that an automorphism τ of a group G is *unipotent* if the mapping $\xi\colon G\longrightarrow G$ defined by $\xi(a)=\tau(a)a^{-1}$, $a\in G$, satisfies $\xi^{\circ q}\equiv\mathbf{1}_G$ for $q\in\mathbb{N}$ large enough.

3.8. Proposition. *Let τ be an automorphism of a nilpotent group G and let $G_2=[G,G]$. Then τ is unipotent iff the automorphism induced by τ on G/G_2 is unipotent.*

Proof. Let $G=G_1\supset G_2\supset\ldots\supset G_r\supset G_{r+1}=\{\mathbf{1}_G\}$ be the lower central series of G. By induction on the nilpotency class r of G, assume that τ is unipotent on G/G_r, that is, $\xi^{\circ q}(G)\subseteq G_r$ for q large enough. We only have to check that τ is unipotent on G_r. Let $A_i=\xi^{\circ i}(G)$, $B_i=A_i\cap G_{r-1}$, $i=0,\ldots,q-1$, and let $C_k=\big\langle [B_j,A_i],\ j+i\geq k\big\rangle$, $k=0,\ldots,2q-1$. We claim that $\xi(C_k)\subseteq C_{k+1}$, $k=0,\ldots,2q-2$, and so, $\xi^{\circ(2q-1)}(G_r)=\xi^{\circ(2q-1)}(C_0)\subseteq C_{2q-1}=\{\mathbf{1}_G\}$. Indeed, the mapping $G_{r-1}\times G\longrightarrow G_r$, $(b,a)\mapsto[b,a]$, is bilinear; since τ and ξ commute, $\tau(A_i)=A_i$ for all i; so, if $b\in B_j$ and $a\in A_i$, then

$$\begin{aligned}\xi([b,a])&=\tau([b,a])\cdot[b,a]^{-1}=\big[\tau(b),\tau(a)\big]\cdot\big[b,\tau(a)\big]^{-1}\cdot\big[b,\tau(a)\big]\cdot[b,a]^{-1}\\&=\big[\tau(b)b^{-1},\tau(a)\big]\cdot\big[b,\tau(a)a^{-1}\big]=\big[\xi(b),\tau(a)\big]\cdot\big[b,\xi(a)\big]\\&\qquad\in[B_{j+1},\tau(A_i)]\cdot[B_j,A_{i+1}]=[B_{j+1},A_i]\cdot[B_j,A_{i+1}]\subseteq C_{j+i+1}.\end{aligned}$$

■

3.9. Proposition. *Let G be a finitely generated nilpotent Lie group and let τ be a unipotent automorphism of G. Then the extension $\widehat{G}$ of G by τ is a nilpotent Lie group.*

Proof. $\widehat{G}$ is a solvable Lie group ($G\triangleleft\widehat{G}$ and $\widehat{G}/G\simeq\mathbb{Z}$); it therefore suffices to show that $\widehat{G}$ is generated by Engel elements. (An element a of a group H is said to be *Engel* if for any $b\in H$, $\big[\ldots\big[[b,a],a\big],\ldots\big]=\mathbf{1}_G$ if the number of brackets is large enough. Engel elements in a finitely generated solvable Lie group form a nilpotent subgroup.) $\widehat{G}$ is generated by G and the element $\hat{\tau}$ representing τ; $\hat{\tau}$ is Engel since τ is a unipotent automorphism of G, and each $b\in G$ is Engel since G is nilpotent and normal in $\widehat{G}$. ■

3.10. Starting from this point, let, again, G be a nilpotent Lie group, G^o be the identity component of G, Γ be a discrete uniform subgroup of G and $X=G/\Gamma$. Any polynomial sequence $g(n)=a_1^{p_1(n)}\cdots a_m^{p_m(n)}$ in G is contained in the group of G generated by the finite set $\{a_1,\ldots,a_m\}$. Studying the action of g on X we may, therefore, assume that G is a finitely generated Lie group.

3.11. We now deduce from Theorem 2.21 the following fact:

Theorem. *Let τ be a unipotent measure-preserving automorphism of G with $\tau(\Gamma)=\Gamma$; then τ acts on X. For any $x\in X$ there exist a connected closed subgroup H of G and points $x_1,x_2,\ldots,x_k\in X$ such that $Y_j=Hx_j$, $j=1,\ldots,k$, are closed sub-nilmanifolds of X, and for each $j=1,\ldots,k$ the sequence $\{\tau^{j+kn}x\}_{n\in\mathbb{Z}}$ is well distributed on Y_j.*

