Rational sub-nilmanifolds of a compact nilmanifold

A. Leibman
Department of Mathematics
The Ohio State University
Columbus, OH 43210, USA
e-mail: leibman@math.ohio-state.edu

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Abstract

We introduce the notions of a rational point, a rational sub-nilmanifold and a rational homomorphism of compact nilmanifolds and describe elementary properties thereof. We also show that the closure of the orbit of a point of a nilmanifold X under a polynomial action of the group $\mathbb{Z}^d$ by translations is, up to a shift, a union of rational sub-nilmanifolds of X.

0. Introduction

A nilmanifold is a compact homogeneous space of a nilpotent Lie group. Thanks to recent developments of Host and Kra ([HK1], [HK2]) and Ziegler ([Z]), nilmanifolds started to play a crucial role in the ergodic combinatorial number theory. Studying orbits of points of a nilmanifold under the action of a polynomial sequence of translations becomes an extremely important task. It is known that the closure of such an orbit is the union of several subnilmanifolds (see [L1] and [L2]). This paper is devoted to the property of *rationality* of these subnilmanifolds.

A point x of a standard torus $X=\mathbb{R}^k/\mathbb{Z}^k$ is *rational* if $nx=0$ for some $n\in\mathbb{N}$, or, equivalently, if all coordinates of x are rational numbers. An (affine) subtorus Y of X is *rational* if it contains at least one rational point of X; in this case rational points of X are dense in Y, and in coordinates on X, Y is described by a system of linear equations with rational coefficients. More general, an (affine) homomorphism $\psi\colon Y\longrightarrow X$ of two tori is *rational* if $\psi(Y)$ contains a rational point of X; in this case, in coordinates on X and Y, ψ is given by linear equations with rational coefficients. We transfer this simple theory to the case where X is a nilmanifold. We do this in the first two sections of the paper; the facts gathered there are quite elementary and we often skip their proofs. Finally, in the third section we show that all components of the closure of the orbit of any rational point of a nilmanifold X under a polynomial action of $\mathbb{Z}^d$ by translations on X are rational subnilmanifolds of X.

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1. Rational elements and subgroups of a nilpotent Lie group

1.1. Let G be a nilpotent Lie group with a discrete uniform (that is, cocompact) subgroup Γ. We will denote by G^o the identity component of G and assume that the group G/G^o is finitely generated.

1.2. We will say that an element $a \in G$ is *rational* (with respect to Γ) if $a^n \in \Gamma$ for some $n \in \mathbb{N}$. We will denote the set of rational elements of G by $\mathbb{Q}(G)$.

1.3. Lemma. *$\mathbb{Q}(G)$ is a subgroup of G.*

Proof. Let $a, b \in \mathbb{Q}(G)$ let H be the group generated by a and b, $H = \langle a, b\rangle$, and let $L = \Gamma \cap H$; we have to show that $H^n = \big\{c^n,\ c \in H\big\} \subseteq L$ for a certain $n \in \mathbb{N}$. Let $H = H_1 \supset H_2 \supset \ldots \supset H_r \supset H_{r+1} = \mathbf{1}_G$ be the lower central series of H. H/H_2L is finite, so there is $m \in \mathbb{N}$ such that $H^m = \{c^m,\ c \in H\} \subseteq H_2L$. Assume by induction on i that $H_i^{m^i} \subseteq H_{i+1}L$, then $H_i^{m^i} \subseteq H_{i+1}(L \cap H_i)$, and we have

$$H_{i+1}^{m^{i+1}} \subseteq [H^m, H_i^{m^i}]H_{i+2} \subseteq \big[H_2L, H_{i+1}(L \cap H_i)\big]H_{i+2} \subseteq H_{i+2}L.$$

Now assume by induction that $H^{m^{i(i+1)/2}} \subseteq H_{i+1}L$ for some i; then

$$H^{m^{(i+1)(i+2)/2}} = (H^{m^{i(i+1)/2}})^{m^{i+1}} \subseteq (H_{i+1}L)^{m^{i+1}} \subseteq H_{i+1}^{m^{i+1}} L^{m^{i+1}} H_{i+2} \subseteq H_{i+2}L.$$

So, $H^{m^{r(r+1)/2}} \subseteq L$. ∎

1.4. G^o is an exponential group (that is, the exponential mapping $\mathcal{G} \longrightarrow G$ from the Lie algebra $\mathcal{G}$ of G is surjective), thus for any $a \in G^o$ there exists a one-parameter subgroup $\{\alpha(t)\}_{t\in\mathbb{R}}$ with $\alpha(1) = a$. We will write a^t for $\alpha(t)$, $t \in \mathbb{R}$ (ignoring the fact that, in the case G^o is not simply connected, the subgroup $\{\alpha(t)\}_{t\in\mathbb{R}}$ corresponding to a may not be unique). Clearly,

Lemma. *If $a \in \mathbb{Q}(G^o)$ then $a^t \in \mathbb{Q}(G^o)$ for all $t \in \mathbb{Q}$.*

1.5. Let $\bar{G}$ be the universal covering of G^o, $\rho\colon \bar{G} \longrightarrow G^o$ be the projection mapping and $\bar{\Gamma} = \rho^{-1}(\Gamma)$. The connected simply-connected nilpotent Lie group $\bar{G}$ posesses *a Malcev basis* compatible with $\bar{\Gamma}$: a minimal finite set $\{\bar{a}_1, \ldots, \bar{a}_{k_1}\} \subseteq \bar{\Gamma}$ such that $\{\bar{a}_1, \ldots, \bar{a}_{k_1}\}$ generates $\bar{\Gamma}$, $\{\bar{a}_1^{t_1}, \ldots, \bar{a}_l^{t_{k_1}}\}_{t_1,\ldots,t_{k_1}\in\mathbb{R}}$ generates G and for each $i \in \{1, \ldots, k_1\}$ the group $\bar{A}_i$ generated by $\{\bar{a}_i^{t_i}, \ldots, \bar{a}_{k_1}^{t_{k_1}}\}_{t_i,\ldots,t_{k_1}\in\mathbb{R}}$ is closed and normal in $\bar{G}$. Any $\bar{a} \in \bar{G}$ is uniquely representable in the form $\bar{a} = \bar{a}_1^{t_1} \cdot\ldots\cdot \bar{a}_{k_1}^{t_{k_1}}$ with $t_1, \ldots, t_{k_1} \in \mathbb{R}$. (See [M].) Let $a_i = \rho(\bar{a}_i)$, $i = 1, \ldots, k_1$. Any $a \in G$ is then representable in the form $a = a_1^{t_1} \cdot\ldots\cdot a_{k_1}^{t_{k_1}}$ with $(t_1, \ldots, t_{k_1}) \in \mathbb{R}^k$. Such a representation may not be unique, but distinct representations of any element of G^o form a discrete set in $\mathbb{R}^{k_1}$. One has $a \in \Gamma$ if and only if $t_1, \ldots, t_{k_1} \in \mathbb{Z}$.

