Multiple polynomial correlation sequences and nilsequences

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Abstract

A *basic nilsequence* is a sequence of the form $\psi(n)=f(T^n x)$, where x is a point of a compact nilmanifold X, T is a translation on X, and $f\in C(X)$; a *nilsequence* is a uniform limit of basic nilsequences. Let $X=G/\Gamma$ be a compact nilmanifold, Y be a subnilmanifold of X, $g(n)$ be a polynomial sequence in G, and $f\in C(X)$; we show that the sequence $\int_{g(n)Y} f$, $n\in\mathbb{Z}$, is the sum of a basic nilsequence and a sequence that converges to 0 in uniform density. This implies that, given an ergodic invertible measure preserving system $(W,\mathcal{B},\mu,T)$, with $\mu(W)<\infty$, polynomials $p_1,\ldots,p_k\in\mathbb{Z}[n]$, and sets $A_1,\ldots,A_k\in\mathcal{B}$, the sequence $\mu(T^{p_1(n)}A_1\cap\ldots\cap T^{p_k(n)}A_k)$ is the sum of a nilsequence and a sequence that converges to 0 in uniform density. We also get a version of this result for the case where p_i are polynomials in several variables.

0. Introduction

A (d-step) *nilmanifold* is a compact homogeneous space of a (d-step) nilpotent Lie group; one can show that any d-step nilmanifold has the form G/Γ, where G is a d-step nilpotent (not necessarily connected) Lie group and Γ is a discrete co-compact subgroup of G. Elements of G act on X by *translations*; a (d-step) *nilsystem* is a (d-step) nilmanifold $X=G/\Gamma$ with a translation $a\in G$ on it. Nilsystems play an important role in studying "non-conventional", or "multiple", ergodic averages $\frac{1}{N}\sum_{n=1}^{N}T^{p_1(n)}h_1\cdot\ldots T^{p_k(n)}h_k$, where T is a transformation of a finite measure space (W,μ), $p_1,\ldots,p_k\in\mathbb{Z}[n]$, and $h_1,\ldots,h_k\in L^\infty(W)$. (See [HK1], [Z], [HK2].)

Let $X=G/\Gamma$ be a nilmanifold and Y be a subnilmanifold of X. Let g be a polynomial sequence in G, that is, a sequences of the form $g(n)=a_1^{p_1(n)}\ldots a_r^{p_r(n)}$, where $a_1,\ldots,a_r\in G$ and $p_1,\ldots,p_r$ are polynomials taking on integer values on the integers. It is shown in [L1] that the closure of the sequence $g(n)Y$, $X'=\overline{\bigcup_{n\in\mathbb{Z}}g(n)Y}$, is a disjoint finite union of subnilmanifolds of X, and, if X' is a single subnilmanifold, the sequence $g(n)Y$ is well distributed in X'. (That is, for every $f\in C(X')$,

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$\frac{1}{N_2-N_1}\sum_{n=N_1+1}^{N_2}\int_{g(n)Y} f\,d(g(n)\mu_Y) \underset{N_2-N_1\to\infty}{\longrightarrow} \int_{X'} f\,d\mu_{X'}$, where μ_Y and $\mu_{X'}$ are the normalized Haar measures on Y and on X' respectively.)

We were inspired by the following example. Let X be the 2-dimensional torus $\mathbb{T}^2=(\mathbb{R}/\mathbb{Z})^2$ and G be the group generated by the ordinary rotations of X and by the transformation $a(x,y)=(x,y+x)$; then G is a nilpotent Lie group acting on X transitively, which turns X to a nilmanifold. Choose an irrational $\alpha\in\mathbb{T}$ and put $b(x,y)=(x+\alpha,y+x)$, then $b\in G$. Let $Y_1=\{(0,t),\ t\in\mathbb{T}\}$ and $Y_2=\{(t,0),\ t\in\mathbb{T}\}$. Then $b^nY_1=\{(n\alpha,t),\ t\in\mathbb{T}\}$ and $b^nY_2=\bigl\{\bigl(t+n\alpha,nt+\frac{n(n-1)}{2}\alpha\bigr),\ t\in\mathbb{T}\bigr\}$, $n\in\mathbb{Z}$. Both sequences b^nY_1 and b^nY_2, $n\in\mathbb{Z}$, are dense in X, but their behaviors are different: the sequence b^nY_1 consists of congruent subtori that simply "rotate" along X, whereas the members of the sequence b^nY_2, $n\in\mathbb{Z}$, become more and more dense in X. We can say that the sequence b^nY_2 *converges* to X: $\int_{g(n)Y_2} f\,d\mu_{Y_2}\longrightarrow\int_X f\,d\mu_X$ for any $f\in C(X)$, whereas the sequence b^nY_1 converges to X only in average: $\frac{1}{N_2-N_1}\sum_{n=N_1+1}^{N_2}\int_{g(n)Y_1} f\,d\mu_{Y_1}\longrightarrow\int_X f\,d\mu_X$ for any $f\in C(X)$. It is clear what difference between Y_1 and Y_2 causes this effect: Y_1 is a normal subgroup of G whereas Y_2 is not.

Our goal was to show that in the general situation the sequence $g(n)Y$ has a "mixed" behavior: $g(n)Y$ converges to a subnilmanifold Z (the normal closure of Y), which, in its turn, rotates along X. We, however, have been unable to prove this, and only prove the weaker fact that $g(n)Y$ converges to Z "in uniform density" (see Proposition 2.1). Our proof essentially uses a result from a recent paper by Green and Tao ([GT]) about the "uniform distribution" of subnilmanifolds (see Appendix).