Proof. Let $\widehat{G}$ be the extension of G by τ; by Proposition 3.9 $\widehat{G}$ is a nilpotent Lie group. Let $\hat{\tau}$ be the element in $\widehat{G}$ representing τ, so that $\tau(a)=\hat{\tau}a\hat{\tau}^{-1}$ for any $a\in G$. Let $\widehat{\Gamma}=\langle\Gamma,\hat{\tau}\rangle\subseteq\widehat{G}$. Since $\tau(\Gamma)=\Gamma$ one has $\widehat{\Gamma}\cap G=\Gamma$, so $\widehat{\Gamma}$ is a discrete subgroup of $\widehat{G}$ and $X=\widehat{G}/\widehat{\Gamma}$. For any $a\in G$ and $x=a\widehat{\Gamma}\in X$ one has $\tau(x)=\tau(a)\widehat{\Gamma}=\hat{\tau}a\hat{\tau}^{-1}\widehat{\Gamma}=\hat{\tau}a\widehat{\Gamma}=\hat{\tau}x$. By Theorem 2.21, there exists a closed subgroup E of $\widehat{G}$ such that Ex is closed and $\{\tau^n x\}_{n\in\mathbb{Z}}$ is well distributed on Ex. Let H be the identity component of E; since $\widehat{G}/G$ is discrete, $H\subseteq G$. Hx is a connected component of Ex; since Ex is compact, it consists of finitely many translates of Hx and so, the stabilizer $\mathrm{Stab}(Hx)$ of Hx has finite index in E. Let $b_1,\ldots,b_k\in E$ be a set of representatives of $E/\mathrm{Stab}(Hx)$ and let $x_j=b_jx$, $j=1,\ldots,k$. Since H is normal in E, $b_jHx=Hb_jx=Hx_j$, $j=1,\ldots,k$. Put $Y_j=Hx_j$, $j=1,\ldots,k$, these are connected disjoint subnilmanifolds of X and we have $Ex=\bigcup_{j=1}^k b_jHx=\bigcup_{j=1}^k Y_j$.

τ transitively acts on the set $\{Y_1,\ldots,Y_k\}$ and thus, cyclically permutes these sub-nilmanifolds. Reorder $Y_1,\ldots,Y_k$ so that $\tau x\in Y_1$ and $\tau(Y_j)=Y_{j+1}$, $j=1,\ldots,k-1$. Then $\tau^{j+kn}x\in Y_j$ for all j and all $n\in\mathbb{Z}$. The sequence $\{\tau^{j+kn}x\}_{n\in\mathbb{Z}}=\{(\tau^k)^n(\tau^jx)\}_{n\in\mathbb{Z}}$ is therefore well distributed on Y_j for each j. ∎

3.12. The following simple example demonstrates that in Theorem 3.11, $\overline{\{\tau^n(x)\}}_{n\in\mathbb{Z}}$ need not be of the form Hx where H is a subgroup of G.

Example. Let $G=X=(\mathbb{Z}_3)^3$ and a unipotent automorphism τ of X be defined by $\tau(a,b,c)=(a,b+a,c+b)$, then $\tau^3=\mathrm{Id}_X$. Take $x=(1,0,0)\in X$. Then $\tau(x)=(1,1,0)$, $\tau^2(x)=(1,2,1)$, and $\{\tau^n(x)\}_{n\in\mathbb{Z}}=\big\{(1,0,0),(1,1,0),(1,2,1)\big\}$ is not a coset of a subgroup of X.

3.13. Following 2.6 and 2.7, choose a basis $\{a_1,\ldots,a_l,e_1,\ldots,e_m\}$ in G, where $a_1,\ldots,a_l\in\Gamma\cap G^o$ and $e_1^d,\ldots,e_m^d\in\Gamma$ for some $d\in\mathbb{N}$, such that every element a of G can be written in the form $a=a_1^{t_1}\cdots a_l^{t_l}e_1^{n_1}\cdots e_m^{n_m}$ with $t_1,\ldots,t_l\in\mathbb{R}$ and $n_1,\ldots,n_m\in\mathbb{Z}$. Any polynomial sequence g in G is then representable in the form $g(n)=a_1^{p_1(n)}\cdots a_l^{p_l(n)}e_1^{q_1(n)}\cdots e_m^{q_m(n)}$, where $p_1,\ldots,p_l$ are polynomials $\mathbb{Z}\longrightarrow\mathbb{R}$ and $q_1,\ldots,q_m$ are polynomials $\mathbb{Z}\longrightarrow\mathbb{Z}$.

Let $D=\langle a_1,\ldots,a_l,e_1,\ldots,e_m\rangle$. In a finitely generated nilpotent group any subgroup generated by nontrivial powers of the generators has finite index, so $\Gamma\cap D$ has finite index in D. Thus, there exists $s\in\mathbb{N}$ such that $b^s\in\Gamma$ for any $b\in D$.