If G is not connected, then the finitely generated group G/G^o also has *a basis*, that is, a subset $\{e_1, \ldots, e_{k_2}\} \subseteq G$ such that, for every j, the group generated by $e_j, \ldots, e_{k_2}$ and G^o is normal in G. Every element of G/G^o is then representable in the form $e_1^{n_1} \cdot\ldots\cdot e_{k_2}^{n_{k_2}} G^o$ with $n_1, \ldots, n_{k_2} \in \mathbb{Z}$. If $G = G^o\Gamma$ then $e_1, \ldots, e_{k_2}$ can be chosen from Γ; otherwise $G^o\Gamma$ has finite index in G and so, there exists $\delta \in \mathbb{N}$ such that $b^\delta \in G^o\Gamma$ for any $b \in G$.

Lemma. *For any $b \in G$ there exists $c \in G^o$ such that $(bc)^\delta \in \Gamma$.*

Proof. Let $G = G_1 \supset G_2 \supset \ldots \supset G_r \supset G_{r+1} = \{\mathbf{1}_G\}$ be the lower central series of G, and let G_i^o be the identity component of G_i, $i = 1, \ldots, r$. Assume that $b^\delta = c\gamma$ with $c \in G_i^o$ and $\gamma \in \Gamma$. Then $(bc^{-1/\delta})^\delta = c'\gamma$ with $c' \in G_{i+1}^o$. By the descending induction on i we are done. ■

Now let $\{e_1, \ldots, e_{k_2}\} \subseteq G$ be a basis of G/G^o; after replacing each e_j by $e_j c_j$ with an appropriate $c_j \in G^o$ we will have $e_j^\delta \in \Gamma$, $j = 1, \ldots, k_2$. Put $k = k_1 + k_2$, $e_{k_2+1} = a_1, \ldots, e_k = a_{k_1}$. We will call $\{e_1, \ldots, e_{k_2}, e_{k_2+1}, \ldots, e_k\}$ *a basis* in G. Every element a of G is now representable in the form $a = e_1^{t_1} \ldots e_k^{t_k}$ with $t_1, \ldots, t_{k_2} \in \mathbb{Z}$ and $t_{k_2+1}, \ldots, t_k \in \mathbb{R}$; we will call $t_1, \ldots, t_k$ *the coordinates* of a. We get a continuous surjective *coordinate mapping* $\tau_G \colon \mathbb{Z}^{k_2} \times \mathbb{R}^{k_1} \longrightarrow G$, $\tau_G(t_1, \ldots, t_k) = e_1^{t_1} \ldots e_k^{t_k}$, so that $\tau_G(\mathbb{Z}^{k_2} \times \mathbb{Z}^{k_1}) \subseteq E$. For any $a \in G$ the set $\tau_G^{-1}(a)$ is discrete in $\mathbb{Z}^{k_2} \times \mathbb{R}^{k_1}$.

1.6. In coordinates, the multiplication in G is given by polynomial formulas: there are polynomials $\xi_i \colon (\mathbb{R}^{i-1})^2 \longrightarrow \mathbb{R}$ and $\zeta_i \colon \mathbb{R}^{i-1} \times \mathbb{R} \longrightarrow \mathbb{R}$, $i = 1, \ldots, k$, with rational coefficients and vanishing on $(\{0\} \times \mathbb{R}^{i-1}) \cup (\mathbb{R}^{i-1} \times \{0\})$ and $(\{0\} \times \mathbb{R}) \cup (\mathbb{R}^{i-1} \times \{0\})$ respectively, such that for $a = \tau_G(u_1, \ldots, u_k)$ and $b = \tau_G(v_1, \ldots, v_k)$ the i-th coordinate of ab is

$$u_i + v_i + \xi_i(u_1, \ldots, u_{i-1}, v_1, \ldots, v_{i-1}), \tag{1.1}$$

and for $a = \tau_G(u_1, \ldots, u_k)$ and $t \in \mathbb{R}$ if $a \in G^o$ and $t \in \mathbb{Z}$ otherwise, the i-th coordinate of a^t is

$$tu_i + \zeta_i(u_1, \ldots, u_{i-1}, t). \tag{1.2}$$

1.7. Let us say that a uniform subgroup Γ' of G is *rationally equivalent* to Γ if $\Gamma' \cap \Gamma$ has finite index in Γ'. In this case Γ' is discrete, $G/(\Gamma' \cap \Gamma)$ is compact, and since Γ is discrete, $\Gamma' \cap \Gamma$ has finite index in Γ as well. Clearly, if Γ' is rationally equivalent to Γ, the rationality of an element $a \in G$ with respect to Γ is equivalent to the rationality of a with respect to Γ'.

1.8. Lemma. *$a \in G$ is a rational element of G if and only if all coordinates of a are rational numbers.*

(The coordinates of a may not be uniquely defined; we claim that $\tau_G^{-1}(\mathbb{Q}(G)) = \mathbb{Z}^{k_2} \times \mathbb{Q}^{k_1}$.)

Proof. The group E generated by $\{e_1, \ldots, e_k\}$ is uniform in G and rationally equivalent to Γ. Let $a = \tau_G(u_1, \ldots, u_k)$. If $u_1, \ldots, u_k \in \mathbb{Q}$, then since ζ_i in formula (1.2) have rational coefficients and vanish on $\mathbb{R}^{i-1} \times \{0\}$, for certain n one has $nu_i + \zeta_i(u_1, \ldots, u_{i-1}, n) \in \mathbb{Z}$ for all $i = 1, \ldots, k$. Hence, $a^n \in \tau_G(\mathbb{Z}^k) \subseteq E$, thus a is rational with respect to E, and so, with respect to Γ.