In the terminology introduced in [BHK], *a basic d-step nilsequence* is a sequence of the form $\psi(n)=h(R^nw)$, where w is a point of a d-step nilmanifold M, R is a translation on M, and $h\in C(M)$; a *d-step nilsequence* is a uniform limit of basic d-step nilsequences. The algebra of nilsequences is a natural generalization of Weyl's algebra of almost periodic sequences, which are just 1-step nilsequences. We obtain, as a corollary, that for any $f\in C(X)$ the sequence $\int_{g(n)Y} f\,d\mu_{g(n)Y}$ is a sum of a basic nilsequence and a sequence that tends to 0 in uniform density (Theorem 2.5 below). We apply this fact to show that for any ergodic invertible measure preserving system $(W,\mathcal{B},\mu,T)$ with $\mu(W)<\infty$, polynomials $p_1,\dots,p_k\in\mathbb{Z}[n]$, and sets $A_1,\dots,A_k\in\mathcal{B}$, the "multiple polynomial correlation sequence" $\varphi(n)=\mu\bigl(T_1^{p_1(n)}A_1\cap\ldots\cap T_k^{p_k(n)}A_k\bigr)$, $n\in\mathbb{Z}$, is a sum of a nilsequence and a sequence that tends to 0 in uniform density (Theorem 3.1 below). (A special case of this theorem, when $p_i(n)=in$, $i=1,\dots,k$, was established in [BHK].) The question whether this is true for non-ergodic systems remains open to us. We also formulate and sketch the proof of a "multiparameter" version of this result: when $p_1,\dots,p_k$ are polynomials of m integer variables, then the sequence $\varphi(n)=\mu\bigl(T_1^{p_1(n)}A_1\cap\ldots\cap T_k^{p_k(n)}A_k\bigr)$, $n\in\mathbb{Z}^m$, is a sum of an (m-parameter) nilsequence and a sequence that tends to 0 in (ordinary) density (Theorem 4.3).

1. Nilmanifolds and subnilmanifolds

We will now give necessary definitions and list some facts that we will need below; details and proofs can be found in [M], [L1], [L2], [L4], and [L5]. Throughout the paper, let

$X = G/\Gamma$ be a compact nilmanifold, where G is a nilpotent Lie group and Γ is a discrete subgroup of G, and let $\pi: G \longrightarrow X$ be the natural projection. By $\mathbf{1}_X$ we will denote the point $\pi(\mathbf{1}_G)$ of X.

By G^o we will denote the identity component of G. We will assume that the group G/G^o is finitely generated (which is enough for our goals).

Note that if G is disconnected, X can be interpreted as a nilmanifold, $X = G'/\Gamma'$, in different ways; for example, if X is connected, $X = G^o/(\Gamma \cap G^o)$. If X is connected and we study the action on X of a sequence $g(n)$ in G, we may always assume that G is generated by G^o and the elements of g.

Every nilpotent Lie group G is a factor of a simply-connected (not necessarily connected) torsion free nilpotent Lie group. (As such, a suitable "free nilpotent Lie group" F can be taken. If G^o has l_1 generators, G/G^o has l_2 generators, and G is d-step nilpotent, then $F = \mathcal{F}/\mathcal{F}_{d+1}$, where $\mathcal{F}$ is the free product of l_1 copies of $\mathbb{R}$ and l_2 copies of $\mathbb{Z}$, and $\mathcal{F}_{d+1}$ is the $(d+1)$st term of the lower central series of $\mathcal{F}$.) Thus, we may and will assume that G is simply connected and torsion-free. The identity component G^o of G is then an exponential Lie group, which means that for every element $a \in G^o$ there exists a (unique) one-parametric subgroup a^t such that $a^1 = a$.

A *Malcev basis* of G is a finite set $\{e1, \dots, ek\}$ of elements of Γ, with $e_1, \dots, e_{k_1} \in G^o$ and $e_{k_1+1}, \dots, e_k \notin G^o$, that generates Γ and is such that every element $a \in G$ can be uniquely written in the form $a = e_1^{u_1} \dots e_k^{u_k}$ with $u_1, \dots, u_{k_1} \in \mathbb{R}$ and $u_{k_1+1}, \dots, u_k \in \mathbb{Z}$; we call $u_1, \dots, u_k$ *the coordinates* of a. Thus, Malcev coordinates define a homeomorphism $G \simeq \mathbb{R}^{k_1} \times \mathbb{Z}^{k-k_1}$, $a \leftrightarrow (u_1, \dots, u_k)$, and we may identify G with $\mathbb{R}^{k_1} \times \mathbb{Z}^{k-k_1}$.

If L is a connected closed normal subgroup of G of dimension l such that the lattice $L \cap \Gamma$ is co-compact in L, the Malcev coordinates on G can be chosen so that $e_1, \dots, e_l \in L \cap \Gamma$; then $e_1^{u_1} \dots e_k^{u_k} \in L$ iff $u_{l+1}, \dots, u_k = 0$, and L is identified with the subspace $\mathbb{R}^l \times \{0\}^{k-l} \subseteq \mathbb{R}^{k_1} \times \mathbb{Z}^{k-k_1}$. We will call such coordinates on G *compatible with* L.