3.14. Proposition. *Let g be a polynomial sequence in G. There exists a nilpotent Lie group $\widetilde{G}$ with a discrete uniform subgroup $\widetilde{\Gamma}$, an epimorphism $\eta\colon\widetilde{G}\longrightarrow G$ with $\eta(\widetilde{\Gamma})\subseteq\Gamma$, a unipotent automorphism τ of $\widetilde{G}$ with $\tau(\widetilde{\Gamma})=\widetilde{\Gamma}$, and an element $c\in\widetilde{G}$ such that $g(n)=\eta\big(\tau^n(c)\big)$, $n\in\mathbb{Z}$.*

Proof. Let $\{a_1,\ldots,a_l,e_1,\ldots,e_m\}$ be a basis of G described in 3.13 and let $s\in\mathbb{N}$ be such that $b^s\in\Gamma$ for any b from the (discrete) group generated by $\{a_1,\ldots,a_l,e_1,\ldots,e_m\}$. Let F be a free nilpotent Lie group with continuous generators $a_1,\ldots,a_l$ and discrete generators $e_1,\ldots,e_m$, and let $\eta'\colon F\longrightarrow G$ be the natural epimorphism. Then $\eta'(b^s)\in\Gamma$ for any $b\in\Gamma(F)$.

Let $g(n)=a_1^{p_1(n)}\cdots a_l^{p_l(n)}e_1^{q_1(n)}\cdots e_m^{q_m(n)}$, where $p_1,\ldots,p_l$ are polynomials $\mathbb{Z}\longrightarrow\mathbb{R}$ and $q_1,\ldots,q_m$ are polynomials $\mathbb{Z}\longrightarrow\mathbb{Z}$. Let $\widetilde{G}$ be the free nilpotent Lie group with continuous generators $\{b_{i,0}=a_i,b_{i,1},\ldots,b_{i,\deg p_i}\}_{i=1,\ldots,l}$ and discrete generators $\{d_{j,0}=e_j,d_{j,1},\ldots,d_{j,\deg q_j}\}_{j=1,\ldots,m}$. Let B be the normal closure in $\widetilde{G}$ of the group generated by $\{b_{i,1}^t,\ldots,b_{i,\deg p_i}^t\}_{\substack{i=1,\ldots,l\\ t\in\mathbb{R}}}$ and $\{d_{j,1},\ldots,d_{j,\deg q_j}\}_{j=1,\ldots,m}$; then $F\simeq\widetilde{G}/B$. Let $\eta''\colon\widetilde{G}\longrightarrow F$ be the factorization mapping and let $\eta=\eta''\circ\eta'$.

Let $\widetilde{\Gamma}$ be the subgroup of $\Gamma(\widetilde{G})$ generated by the s-th powers of the elements of $\widetilde{G}$, $\widetilde{\Gamma}=\big\langle\{\gamma^s,\ \gamma\in\Gamma(\widetilde{G})\}\big\rangle$. Then $\widetilde{\Gamma}$ has finite index in $\Gamma(\widetilde{G})$ and so, is uniform in $\widetilde{G}$. One has $\eta(\widetilde{\Gamma})\subseteq\Gamma$ and $\tau(\widetilde{\Gamma})=\widetilde{\Gamma}$ for any automorphism τ of $\Gamma(\widetilde{G})$.

We define $\tau\colon\widetilde{G}\longrightarrow\widetilde{G}$ by $\tau(a_i)=a_i$ $(i=1,\ldots,l)$, $\tau(b_{i,k})=b_{i,k}b_{i,k-1}$ $(k=1,\ldots,\deg p_i,\ i=1,\ldots,l)$,

$\tau(e_j) = e_j$ $(j = 1, \ldots, m)$, $\tau(d_{j,k}) = d_{j,k}d_{j,k-1}$ $(k = 1, \ldots, \deg q_j,\ j = 1, \ldots, m)$. So defined, τ induces a unipotent automorphism of $\widetilde{G}/\widetilde{G}_2$. By Propositions 3.2, 3.5 and 3.8, τ is a unipotent automorphism of $\widetilde{G}$.