Now assume that a is rational with respect to E whereas not all of u_i are rational. Let $a^n \in E$. Let i be the minimal index for which $u_i \notin \mathbb{Q}$, then the i-th coordinates $mnu_i + \zeta_i(u_1, \ldots, u_{i-1}, mn)$ of $a^{mn} \in E$ are all different modulo 1 for distinct $m \in \mathbb{Z}$. This implies that there are infinitely many elements of $\tau_G^{-1}(E)$ in the compact set $\{0\}^{k_2} \times [0,1]^{k_1}$, which contradicts the discreteness of $\tau_G^{-1}(E)$. ■

1.9. Corollary. *$\mathbb{Q}(G)$ is dense in G.*

1.10. A closed subgroup H of G will be said to be *rational* if it has a basis consisting of rational elements of G.

1.11. Lemma. *A closed subgroup H of G is rational if and only if $\Gamma \cap H$ is uniform in H.*

Proof. Let H be rational and let $\{b_1, \dots, b_l\} \subset \mathbb{Q}(G)$ be a basis in H. The group D generated by $b_1, \dots, b_l$ is uniform in H. For each $i = 1, \dots, l$ let $n_i \in \mathbb{N}$ be such that $b_i^{n_i} \in \Gamma$. Then the subgroup of Γ generated by $b_1^{n_1}, \dots, b_l^{n_l}$ has finite index in D and so, is uniform in H.

Conversely, let $\Gamma \cap H$ be uniform in H. Then H has a basis compatible with $\Gamma \cap H$, and elements of this basis are rational elements of G. ∎

1.12. Let H be a closed subgroup of G and let $\{b_1, \dots, b_l\}$ be a basis in H with $b_1, \dots, b_{l_2} \notin H^o$ and $b_{l_2+1}, \dots, b_l \in H^o$, so that any element $b \in H$ is representable in the form

$$b = b_1^{t_1} \dots b_{l_2}^{t_l} \tag{1.3}$$

with $t_1, \dots, t_{l_2} \in \mathbb{Z}$ and $t_{l_2+1}, \dots, t_l \in \mathbb{R}$. Let $\tau_H: \mathbb{Z}^{l_2} \times \mathbb{R}^{l_1} \longrightarrow H$ be the corresponding coordinate mapping: $\tau_H(t_1, \dots, t_l) = b$. For each of $b_1, \dots, b_l$ choose a coordinate representation in terms of coordinates of G; applying formulas (1.2) and (1.1) to the product (1.3) we obtain a polynomial mapping $P_H: \mathbb{Z}^{l_2} \times \mathbb{R}^{l_1} \longrightarrow \mathbb{Z}^{k_2} \times \mathbb{R}^{k_1}$ for which the diagram

$$\begin{array}{ccc} \mathbb{Z}^{l_2} \times \mathbb{R}^{l_1} & \xrightarrow{P_H} & \mathbb{Z}^{k_2} \times \mathbb{R}^{k_1} \\ \tau_H \downarrow & & \downarrow \tau_G \\ H & \hookrightarrow & G \end{array}$$

is commutative.

1.13. Since the polynomials ξ_i, ζ_i occuring in (1.1) and (1.2) are rational, we have

Lemma. *H is a rational subgroup if and only if the polynomial mapping P_H is rational (that is, has rational coefficients).*

1.14. Corollary. *A closed subgroup H of G is rational if and only if rational elements of G are dense in H.*

1.15. Corollary. *If H is a rational closed subgroup of G then for any $b \in \mathbb{Q}(G)$ the subgroup $b^{-1}Hb$ is also rational.*

1.16. Let K be a (left or right) coset of a closed subgroup H of G. The natural isomorphism $H \longrightarrow K$ induces a coordinate mapping $\tau_K: \mathbb{Z}^{l_2} \times \mathbb{R}^{l_1} \longrightarrow K$. Since the (left or right) translation in G in coordinates is given by polynomial formulas, the mapping P_H induces a polynomial mapping $P_K: \mathbb{Z}^{l_2} \times \mathbb{R}^{l_1} \longrightarrow \mathbb{Z}^{k_2} \times \mathbb{R}^{k_1}$ such that the diagram

$$\begin{array}{ccc} \mathbb{Z}^{l_2} \times \mathbb{R}^{l_1} & \xrightarrow{P_K} & \mathbb{Z}^{k_2} \times \mathbb{R}^{k_1} \\ \tau_K \downarrow & & \downarrow \tau_G \\ K & \hookrightarrow & G \end{array}$$

is commutative.

1.17. We will say that a coset K of a closed subgroup H of G is *rational* if H is a rational subgroup and K contains a rational element of G. From Lemma 1.13 we obtain

Lemma. *K is rational if and only if the polynomial mapping P_K is rational.*

1.18. Corollary. *A coset K of a closed subgroup H of G is rational if and only if rational elements of G are dense in K.*

1.19. Corollary. *A coset K of a closed subgroup H of G is rational if and only if all connected components of K are rational cosets of H^o.*

1.20. Let F be a nilpotent Lie group with a discrete uniform subgroup Λ. A homomorphism $\varphi: F \longrightarrow G$ will be said to be *rational* if $\varphi(\Lambda) \subseteq \mathbb{Q}(G)$. In this case, $\varphi(\mathbb{Q}(F)) \subseteq \mathbb{Q}(G)$. A rational homomorphism maps rational subgroups and cosets in F onto rational subgroups and cosets in G. It follows from the formulas (1.1) and (1.2) that in coordinates a rational homomorphism is given by a rational polynomial mapping.

2. Rational points, sub-nilmanifolds and homomorphisms of nilmanifolds

2.1. Let X be the compact *nilmanifold* G/Γ and let $\pi: G \longrightarrow X$ be the factorization mapping. The group G acts on X by left translations: $x \mapsto ax$, $a \in G$, $x \in X$.

2.2. We will say that $x \in X$ is rational if $x \in \pi(\mathbb{Q}(G))$. In this case, $\pi^{-1}(x) \subseteq \mathbb{Q}(G)$. We will denote the set of rational points of X by $\mathbb{Q}(X)$.