Let X be connected. Then, under the identification $G^o \leftrightarrow \mathbb{R}^{k_1}$, the cube $[0,1)^{k_1}$ is the fundamental domain of X. We will call the closed cube $Q = [0,1]^{k_1}$ *the fundamental cube of* X *in* G^o and identify X with Q. When X is identified with its fundamental cube Q, the normalized Haar measure μ_X on X coincides with the standard Lebesgue measure μ_Q on Q.

In Malcev coordinates, multiplication in G is a polynomial operation: there are polynomials $q_1, \dots, q_k$ in $2k$ variables with rational coefficients such that for $a = e_1^{u_1} \dots e_k^{u_k}$ and $b = e_1^{v_1} \dots e_k^{v_k}$ we have $ab = e_1^{q_1(u_1,v_1,\dots,u_k,v_k)} \dots e_k^{q_k(u_1,v_1,\dots,u_k,v_k)}$. This implies that "life is polynomial" in nilpotent Lie groups: homomorphisms are polynomial mappings, connected closed subgroups are images of polynomial mappings and are defined by systems of polynomial equations.

A subnilmanifold Y of X is a closed subset of the form $Y = Hx$, where H is a closed subgroup of G and $x \in X$. For a closed subgroup H of G, the set $\pi(H) = H\mathbf{1}_X$ is closed, and so is a subnilmanifold, iff the subgroup $\Gamma \cap H$ is co-compact in H; we will call the subgroup H with this property *rational*.

If Y is a subnilmanifold of X such that $\mathbf{1}_X \in Y$, then $H = \pi^{-1}(Y)$ is a closed subgroup of G, and $Y = \pi(H) = H\mathbf{1}_X$. H, however, does not have to be the minimal subgroup with this property: if Y is connected, then the identity component H^o of H also satisfies

$Y = \pi(H^o)$.

Given a subnilmanifold Y of X, by μ_Y we will denote the normalized Haar measure on Y; we have $a\mu_Y = \mu_{aY}$ for all $a \in G$.

Let Z be a subnilmanifold of X, $Z = Lx$, where L is a closed subgroup of G. We say that Z is *normal* if L is normal. In this case the nilmanifold $\widehat{X} = X/Z = G/(L\Gamma)$ is defined, and X splits into a disjoint union of fibers of the projection mapping $X \longrightarrow \widehat{X}$. (Note that if L is normal in G^o only, then the factor $X/Z = G^o/(L\Gamma)$ is also defined, but the elements of $G \setminus G^o$ do not act on it.)

One can show that a subgroup L is normal iff $\gamma L \gamma^{-1} = L$ for all $\gamma \in \Gamma$; hence, $Z = \pi(L)$ is normal iff $\gamma Z = Z$ for all $\gamma \in \Gamma$.

If H is a closed rational subgroup of G then its normal closure L (the minimal normal subgroup of G containing H) is also closed and rational, thus $Z = \pi(L)$ is a subnilmanifold of X. We will call Z *the normal closure* of the subnilmanifold $Y = \pi(H)$. If L is normal then the identity component of L is also normal; this implies that the normal closure of a connected subnilmanifold is connected.

Let X be connected and k-dimensional, and let Z be an l-dimensional connected normal subnilmanifold of X. Let L be the connected normal closed subgroup of G such that $Z = Lx$; choose Malcev coordinates on G compatible with L, and let Q be the fundamental cube of X in G^o associated with these coordinates. Then the fundamental cube of Z is the subcube $[0,1]^l \times \{0\}^{k-l}$ of Q, and the fundamental cube of X/Z is the orthogonal projection of Q to the $(k-l)$-dimensional subspace associated with the last $k-l$ coordinates on Q.

Let X be connected. We will need the fact that "almost all" subnilmanifolds of X are "quite uniformly" distributed in X. (This is in complete analogy with the situation on tori: if X is a torus, for any $\varepsilon > 0$ there are only finitely many subtori $V_1, \ldots, V_r$, of codimension 1 in X, such that any subtorus Y of X that contains 0 and is not contained in $\bigcup_{i=1}^r V_i$ is ε-dense and "ε-uniformly distributed" in X.) The following proposition is a corollary (of a special case) of the result obtained in [GT] (see Appendix for details):

Proposition 1.1. *For any $f \in C(X)$ and any $\varepsilon > 0$ there are finitely many subnilmanifolds $V_1, \ldots, V_r$ of X, connected, of codimension 1, and containing $\mathbf{1}_X$, such that for any connected subnilmanifold Y of X with $\mathbf{1}_X \in Y$, either $Y \in V_i$ for some $i \in \{1, \ldots, r\}$, or $\left|\int_Y f\,d\mu_Y - \int_X f\,d\mu_X\right| < \varepsilon$, (or both).*

Identifying a subnilmanifold Y of X with the measure μ_Y on X, we introduce the weak* topology on the set of subnilmanifolds of X; in this topology, given subnilmanifolds $Z, Y_1, Y_2, \ldots$ of X, we write $Y_n \longrightarrow Z$ if $\int_{Y_n} f\,d\mu_{Y_n} \longrightarrow \int_Z f\,d\mu_Z$ for every $f \in C(X)$. It now follows from Proposition 1.1 that if connected subnilmanifolds $Y_1, Y_2, \ldots$ of X, with $\mathbf{1}_X \in Y_n$ for all n, are such that for any proper subnilmanifold V of X (connected, of codimension 1, and with $\mathbf{1}_X \in V$) the set $\{n \in \mathbb{Z} : Y_n \subseteq V\}$ is finite, then $Y_n \longrightarrow X$.