For $i \in \{1, \ldots, l\}$ let $p_i(n) = \alpha_0 + \alpha_1\binom{n}{1} + \alpha_2\binom{n}{2} + \ldots + \alpha_k\binom{n}{k}$, $\alpha_0, \ldots, \alpha_k \in \mathbb{R}$. Define $u_i = a_i^{\alpha_0} b_{i,1}^{\alpha_1} \ldots b_{i,k}^{\alpha_k}$, then $\tau^n(u_i) = a_i^{\alpha_0+\alpha_1\binom{n}{1}+\alpha_2\binom{n}{2}+\ldots+\alpha_k\binom{n}{k}} h(n) = a_i^{p_i(n)} h(n)$, where $h(n) \in B$, $n \in \mathbb{Z}$. Similarly, if for $j \in \{1, \ldots, m\}$, $q_j(n) = \beta_0 + \beta_1\binom{n}{1} + \beta_2\binom{n}{2} + \ldots + \beta_k\binom{n}{k}$, $\beta_0, \ldots, \beta_k \in \mathbb{Z}$, define $v_j = e_j^{\beta_0} d_{j,1}^{\beta_1} \ldots d_{j,k}^{\beta_k}$; then $\tau^n(v_j) = e_j^{q_j(n)} h'(n)$ with $h'(n) \in B$, $n \in \mathbb{Z}$. Put $c = u_1 \ldots u_l v_1 \ldots v_m$, then $\eta(\tau^n(c)) = g(n)$, $n \in \mathbb{Z}$. ∎

3.15. Proof of Theorem B. Let $\pi: G \longrightarrow X$ be the factorization mapping. Let us assume that $x = \pi(\mathbf{1}_G)$; otherwise, if $x = \pi(a)$ for $a \in G$, we write $g(n)x = g(n)a\pi(\mathbf{1}_G)$ and replace $g(n)$ by $g(n)a$. Find $\widetilde{G}$, $\widetilde{\Gamma}$ and c as in Proposition 3.14 and let $\widetilde{X} = \widetilde{G}/\widetilde{\Gamma}$. The epimorphism $\eta: \widetilde{G} \longrightarrow G$ factors to $\eta: \widetilde{X} \longrightarrow X$, so that if $\tilde{\pi}: \widetilde{G} \longrightarrow \widetilde{X}$ is the factorization mapping, then $\pi \circ \eta = \eta \circ \tilde{\pi}$. Let $\widetilde{x} = \tilde{\pi}(\mathbf{1}_{\widetilde{G}})$, then $\eta\big(\tau^n(c\widetilde{x})\big) = g(n)x$, $n \in \mathbb{Z}$. By Theorem 3.11, there exist a connected closed subgroup $\widetilde{H}$ of $\widetilde{G}$ and points $\widetilde{x}_1, \widetilde{x}_2, \ldots, \widetilde{x}_k \in \widetilde{X}$ such that, for each $j = 1, \ldots, k$, $\{\tau^{j+kn}(c\widetilde{x})\}_{n\in\mathbb{Z}}$ is well distributed on $\widetilde{H}\widetilde{x}_j$. Let $H = \eta(\widetilde{H})$ and $x_j = \eta(\widetilde{x}_j)$, $j = 1, \ldots, k$. Since, for each $j = 1, \ldots, k$, $\widetilde{H}\widetilde{x}_j$ is compact, $Y_j = Hx_j = \eta(\widetilde{H}\widetilde{x}_j)$ is a connected sub-nilmanifold of X, and the H-invariant measure on Y_j is the η-image of the $\widetilde{H}$-invariant measure on $\widetilde{H}\widetilde{x}_j$. Hence, for each $j = 1, \ldots, k$, $\big\{\eta(\tau^{j+kn}(c\widetilde{x}))\big\}_{n\in\mathbb{Z}} = \big\{g(j + kn)x\big\}_{n\in\mathbb{Z}}$ is well distributed on Y_j. ∎

3.16. Proof of Theorem C. Let, in accordance with Theorem B, a connected closed subgroup H of G and points $x_1, \ldots, x_k \in X$ be such that $\overline{\{g(n)x\}_{n\in\mathbb{Z}}} = \bigcup_{j=1}^k Hx_j$. If (i) holds, then $\overline{\{g(n)x\}_{n\in\mathbb{Z}}} = X$ and since X is connected, $Hx_1 = \ldots = Hx_k = X$. So, $Hx = X$ and $\{g(n)x\}_{n\in\mathbb{Z}}$ is well distributed on X. Hence (i) implies (ii).

Let $T = [G^o, G^o]\backslash X$ and $p: X \longrightarrow T$ be the factorization mapping. Assume that the sequence $\{g(n)p(x)\}_{n\in\mathbb{Z}}$ is dense in T. Then $T = \bigcup_{j=1}^k Hp(x_j)$, and since T is connected, $Hp(x_j) = T$ for some j. Hence, $H[G^o, G^o](\Gamma \cap G^o) = G^o$, and since Γ is countable, $H[G^o, G^o] = G^o$. By Lemma 3.4, $H = G^o$, so $\overline{\{g(n)x\}_{n\in\mathbb{Z}}} = Hx_1 = X$, and (iii) implies (i). ∎

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