2.3. It follows from Corollary 1.9 that

Lemma. *$\mathbb{Q}(X)$ is dense in X.*

2.4. We will now introduce coordinates on X. Let $\{e_1, \ldots, e_k\}$ be a basis in G compatible with Γ and let $\tau_G: \mathbb{Z}^{k_2} \times \mathbb{R}^{k_1} \longrightarrow G$ be the corresponding coordinate mapping. Let S be a finite set in $\mathbb{Z}^{k_2}$ such that $\tau_G|_S$ is a bijection between S and a set of representatives of $G/(G^o\Gamma)$; then $\tau_G\big(S\times\{0\}\big)\cdot G^o\Gamma = G$. Define $\tau_X: S\times[0,1)^{k_1} \longrightarrow X$ by $\tau_X = \pi\circ\tau_G|_{S\times[0,1)^{k_1}}$.

We will call τ_X *the coordinate mapping* for X. If $x \in X$ and $x = \tau_X(t_1, \ldots, t_k)$, we will call $t_1, \ldots, t_k$ *the coordinates* of x.

2.5. Lemma. *τ_X is bijective.*

Proof. Let $x \in X$, $x = \pi(a) = a\Gamma$. By the definition of S, there exists a unique $\gamma_0 \in \Gamma$ such that $a\gamma_0 \in \tau_G\big(S \times \{0\}\big) \cdot G^o$. Then $a\gamma_0 = e_1^{t_1} \ldots e_{k_2}^{t_{k_2}} e_{k_2+1}^{t_{k_2+1}} e_k^{t_k}$ with $(t_1, \ldots, t_{k_2}) \in S$ and $t_{k_2+1}, \ldots, t_k \in \mathbb{R}$. If, for $k_2 < i \leq k$, $\gamma_{i-1} \in \Gamma$ has already been found so that $a\gamma_{i-1} = e_1^{t_1} \ldots e_{k_2}^{t_{k_2}} e_{k_2+1}^{v_{k_2+1}} \ldots e_{i-1}^{v_{i-1}} e_i^{u_i} e_{i+1}^{u_{i+1}} \ldots e_k^{u_k}$ with $v_1, \ldots, v_{i-1} \in [0,1)$, put $\gamma_i = \gamma_{i-1}e_i^{-[u_i]}$ (where $[u]$ denotes the integer part of $u \in \mathbb{R}$). Then by (1.1), $a\gamma_{i-1} = e_1^{t_1} \ldots e_{k_2}^{t_{k_2}}$ $e_{k_2+1}^{v_{k_2+1}} \ldots e_{i-1}^{v_{i-1}} e_i^{v_i} e_{i+1}^{u'_{i+1}} \ldots e_k^{u'_k}$ where $v_i = u_i - [u_i] \in [0,1)$. By induction, for $\gamma = \gamma_k \in \Gamma$ we will have $a\gamma = e_1^{t_1} \ldots e_{k_2}^{t_{k_2}} e_{k_2+1}^{v_{k_2+1}} \ldots e_k^{v_k}$ with $v_{k_2+1}, \ldots, v_k \in [0,1)$. Thus, $a\gamma \in \tau_G\big(S \times [0,1)^{k_1}\big)$ and $x = \pi(a\gamma) \in \operatorname{Range}\tau_X$, so τ_X is surjective. Since $\gamma_0, \gamma_1, \ldots, \gamma_k \in \Gamma$ were defined uniquely, τ_X is one-to-one. ∎

Though the mapping τ_X is continuous, its inverse is discontinuous at the points of $\tau_X\big((S\times[0,1)^{k_1})\setminus(S\times(0,1)^{k_1})\big)$.

2.6. It follows from Lemma 1.8 that

Lemma. *A point x of X is rational if and only if all coordinates of x are rational numbers.*

2.7. Proposition. *A closed subgroup H of G is rational if and only if $\pi(H)$ is closed in X.*

Proof. By Lemma 1.11, H is rational if and only if $\Gamma\cap H$ is uniform in H. We have a continuous mapping $\chi\colon H/(\Gamma\cap H)\longrightarrow\pi(H)=(H\Gamma)/\Gamma$. If $\Gamma\cap H$ is uniform in H then $H/(\Gamma\cap H)$ is compact, χ is a homeomorphism and $\pi(H)$ is closed. On the other hand, H is locally compact and separable, so when $\pi(H)$ is locally compact χ is a homeomorphism ([MZ] Theorem 2.13). Thus, if $\pi(H)$ is closed then χ is a homeomorphism, so $H/(\Gamma\cap H)$ is compact and $\Gamma\cap H$ is uniform in H. ∎

2.8. *A sub-nilmanifold* Y of X is a closed subset of X of the form $Y=\pi(bH)$ where H is a closed subgroup of G and $b\in G$. We see from Proposition 2.7 that for Y be closed H must be a rational subgroup of G. We will say that Y is a *rational* sub-nilmanifold if $b\in\mathbb{Q}(G)$; in other words, Y is a rational sub-nilmanifold of X if $Y=\pi(K)$ where $K=bH$ is a rational left coset in G.

2.9. Let H be a closed subgroup of G and $b\in G$; define $Y=\pi(Hb)$. Then $Y=\pi(b(b^{-1}Hb))$ and so, Y is a sub-nilmanifold of X if and only if the group $b^{-1}Hb$ is rational. If $b\in\mathbb{Q}(G)$, by Corollary 1.15 the subgroups H and $b^{-1}Hb$ are rational simultaneously, so Y is a rational sub-nilmanifold if and only if H is a rational subgroup. Hence, Y is a rational sub-nilmanifold of X if and only if $Y=\pi(L)$ where $L=Hb$ is a rational right coset in G.