For a set $S \in \mathbb{Z}$, *the uniform* (or *Banach*) *density* of S is $\mathcal{D}(S) = \lim_{N_2 - N_1 \to \infty} \frac{|S \cap [N_1, N_2]|}{N_2 - N_1}$ (if it exists). We will say that a sequence of points $(\omega_n)_{n \in \mathbb{Z}}$ of a topological space Ω converges to $\omega \in \Omega$ *in uniform density* if for every neighborhood U of ω one has $\mathcal{D}\big(\{n \in \mathbb{Z} : \omega_n \notin U\}\big) = 0$. It follows from Proposition 1.1 that, given connected subnilmanifolds $Y_1, Y_2, \ldots$ of X with $\mathbf{1}_X \in Y_n$ for all n, if for any proper subnilmanifold V of X

(connected, of codimension 1, and with $\mathbf{1}_X \in V$) one has $\mathcal{D}\big(\{n \in \mathbb{Z} : Y_n \subseteq V\}\big) = 0$, then $Y_n \longrightarrow X$ in uniform density.

2. Polynomial orbits of subnilmanifolds and nilsequences

Our main technical result is the following proposition.

Proposition 2.1. *Let X be connected and let $Y = \pi(H)$ be a connected subnilmanifold of X, where H is a connected closed subgroup of G. Let g be a polynomial sequence in G with $g(0) = \mathbf{1}_G$ such that $g(\mathbb{Z})Y$ is dense in X, and assume that G is generated by G^o and the elements of g. Let Z be the normal closure of Y in X; then $g(n)Y - g(n)Z \longrightarrow 0$ in uniform density.*

Remark. We believe that, actually, $g(n)Y - g(n)Z \longrightarrow 0$ (that is, for any $f \in C(X)$, $\big|\int_{g(n)Y} f\,d\mu_{g(n)Y} - \int_{g(n)Z} f\,d\mu_{g(n)Z}\big| \longrightarrow 0$ as $n \to \infty$).

Proof. Let L be the identity component of $\pi^{-1}(Z)$. Choose Malcev's coordinates in G^o compatible with L, and let Q be the corresponding fundamental cube in G^o. Q is compact, and is as well compact with respect to the uniform norm when elements of G are interpreted as transformations of X. Represent $g(n) = t_n\gamma_n$ so that $\gamma_n \in \Gamma$ and $t_n \in Q$, $n \in \mathbb{Z}$. Since Z is normal, $\gamma_n Z = Z$ for all n, so that $g(n)Z = t_n\gamma_n Z = t_n Z$, $n \in \mathbb{Z}$. We have $g(n)Y = t_n\gamma_n Y$, $n \in \mathbb{Z}$, and since Q is compact, we only have to show that $\gamma_n Y \longrightarrow Z$ in uniform density.

Let Q' be the fundamental cube of X/Z and let $\tau\colon Q \longrightarrow Q'$ be the natural projection. Since the sequence $(g(n)Z)$ is well distributed in X, the sequence $(\tau(t_n))$ is well distributed in Q', which means that for any measurable subset U of Q' whose boundary is a null-set, $\mathcal{D}\big(\{n \in \mathbb{Z} : \tau(t_n) \in U\}\big) = \mu_{Q'}(U)$.

Let V be a subnilmanifold of Z, connected, of codimension 1 in Z, and with $\mathbf{1}_X \in V$; based on Proposition 1.1, we only need to show that the set $\big\{n \in \mathbb{Z} : \gamma_n Y \subseteq V\big\}$ has zero uniform density. Let K be the identity component of $\pi^{-1}(V)$; we have $\gamma_n H \gamma_n^{-1} \subseteq L$ for all $n \in \mathbb{Z}$, and have to prove that the set $S = \big\{n \in \mathbb{Z} : \gamma_n H \gamma_n^{-1} \subseteq K\big\}$ has zero uniform density.

Since K is a proper subgroup of L, there exists $b \in G$ such that $bHb^{-1} \not\subseteq K$. By assumption, G is generated by G^o and g. The group G^o is generated by Q, thus $tHt^{-1} \not\subseteq K$ for some $t \in Q$ or $g(n)Hg(n)^{-1} \not\subseteq K$ for some $n \in \mathbb{Z}$. So, there exists $a \in H$ such that $tat^{-1} \notin K$ for some $t \in Q$ or $g(n)ag(n)^{-1} \notin K$ for some $n \in \mathbb{Z}$. Let $S' = \big\{n \in \mathbb{Z} : \gamma_n a \gamma_n^{-1} \in K\big\}$; since $S \subseteq S'$, it suffices to show that $\mathcal{D}(S') = 0$. (This would not be a problem if γ_n were a polynomial sequence, but it is not.)

Consider the mapping $\eta(n,t) = t^{-1}g(n)ag(n)t$ from $\mathbb{Z}^m \times G^o$ to L; this is a polynomial mapping. Let χ be a homomorphism $L \longrightarrow \mathbb{R}$ such that $K = \{\chi = 0\}$. Let $\theta = \chi\circ\eta$; then θ is a polynomial, and it is shown above that $\theta \not\equiv 0$. Since K has codimension 1 in L, it contains $[L,L]$, and so, is normal in L; hence, for any $s \in L$ we have $\theta(n,ts) = \chi(s^{-1}t^{-1}g(n)ag(n)^{-1}ts) = \chi(t^{-1}g(n)ag(n)^{-1}t) = \theta(n,t)$ for all $t \in G^o$, $n \in \mathbb{Z}$. Thus, θ is defined on $\mathbb{Z} \times (G^o/L)$: there exists a polynomial θ' on $\mathbb{Z} \times (G^o/L)$ such that $\theta(n,t) = \theta'(n,\tau(t))$, $t \in G^o$, $n \in \mathbb{Z}$. Let P be the restriction of θ' to $\mathbb{Z} \times Q'$. Now, $n \in S'$ iff

$\gamma_n a \gamma_n^{-1} = t_n^{-1} g(n) a g(n)^{-1} t_n \in K$, iff $\theta(n, t_n) = 0$, iff $P(n, \tau(t_n)) = 0$.

Write P in coordinates on Q', $P(n,u) = \sum_{\alpha \in A} q_\alpha(n) u^\alpha$, $n \in \mathbb{Z}$, $u \in Q'$, where A is a set of multiindices and for each $\alpha \in A$, $q_\alpha(n)$ is a polynomial in n. We want to show that the set of zeroes of the polynomials $P_n(u) = P(n,u)$ in Q' "converges", as $n \to \infty$, to a set of zero measure. Let $d = \max\{\deg q_\alpha,\ \alpha \in A\}$. Then for any $\alpha \in A$, a finite limit $b_\alpha = \lim_{n\to\infty} n^{-d} q_\alpha(n)$ exists, and is nonzero for some α. Thus, as $n \to \infty$, the polynomials $n^{-d} P_n(u)$ converge uniformly on Q' to the nonzero polynomial $p(u) = \sum_{\alpha \in A} b_\alpha u^\alpha$. The set $N = \{u \in Q' : p(u) = 0\}$ has zero measure. Given $\varepsilon > 0$, find $\delta > 0$ such that the set $N_\delta = \{u \in Q' : |p(u)| < \delta\}$ has measure $< \varepsilon$. Let n_0 be such that $|P(n,u) - p(u)| < \delta$ on Q' for $|n| > n_0$; then for $|n| > n_0$ the set $D_n = \{u \in Q' : P(n,u) = 0\}$ is contained in N_δ. The sequence $u_n = \tau(t_n)$, $n \in \mathbb{Z}$, is well distributed in Q' and the boundary of N_δ is a null-set, so $\mathcal{D}\{n \in \mathbb{Z} : u_n \in N_\delta\} = \mu_{Q'}(N_\delta) < \varepsilon$. Now,

$$S' = \{n \in \mathbb{Z} : P(n, u_n) = 0\} \subseteq \{n \in \mathbb{Z} : u_n \in D_n\} \subseteq \{-n_0, \dots, n_0\} \cup \{n \in \mathbb{Z} : u_n \in N_\delta\},$$

thus $\mathcal{D}(S') < \varepsilon$. Hence, $\mathcal{D}(S') = 0$. ∎

Corollary 2.2. *Let X be connected, let Y be a connected subnilmanifold of X, let g be a polynomials sequence in G, let $g(\mathbb{Z})Y$ be dense in X, and let $f \in C(X)$. There exists a factor-nilmanifold $\widehat{X}$ of X, a point $\hat{x} \in \widehat{X}$, and a function $\hat{f} \in C(\widehat{X})$ such that $\int_{g(n)Y} f \, d\mu_{g(n)Y} - \hat{f}(g(n)\hat{x}) \longrightarrow 0$ in uniform density.*

Proof. We may assume that $g(0) = \mathbf{1}_G$, that G is generated by G^o and the elements of g, and that $Y \ni \mathbf{1}_X$. Let Z be the normal closure of Y in X, then $\int_{g(n)Y} f \, d\mu_{g(n)Y} - \int_{g(n)Z} f \, d\mu_{g(n)Z} \longrightarrow 0$ in uniform density. Let $\widehat{X} = X/Z$, $\hat{x} = \{Z\} \in \widehat{X}$, and $\hat{f} = E(f|\widehat{X}) \in C(\widehat{X})$; then $\int_{g(n)Y} f \, d\mu_{g(n)Y} - \int_{g(n)Z} f \, d\mu_{g(n)Z} \longrightarrow 0$ in uniform density, and $\int_{g(n)Z} f \, d\mu_{g(n)Z} = \hat{f}(g(n)\hat{x})$ for all n. ∎

We now involve nilsequences into our consideration. Recall that a basic d-step nilsequence is a sequence of the form $\psi(n) = h(R^n w)$, where w is a point of a d-step nilmanifold M, R is a translation on M, and $h \in C(M)$. We find it worthy to expand this notion. Given a polynomial sequence $g(n) = a_1^{p_1(n)} \cdots a_r^{p_r(n)}$ in a nilpotent group with $\deg p_i \leq s$ for all i, we will say that g has *naive degree* $\leq s$. (The term "degree" had already been reserved for another parameter of a polynomial sequence.) Let us call a sequence of the form $\psi(n) = h(g(n)w)$, where w is a point of a d-step nilmanifold $M = J/\Lambda$, g is a polynomial sequence of naive degree $\leq s$ in J, and $h \in C(M)$, *a basic polynomial d-step nilsequence of degree $\leq s$*. Actually, any basic polynomial nilsequence is a basic nilsequence, as the following proposition says; the reason why we introduce this notion is that we do not want to loose the valuable information about the way a nilsequence was produced.