2.10. Proposition. *A sub-nilmanifold Y of X is rational if and only if Y contains a rational point of X, and if and only if rational points of X are dense in Y.*

Proof. If Y contains a point $\pi(b)$ with $b\in\mathbb{Q}(G)$ then $Y=\pi(bH)$ where H is a rational closed subgroup of G. By Corollary 1.18, rational elements of G are dense in bH and so, rational points are dense in Y. ∎

2.11. We also get

Proposition. *A sub-nilmanifold Y of X is rational if and only if all connected components of Y are rational sub-nilmanifolds of X.*

2.12. Let X be the nilmanifold G/Γ and Y be a nilmanifold F/Λ. A mapping $\psi\colon Y\longrightarrow X$ will be called *a homomorphism* if there exists a Lie group homomorphism $\varphi\colon F\longrightarrow G$ such that $\psi(cy)=\varphi(c)\psi(y)$ for all $y\in Y$ and $c\in F$.

Let $\psi\colon Y\longrightarrow X$ be a homomorphism and $\varphi\colon F\longrightarrow G$ be the corresponding Lie group homomorphism. Let $\pi_X\colon G\longrightarrow X$ and $\pi_Y\colon F\longrightarrow Y$ be the factorization mappings; put

$\mathbf{0}_X = \pi_X(\mathbf{1}_G)$ and $\mathbf{0}_Y = \pi_Y(\mathbf{1}_F)$. Assume that $\psi(\mathbf{0}_Y) = \mathbf{0}_Y$. Then for any $d \in \Lambda$ we have $\varphi(d)\mathbf{0}_X = \psi(d\mathbf{0}_Y) = \psi(\mathbf{0}_Y) = \mathbf{0}_X$ and thus $\varphi(d) \in \Gamma$. Hence, the homomorphism φ is rational.

Now let $\psi(\mathbf{0}_Y) = x_0 \in X$. Take $b \in \pi_X^{-1}(x_0)$, so that $x_0 = b\mathbf{0}_X$, and define $\psi_0 = b^{-1}\psi$. Then for any $c \in F$ and $y \in Y$ we have $\psi_0(cy) = b^{-1}\varphi(c)\psi(y) = b^{-1}\varphi(c)b\psi_0(y)$. Hence, ψ_0 is a homomorphism from Y to X corresponding to the Lie group homomorphism $\varphi_0\colon F \longrightarrow G$ defined by $\varphi_0(c) = b^{-1}\varphi(c)b$, $c \in F$. ψ_0 satisfies $\psi_0(\mathbf{0}_F) = \mathbf{0}_G$, thus φ_0 is a rational homomorphism. Hence, φ is rational if and only if b is a rational element of G, which is true if and only if x_0 is a rational element of X. We will say that a homomorphism $\psi\colon Y \longrightarrow X$ is *rational* if $\psi(\mathbf{0}_Y) \in \mathbb{Q}(X)$. We have proven the following:

Proposition. *A homomorphism $\psi\colon Y \longrightarrow X$ is rational if and only if the corresponding homomorphism $\varphi\colon F \longrightarrow G$ is rational.*

2.13. We also have

Proposition. *A homomorphism $\psi\colon Y \longrightarrow X$ is rational if and only if there exists $y \in \mathbb{Q}(Y)$ such that $\psi(y) \in \mathbb{Q}(X)$. If this is the case, $\psi(\mathbb{Q}(Y)) \subseteq \mathbb{Q}(X)$, and for any rational sub-nilmanifold Z of Y its image $\psi(Z)$ is a rational sub-nilmanifold of X.*

2.14. Corollary. *If Y is a sub-nilmanifold of a nilmanifold X, the inclusion homomorphism $Y \longrightarrow X$ is rational if and only if Y is a rational sub-nilmanifold of X.*

2.15. Corollary. *X has only countably many rational subnilmanifolds. For any $x \in X$, there are only countably many subnilmanifolds of X containing x.*

Proof. If Y is a rational subnilmanifold of X, the inclusion homomorphism $Y \longrightarrow X$ is defined by a rational homomorphism $F \longrightarrow G$, which is a polynomial mapping with rational coefficients, and there are only countably many of those. In particular, there are only countably many subnilmanifolds of X containing $\mathbf{0}_X$. For any other point $x \in X$, subnilmanifolds containing x are shifts of subnilmanifolds containing $\mathbf{0}_X$. ■

2.16. We will now describe how homomorphisms of nilmanifolds "look in coordinates". Let $\psi\colon Y \longrightarrow X$ be a homomorphism from a nilmanifold $Y = F/\Lambda$ to a nilmanifold $X = G/\Gamma$, let $\varphi\colon F \longrightarrow G$ be the corresponding Lie group homomorphism and let $\pi_Y\colon F \longrightarrow Y$ and $\pi_X\colon G \longrightarrow X$ be the factorization mappings. We will first assume that both Y and X are connected; we may then also assume that both F and G are connected. Let $\tau_G\colon \mathbb{R}^k \longrightarrow G$ and $\tau_F\colon \mathbb{R}^l \longrightarrow F$ be coordinate mappings for G and F, and let $\tau_X\colon [0,1)^k \longrightarrow X$ and $\tau_Y\colon [0,1)^l \longrightarrow Y$ be the corresponding coordinate mappings for X and Y. In the commutative diagram

$$\begin{array}{ccc} F & \xrightarrow{\tilde{\varphi}} & G \\ {\scriptstyle\pi_Y}\downarrow & & \downarrow{\scriptstyle\pi_X} \\ Y & \xrightarrow{\psi} & X \end{array}$$

the mapping $\tilde{\varphi}$ is the composition of φ and the left translation by an element $b \in \pi_X^{-1}\big(\psi(\pi_Y(\mathbf{1}_F))\big)$. Since, in coordinates, both φ and the translation by b are polynomial mappings, in the commutative diagram

$$\begin{array}{ccc} \mathbb{R}^l & \xrightarrow{P_F} & \mathbb{R}^k \\ \tau_F \downarrow & & \downarrow \tau_G \\ F & \xrightarrow{\psi} & G \end{array}$$

P_F is a polynomial maping. P_F is rational if b and so, ψ is rational.