Proposition 2.3. (See [L1], Proposition 3.14) *Any basic polynomial d-step nilsequence of degree $\leq s$ is a ds-step basic nilsequence.*

Clearly, basic polynomial d-step nilsequences of degree $\leq s$ form an algebra; we will also need the following fact:

Lemma 2.4. *Let $\psi_0, \ldots, \psi_{m-1}$ be basic polynomial d-step nilsequences of degree $\leq s$. Then the sequence $\big(\ldots, \psi_0(0), \ldots, \psi_{m-1}(0), \psi_0(1), \ldots, \psi_{m-1}(1), \psi_0(2), \ldots, \psi_{m-1}(2), \ldots\big)$ is also a basic polynomial d-step nilsequence of degree $\leq s$.*

Proof. For each $i = 0, \ldots, m-1$, let $M_i = J_i/\Lambda_i$ be the d-step nilmanifold, g_i be the polynomial sequence in J_i, $w_i \in M_i$ be the point, and $h_i \in C(M_i)$ be the function such that $\psi_i(n) = h(g_i(n)w_i)$, $n \in \mathbb{Z}$. If, for some i, J_i is not connected, it is a factor-group of a free d-step nilpotent group with continuous and discrete generators, which , in its turn, is a subgroup of a free d-step nilpotent group with only continuous generators (see [L1]); thus after replacing, if needed, M_i by a larger nilmanifold and extending h_i to a continuous function on this nilmanifold we may assume that every J_i is connected. In this case for any element $b \in J_i$ and any $r \in \mathbb{N}$ a r-th root $b^{1/r}$ exists in J_i, and thus the polynomial sequence $b^{p(n)}$ in J_i makes sense even if a polynomial p has non-integer rational coefficients. Thus, for each i, we may construct a polynomial sequence g'_i in J_i, of the same naive degree as g_i, such that $g'_i(mn+i) = g_i(n)$ for all $n \in \mathbb{Z}$. Put $M = \mathbb{Z}_m \times \prod_{i=0}^{m} M_i$, $g = (1, g'_0, \ldots, g'_{m-1})$, $w = (0, w_0, w_1, \ldots, w_{m-1}) \in M$, and $h(i, v_0, \ldots, v_{m-1}) = h_i(v_i)$, $(i, v_0, \ldots, v_{m-1}) \in M$. Then M is a d-step nilmanifold, $h \in C(M)$, and the basic polynomial nilsequence $\psi(n) = h(g(n)w) = h_i(g'_i(n)w_i) = h_i(g_i(k)w_i) = \psi_i(k)$ whenever $n = km+i$, $i = 0, 1, \ldots, m-1$. ∎

We now get:

Theorem 2.5. *Let $X = G/\Gamma$ be a d-step nilmanifold, let Y be a subnilmanifold of X, let g be a polynomial sequence in G of naive degree $\leq s$, let $f \in C(X)$, and let $\varphi(n) = \int_{g(n)Y} f \, d\mu_{g(n)Y}$, $n \in \mathbb{Z}$. There exists a basic polynomial d-step nilsequence ψ of degree $\leq s$ such that $\varphi(n) - \psi(n) \longrightarrow 0$ in uniform density.*

Proof. If both Y and $\overline{g(\mathbb{Z})Y}$ are connected (in which case $\overline{g(\mathbb{Z})Y}$ is a nilmanifold), the assertion follows from Corollary 2.2.

Now assume that Y is connected but $\overline{g(\mathbb{Z})Y}$ is not. Then, by Theorem B in [L1], there exists $m \in \mathbb{N}$ such that $\overline{g(m\mathbb{Z}+j)Y}$ is connected for every $i = 0, \ldots, m-1$. Thus, for every $i = 0, \ldots, m-1$, there exists a basic polynomial d-step nilsequence ψ_i of degree $\leq s$ such that $\varphi(mn+i) - \psi_i(n) \longrightarrow 0$ in uniform density, and the assertion follows from Lemma 2.4.

Finally, if Y is disconnected and $Y_1, \ldots, Y_l$ are the connected components of Y, then $\int_{g(n)Y} f \, d\mu_{g(n)Y} = \sum_{i=1}^{l} \int_{g(n)Y_i} f \, d\mu_{g(n)Y_i}$, $n \in \mathbb{Z}$, and the result holds since it holds for $Y_1, \ldots, Y_l$. ∎

3. Multiple polynomial correlation sequences and nilsequences

Now let $(W, \mathcal{B}, \mu)$ be a probability measure space and let T be an ergodic invertible measure preserving transformation of W. Let $p_1, \ldots, p_k$ be polynomials taking on integer values on the integers. Let $A_1, \ldots, A_k \in \mathcal{B}$ and let $\varphi(n) = \mu\big(T^{p_1(n)}A_1 \cap \ldots \cap T^{p_k(n)}A_k\big)$, $n \in \mathbb{Z}$; or, more generally, let $h_1, \ldots, h_k \in L^\infty(W)$ and $\varphi(n) = \int_W T^{p_1(n)}h_1 \cdot \ldots \cdot T^{p_k(n)}h_k d\mu$, $n \in \mathbb{Z}$. Using results from [HK2] it can be shown (see the argument in [BHK], Corollary 4.5) that, given $\varepsilon > 0$, there exist a d-step nilsystem (X, a), $X = G/\Gamma$, $a \in G$, and functions $f_1, \ldots, f_k \in L^\infty(X)$ such that, for $\phi(n) = \int_X a^{p_1(n)}f_1 \cdot \ldots \cdot a^{p_k(n)}f_k d\mu_X$, $\mathcal{D}\big(\{n \in \mathbb{Z} :$