In the commutative diagram

$$\begin{array}{ccc} [0,1)^l & \xrightarrow{P_Y} & [0,1)^k \\ \tau_Y \downarrow & & \downarrow \tau_X \\ Y & \xrightarrow{\psi} & X \end{array}$$

the mapping P_Y is the composition of $P_F|_{[0,1)^l}$ and the "factorization" mapping $\phi{:}\,\mathbb{R}^k \longrightarrow [0,1)^k$ defined in the following way. For $v = (v_1, \ldots, v_k) \in \mathbb{R}^k$ let $M_v{:}\,\mathbb{R}^k \longrightarrow \mathbb{R}^k$ be the polynomial mapping corresponding, by formula (1.1), to the right translation of G by $\tau_G(v)$. Let $Q = [0,1)^k \in \mathbb{R}^k$ and for $v \in \mathbb{R}^k$ let $Q_v = M_v^{-1}(Q)$. It is seen from the proof of Lemma 2.5 that $\bigcup_{v\in\mathbb{Z}^k} Q_v$ is a partition of $\mathbb{R}^k$; the mapping $\phi{:}\,\mathbb{R}^k \longrightarrow Q$ is defined by $\phi|_{Q_v} = M_v$, $v \in \mathbb{Z}^k$.

Let $v_1, \ldots, v_m \in \mathbb{Z}^k$ be such that $P_F([0,1)^l) \subseteq \bigcup_{j=1}^m Q_{v_j}$ and let $\mathcal{L}_j = P_F^{-1}(Q_{v_j}) \cap [0,1)^l$, $j = 1, \ldots, m$. Then $[0,1)^l$ is partitioned into $\bigcup_{j=1}^m \mathcal{L}_j$. For each $j \in \{1, \ldots, N\}$ the restriction of P_Y on $\mathcal{L}_j$ is the polynomial mapping $P_j = M_{v_j} \circ P_F|_{\mathcal{L}_j}$, and $\mathcal{L}_j$ is defined by $\mathcal{L}_j = P_j^{-1}([0,1)^k) \cap [0,1)^l$. If the homomorphism ψ is rational, then P_F is a rational polynomial mapping, and since M_v are rational for $v \in \mathbb{Z}^d$, the polynomial mappings P_j, $j = 1, \ldots, m$, are also rational.

If X and/or Y are disconnected they consist of finitely many connected components, and our argument is applicable to each component of Y and the corresponding component of X.

2.17. We arrive at the following result:

Proposition. *Let $\psi{:}\,Y \longrightarrow X$ be a homomorphism of nilmanifolds, let $\tau_X{:}\,S \times [0,1)^k \longrightarrow X$ and $\tau_Y{:}\,R \times [0,1)^l \longrightarrow Y$, with S and R being finite sets, be coordinate mappings for X and Y and let $P_Y = \tau_X^{-1} \circ \psi \circ \tau_Y{:}\,R \times [0,1)^l \longrightarrow S \times [0,1)^k$. Then for each $r \in R$ there exist $s \in S$ and polynomial mappings $P_{r,1}, \ldots, P_{r,m_r}{:}\,\mathbb{R}^l \longrightarrow \mathbb{R}^k$ such that the sets $\mathcal{L}_{r,j} = P_{r,j}^{-1}([0,1)^k) \cap [0,1)^l$, $j = 1, \ldots, m_r$, partition $[0,1)^l$ and for each $j = 1, \ldots, m_r$ one has $P_Y|_{\{r\}\times\mathcal{L}_{r,j}} = \{s\} \times P_{r,j}|_{\{r\}\times\mathcal{L}_{r,j}}$. If ψ is rational, then $P_{r,j}$ are rational for all $r \in R$ and all $j \in \{1, \ldots, m_r\}$.*

2.18. Let us say that a mapping $P: U \longrightarrow \mathbb{R}^k$ from a set $U \subseteq \mathbb{R}^l$ is *piecewise polynomial* if U can be partitioned, $U = \bigcup_{j=1}^m \mathcal{L}_j$, into subsets $\mathcal{L}_1, \ldots, \mathcal{L}_m$ such that, for each $j = 1, \ldots, m$, $\mathcal{L}_j$ is defined by a system of polynomial inequalities:

$$\mathcal{L}_j = \bigl\{t \in U : q_1(t) > 0, \ldots, q_i(t) > 0, q_{i+1}(t) \geq 0, \ldots, q_n(t) \geq 0\bigr\}$$

where $q_1, \ldots, q_n$ are polynomials on $\mathbb{R}^l$, and the restriction of P on $\mathcal{L}_j$ is a polynomial mapping. We will say that a piecewise polynomial mapping P is *rational* if, for all j, the polynomials $q_1, \ldots, q_n$ and $P|_{\mathcal{L}_j}$ are rational. If P is a mapping from a union $\bigcup_{r \in R} U_r$ of subsets of distinct copies of $\mathbb{R}^k$, we will say that P is piecewise polynomial if $P|_{U_r}$ is piecewise polynomial for all $r \in R$.

2.19. From Proposition 2.17 we have:

Corollary. *The mapping* $P_Y = \tau_X^{-1} \circ \psi \circ \tau_Y : R \times [0,1)^l \longrightarrow S \times [0,1)^k$ *is piecewise polynomial. If* ψ *is rational, then* P_Y *is rational.*

2.20. Corollary. *Given two coordinate mappings* τ_X *and* τ'_X *for a nilmanifold* X*, the mapping* $\tau_X^{-1} \circ \tau'_X$ *is piecewise polynomial and rational.*

2.21. Let us say that a function $f: X \longrightarrow \mathbb{R}^m$ on a nilmanifold X is *piecewise polynomial* if the function $f \circ \tau_X^{-1}: S \times [0,1)^k \longrightarrow \mathbb{R}^m$ is piecewise polynomial. If this is the case, let us say that f is *rational* if $f \circ \tau_X^{-1}$ is rational. Since the composition of piecewise polynomial mappings is piecewise polynomial, and the composition of rational piecewise polynomial mappings is rational, it follows from Corollary 2.20 that the definitions above do not depend on the choice of the coordinate mapping τ_X.

2.22. Since the composition of piecewise polynomial mappings is piecewise polynomial, we have

Corollary. *If* f *is a piecewise polynomial function on a nilmanifold* X *and* $\psi: Y \longrightarrow X$ *is a homomorphism of nilmanifolds, then* $f \circ \psi$ *is a piecewise polynomial function on* Y*. If both* f *and* ψ *are rational, then* $f \circ \psi$ *is also rational.*

3. Rationality of the closure of a polynomial orbit

3.1. A sequence $\{g(n)\}_{n \in \mathbb{Z}}$ in G of the form $g(n) = a_1^{p_1(n)} \cdots a_m^{p_m(n)}$, where $a_1, \ldots, a_m \in G$ and $p_1, \ldots, p_m$ are polynomials taking on integer values on the integers, will be called *a polynomial sequence.* Let $\{e_1, \ldots, e_k\}$ be a basis in G with $e_1, \ldots, e_{k_2} \notin G^o$ and $e_{k_2+1}, \ldots, e_k \in G^o$; it follows from formulas (1.1) and (1.2) that any polynomial sequence g in G can be written in terms of this basis: $g(n) = e_1^{q_1(n)} \cdots e_k^{q_k(n)}$ where $q_1, \ldots, q_k \in \mathbb{R}[n]$ and $q_1, \ldots, q_{k_2}$ take on integer values on the integers.

3.2. Again, let $X = G/\Gamma$ and $\pi: G \longrightarrow X$ be the factorization mapping. Let $\{g(n)\}_{n \in \mathbb{Z}}$ be a polynomial sequence in G and let $x \in X$. It is proven in [L1] that the closure $\overline{\{g(n)x\}_{n \in \mathbb{Z}}}$ of the "polynomial orbit" $\{g(n)x\}_{n \in \mathbb{Z}}$ of x is a disjoint union $Y_1 \cup \ldots \cup Y_l$ of connected sub-nilmanifolds of X. We now prove:

Theorem. *If $g(0)x$ is rational, then all of $Y_1,\dots,Y_l$ are rational.*

3.3. We will be based on the following fact:

Proposition. *Let g be a polynomial sequence in G with $g(0)=\mathbf{1}_G$. There exists a nilmanifold $\widetilde{X}=\widetilde{G}/\widetilde{\Gamma}$, with the factorization mapping $\tilde{\pi}\colon\widetilde{G}\longrightarrow\widetilde{X}$, an open subgroup $\widehat{G}$ of $\widetilde{G}$ such that $\tilde{\pi}(\widehat{G})=\widetilde{X}$ and so, $\widehat{X}=\widehat{G}/(\widetilde{\Gamma}\cap\widehat{G})\simeq\widetilde{X}$, a rational homomorphism $\psi\colon\widehat{X}\longrightarrow X$ and an element $c\in\widetilde{G}$ such that $\pi(g(n))=\psi(\tilde{\pi}(c^n))$, $n\in\mathbb{Z}$.*

Proof. Let $\{e_1,\dots,e_k\}$ be a basis in G and let $\delta\in\mathbb{N}$ be such that $a^\delta\in\Gamma$ for any a from the (discrete) group generated by $\{e_1,\dots,e_k\}$. Let $g(n)=e_1^{q_1(n)}\dots e_k^{q_k(n)}$, where $q_1,\dots,q_{k_2}$ are polynomials $\mathbb{Z}\longrightarrow\mathbb{Z}$, $q_{k_2+1},\dots,q_k$ are polynomials $\mathbb{Z}\longrightarrow\mathbb{R}$, and $\deg q_j=m_j$, $j=1,\dots,k$. Let $\widehat{G}$ be the free nilpotent Lie group of same nilpotency class as G, with discrete generators $\{b_{j,i}\}_{\substack{i=0,\dots,m_j\\ j=1,\dots,k_2}}$ and continuous generators $\{b_{j,i}\}_{\substack{i=0,\dots,m_j\\ j=k_2+1,\dots,k}}$ (see [L1]). Let $\varphi\colon\widehat{G}\longrightarrow G$ be the rational homomorphism defined by $\varphi(b_{j,0})=e_j$ and $\varphi(b_{j,i})=\mathbf{1}_G$, $i=1,\dots,m_j$, $j=1,\dots,k$. The group B generated by $\{b_{j,i}\}_{\substack{i=1,\dots,m_j\\ j=1,\dots,k_2}}$ and $\{b_{j,i}^t\}_{\substack{t\in\mathbb{R}\\ i=1,\dots,m_j\\ j=k_2+1,\dots,k}}$ is then contained in $\ker\varphi$.

Let $\widehat{E}$ be the subgroup of $\widehat{G}$ generated by $\{b_{j,i}\}_{\substack{i=0,\dots,m_j\\ j=1,\dots,k}}$ and $\widehat{\Gamma}$ be the subgroup generated by the δ-th powers of the elements of $\widehat{E}$, $\widehat{\Gamma}=\bigl\langle\{\gamma^\delta,\ \gamma\in\widehat{E}\}\bigr\rangle$. Then $\widehat{\Gamma}$ is uniform in $\widehat{G}$, $\varphi(\widehat{\Gamma})\subseteq\Gamma$ and one has $\sigma(\widehat{\Gamma})=\widehat{\Gamma}$ for any automorphism σ of $\widehat{E}$. Let $\widehat{X}=\widehat{G}/\widehat{\Gamma}$ and let $\psi\colon\widehat{X}\longrightarrow X$ be the rational homomorphism induced by φ.

We define an automorphism σ of $\widehat{G}$ by $\sigma(b_{j,0})=b_{j,0}$ and $\sigma(b_{j,i})=b_{j,i}b_{j,i-1}$ for $j=1,\dots,k$ and $i=1,\dots,m_j$. σ induces a unipotent automorphism of $\widehat{G}/[\widehat{G},\widehat{G}]$; it is shown in [L1] that σ is a unipotent automorphism of $\widehat{G}$. Let $\widetilde{G}$ be the extension of $\widehat{G}$ by σ and let $s\in\widetilde{G}$ be the element corresponding to σ, so that $s^{-1}as=\sigma(a)$ for any $a\in\widehat{G}$; by [L1], $\widetilde{G}$ is a nilpotent group. Since σ preserves $\widehat{E}$, it also preserves $\widehat{\Gamma}$, and the group $\widetilde{\Gamma}=\langle\widehat{\Gamma},s\rangle$ is uniform in $\widetilde{G}$. The nilmanifold $\widetilde{X}=\widetilde{G}/\widetilde{\Gamma}$ is isomorphic to $\widehat{X}$ as a topological space, and we will identify them. Let $\tilde{\pi}\colon\widetilde{G}\longrightarrow\widetilde{X}$ be the factorization mapping; we obtain the commutative diagram

$$\begin{array}{ccccc}
\widetilde{G} & \supseteq & \widehat{G} & \xrightarrow{\varphi} & G\\
{\scriptstyle\tilde{\pi}}\downarrow & & {\scriptstyle\hat{\pi}}\downarrow & & \downarrow{\scriptstyle\pi}\\
\widetilde{X} & \simeq & \widehat{X} & \xrightarrow{\psi} & X
\end{array}$$