$|\phi(n)-\varphi(n)|<\varepsilon\}\big)=0$; after replacing f_i by L^1-close continuous functions, we may assume that $f_1,\ldots,f_k\in C(X)$. Moreover, there is a universal integer d that works for all systems $(W,\mathcal{B},\mu,T)$, functions h_i, and ε, and depends only on the polynomials p_i; the minimal integer c for which $d=c+1$ has this property is called *the complexity* of the system $\{p_1,\ldots,p_k\}$ (see [L6]). Applying Theorem 2.5 to the nilmanifold $X^k=G^k/\Gamma^k$, the diagonal subnilmanifold $Y=\big\{(x,\ldots,x),\ x\in X\big\}\subseteq X^k$, the polynomial sequence $g(n)=(1_G,a^{p_1(n)},\ldots,a^{p_k(n)})$, $n\in\mathbb{Z}$, in G^k and the function $f(x_0,x_1,\ldots,x_k)=f_1(x_1)\cdot\ldots\cdot f_k(x_k)\in C(X^k)$, we establish the existence of a basic polynomial d-step nilsequence ψ of degree $\leq s=\max_i(\deg p_i)$ such that $\phi(n)-\psi(n)\longrightarrow 0$ in uniform density. Summarizing, we get that $\varphi(n)=\phi(n)+\delta(n)=\psi(n)+\lambda(n)+\delta(n)$, where $\psi(n)$ is a basic polynomial d-step nilsequence of degree $\leq s$, $\lambda(n)\longrightarrow 0$ in uniform density, and $|\delta|<\varepsilon$.

We will say that a numerical sequence ψ is *a polynomial d-step nilsequence of degree $\leq s$* if it is a uniform limit of basic polynomial d-step nilsequences of degree $\leq s$. (It follows from Proposition 2.3 that any polynomial d-step nilsequence of degree $\leq s$ is a ds-step nilsequence.)

We obtain:

Theorem 3.1. *Let $(W,\mathcal{B},\mu,T)$ be an ergodic invertible measure preserving system with $\mu(W)<\infty$, let $h_1,\ldots,h_k\in L^\infty(W)$, let $p_1,\ldots,p_k$ be polynomials taking on integer values on the integers, and let $\varphi(n)=\int_W T^{p_1(n)}h_1\cdot\ldots\cdot T^{p_k(n)}h_k d\mu$, $n\in\mathbb{Z}$. Let the complexity of $\{p_1,\ldots,p_k\}$ be c and $s=\max_i(\deg p_i)$; then there exists a polynomial $(c+1)$-step nilsequence ψ of degree $\leq s$ such that $\varphi(n)-\psi(n)\longrightarrow 0$ in uniform density.*

Proof. We copy the proof of Theorem 1.9 in [BHK]. For each $l\in\mathbb{N}$, let ψ_l be a basic polynomial d-step nilsequence of degree $\leq s$, λ_l be a sequence that tends to 0 in uniform density, and δ_l be a sequence with $|\delta_l|<1/l$, such that $\varphi=\psi_l+\lambda_l+\delta_l$. Then for any l,r, $|\psi_l-\psi_r|\leq\frac{1}{l}+\frac{1}{r}+|\lambda_l-\lambda_r|$, thus $|\psi_l(n)-\psi_r(n)|\leq 2\big(\frac{1}{l}+\frac{1}{r}\big)$ for all $n\in\mathbb{Z}$ but a set of zero uniform density. Nilsystems are distal systems, each point of a nilsystem is uniformly recurrent (which means that it returns to any its neighborhood regularly, see [F] and [L1]), thus any nilsequence visits any interval in $\mathbb{R}$ for $n\in\mathbb{Z}$ from a set of positive uniform density, – or never. Hence, the (polynomial, and just ordinary) nilsequence $\psi_l-\psi_r$ satisfies $|\psi_l(n)-\psi_r(n)|\leq 2\big(\frac{1}{l}+\frac{1}{r}\big)$ for all $n\in\mathbb{Z}$. Hence, the sequence $(\psi_l)_{l=1}^\infty$ of basic polynomial $(c+1)$-step nilsequences of degree $\leq s$ is Cauchy in $l^\infty(\mathbb{Z})$, and has a limit ψ that is a polynomial $(c+1)$-step nilsequence of degree $\leq s$. The sequence $\varphi-\psi$ is the uniform limit of the sequences λ_l, and thus tends to zero in uniform density. ■

Remark. We believe that Theorem 3.1 remains true without the assumption that T is ergodic, but do not see how to prove this. The problem is to show that "an integral of nilsequences is a nilsequence plus a negligible sequence", that is, given a finite measure space Ω and a measurable function $\Psi\colon\Omega\times\mathbb{Z}\longrightarrow\mathbb{C}$ such that for each $\omega\in\Omega$, $\psi(n)=\Psi(\omega,n)$ is a nilsequence, the sequence $\psi(n)=\int_\Omega\Psi(\omega,n)\,d\omega$ is a sum of a nilsequence and a sequence that tends to 0 in uniform density.