For $j\in\{1,\dots,k\}$ let $q_j(n)=\alpha_1\binom{n}{1}+\alpha_2\binom{n}{2}+\dots+\alpha_{m_j}\binom{n}{m_j}$. Define $a_j=b_{j,1}^{\alpha_1}\dots b_{j,m_j}^{\alpha_{m_j}}$, then $\sigma^n(a_j)=b_{j,0}^{\alpha_1\binom{n}{1}+\alpha_2\binom{n}{2}+\dots+\alpha_{m_j}\binom{n}{m_j}}h(n)=b_{j,0}^{p_j(n)}h(n)$ with $h(n)\in B\subseteq\ker\varphi$, $n\in\mathbb{Z}$, and thus $\varphi(\sigma^n(a_j))=e_j^{p_j(n)}$. Put $a=a_1\dots a_k$, then $\varphi(\sigma^n(a))=g(n)$, $n\in\mathbb{Z}$. Define $c=a^{-1}s^{-1}a$, then

$$\tilde{\pi}(\sigma^n(a))=\tilde{\pi}(s^{-n}as^n)=\tilde{\pi}(s^{-n}a)=\tilde{\pi}(a(a^{-1}s^{-n}a))=a\tilde{\pi}((a^{-1}s^{-1}a)^n)=a\tilde{\pi}(c^n).$$

Hence,

$$\pi(g(n))=\pi(\varphi(\sigma^n(a)))=\psi(\tilde{\pi}(\sigma^n(a)))=\psi(a\tilde{\pi}(c^n))=\varphi(a)\psi(\tilde{\pi}(c^n))=\psi(\tilde{\pi}(c^n))$$

since $a \in B \subseteq \ker \varphi$. $\blacksquare$

3.4. Proof of Theorem 3.2. Let $x = \pi(a)$. Then $g(n)x = g(0)a\pi(g'(n))$ where $g'(n) = a^{-1}(g(0)^{-1}g(n))a$ is a polynomial sequence with $g'(0) = \mathbf{1}_G$; since $\pi(g(0)a) = g(0)x \in \mathbb{Q}(X)$, $g(0)a \in \mathbb{Q}(G)$. Replace g by g' and assume that $g(0) = \mathbf{1}_G$ and $x = \pi(\mathbf{1}_G)$. Let $\widetilde{G}$, $\widetilde{X}$, $\tilde{\pi}$, $\widehat{G}$, $\widehat{X}$, ψ and $c \in \widetilde{G}$ be as in Proposition 3.3, so that $g(n)x = \psi(c^n\widetilde{x})$, $n \in \mathbb{Z}$, for $\widetilde{x} = \tilde{\pi}(\mathbf{1}_{\widetilde{G}})$. The closure $Z = \overline{\{c^n\widetilde{x}\}}_{n\in\mathbb{Z}}$ of the orbit of $\widetilde{x}$ under the action of c is a sub-nilmanifold $\widetilde{Y}$ of $\widetilde{X}$ (see, for example, [L1]). Since $\widetilde{Y}$ contains the rational point $\widetilde{x}$, $\widetilde{Y}$ is rational by Proposition 2.10. Let $\widetilde{Y}_1,\ldots,\widetilde{Y}_l$ be the connected components of $\widetilde{Y}$; they all are rational sub-nilmanifolds of $\widetilde{X}$ by Proposition 2.11. Since $\widehat{G}$ is open in $\widetilde{G}$, $\widetilde{Y}_1,\ldots,\widetilde{Y}_l$ are rational sub-nilmanifolds of $\widehat{X}$. We have $\overline{\{g(n)x\}}_{n\in\mathbb{Z}} = \psi(\widetilde{Y}_1)\cup\ldots\cup\psi(\widetilde{Y}_l)$; since ψ is a rational homomorphism, $\psi(\widetilde{Y}_1),\ldots,\psi(\widetilde{Y}_l)$ are rational sub-nilmanifolds of X. $\blacksquare$

3.5. *A polynomial mapping* $\mathbb{Z}^d \longrightarrow G$ is a mapping ω of the form $\omega(n) = a_1^{p_1(n)}\cdot\ldots\cdot a_m^{p_m(n)}$, where $a_1,\ldots,a_m \in G$ and $p_1,\ldots,p_m$ are polynomials $\mathbb{Z}^d \longrightarrow \mathbb{Z}$. One derives from Theorem 3.2 its multiparameter version:

Theorem. *Let $\omega\colon\mathbb{Z}^d \longrightarrow G$ be a polynomial mapping, let $x \in \mathbb{Q}(X)$, and assume that $\omega(0)x \in \mathbb{Q}(X)$. Then $\overline{\{\omega(n)x\}}_{n\in\mathbb{Z}^d} = Y_1\cup\ldots\cup Y_l$ where $Y_1,\ldots,Y_l$ are connected rational sub-nilmanifolds of X.*

Proof. It is proven in [L2] that $\overline{\{\omega(n)x\}}_{n\in\mathbb{Z}^d}$ is a disjoint union of connected sub-nilmanifolds $Y_1,\ldots,Y_l$ of X; we are only going to show that $Y_1,\ldots,Y_l$ are rational. Let $i \in \{1,\ldots,l\}$; find $n \in \mathbb{Z}^d$ such that $\omega(n) \in Y_i$ and consider the polynomial sequence $g(m) = \omega(nm)$, $m \in \mathbb{Z}$ in G. By Theorem 3.2, $\overline{\{g(m)x\}}_{m\in\mathbb{Z}} = Z_1\cup\ldots\cup Z_k$ where $Z_1,\ldots,Z_k$ are connected rational subnilmanifolds of X. We have $Z_j \subseteq Y_i$ for some j; it follows that Y_i contains a rational point and so, is rational by Proposition 2.10. $\blacksquare$

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