4. The multiparameter case

We now switch to the multiparameter case, that is, to the situation where p_i are polynomials of $m \geq 1$ integer variables. We say that a mapping $g: \mathbb{Z}^m \longrightarrow G$ is *an (m-parameter) polynomial sequence in G* if $g(n) = a_1^{p_1(n)} \dots a_r^{p_r(n)}$, where $a_1, \dots, a_r \in G$ and $p_1, \dots, p_r$ are polynomials $\mathbb{Z}^m \longrightarrow \mathbb{Z}$. It is shown in [L2] that, if g is an m-parameter polynomial sequence in G and Y is a connected subnilmanifold of X, then the closure of the sequence $g(n)Y$, $X' = \overline{\bigcup_{n \in \mathbb{Z}^m} g(n)Y}$, is a disjoint finite union of subnilmanifolds of X, and, if X' is a single subnilmanifold, the sequence $g(n)Y$ is well distributed in X'. (That is, for every $f \in C(X')$ and any Følner sequence (Φ_N) in $\mathbb{Z}^m$, $\lim_{N \to \infty} \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} \int_{g(n)Y} f \, d\mu_{g(n)Y} = \int_{X'} f \, d\mu_{X'}$.)

For a subset $S \subseteq \mathbb{Z}^m$, we define *the density* $d(S)$ of S by $d(S) = \lim_{N \to \infty} \frac{|S \cap [-N,N]^m|}{(2N)^m}$, if it exists, and say that a sequence of points $(\omega_n)_{n \in \mathbb{Z}^m}$ of a topological space Ω converges to $\omega \in \Omega$ *in density* if for every neighborhood U of ω, $d\big(\{n \in \mathbb{Z}^m : \omega_n \notin U\}\big) = 0$.

For the case of multiparameter sequences we get a result similar to Proposition 2.1, but weaker since the "ordinary" density instead of the uniform density $\mathcal{D}$ appears in it:

Proposition 4.1. *Let $X = G/\Gamma$ be a connected nilmanifold and let $Y = \pi(H)$ be a connected subnilmanifold of X, where H is a connected closed subgroup of G. Let $g: \mathbb{Z}^m \longrightarrow G$ be a polynomial sequence with $g(0) = \mathbf{1}_G$ such that $g(\mathbb{Z}^m)Y$ is dense in X, and assume that G is generated by G^o and the elements of g. Let Z be the normal closure of Y in X; then $g(n)Y - g(n)Z \longrightarrow 0$ in density.*

Proof. The beginning of the proof is the same as for Proposition 2.1, but we will repeat it. Let L be the identity component of $\pi^{-1}(Z)$. Choose Malcev coordinates in G^o compatible with L, and let Q be the corresponding fundamental cube in G^o. Q is compact, and is as well compact with respect to the uniform norm when elements of G are interpreted as transformations of X. Represent $g(n) = t_n \gamma_n$ so that $\gamma_n \in \Gamma$ and $t_n \in Q$, $n \in \mathbb{Z}^m$. Since Z is normal, $\gamma_n Z = Z$ for all n, so that $g(n)Z = t_n \gamma_n Z = t_n Z$, $n \in \mathbb{Z}^m$. We have $g(n)Y = t_n \gamma_n Y$, $n \in \mathbb{Z}$, and since Q is compact, we only have to show that $\gamma_n Y \longrightarrow Z$ in density. Let Q' be the fundamental cube of X/Z and let $\tau: Q \longrightarrow Q'$ be the natural projection. Since the sequence $(g(n)Z)$ is well distributed in X, the sequence $(\tau(t_n))$ is well distributed in Q'.

Let V be a subnilmanifold of Z, connected, of codimension 1 in Z, and with $\mathbf{1}_X \in V$; based on Proposition 1.1, we only need to show that the set $\{n \in \mathbb{Z}^m : \gamma_n Y \subseteq V\}$ has zero density. Let K be the identity component of $\pi^{-1}(V)$; we have $\gamma_n H \gamma_n^{-1} \subseteq L$ for all $n \in \mathbb{Z}^m$, and have to prove that the set $S = \{n \in \mathbb{Z}^m : \gamma_n H \gamma_n^{-1} \subseteq K\}$ has zero density.

Since K is a proper subgroup of L and L is the normal closure of H in G there exists $b \in G$ such that $bHb^{-1} \nsubseteq K$. By assumption, G is generated by G^o and g. The group G^o is generated by Q, thus $tHt^{-1} \nsubseteq K$ for some $t \in Q$ or $g(n)Hg(n)^{-1} \nsubseteq K$ for some $n \in \mathbb{Z}^m$. So, there exists $a \in H$ such that $tat^{-1} \notin K$ for some $t \in Q$ or $g(n)ag(n)^{-1} \notin K$ for some $n \in \mathbb{Z}^m$. Let $S' = \{n \in \mathbb{Z}^m : \gamma_n a \gamma_n^{-1} \in K\}$; since $S \subseteq S'$, it suffices to show that $d(S') = 0$.

Consider the mapping $\eta(n,t) = t^{-1} g(n) a g(n)^{-1} t$ from $\mathbb{Z}^m \times G^o$ to L; this is a polynomial mapping. Let χ be a homomorphism $L \longrightarrow \mathbb{R}$ such that $K = \{\chi = 0\}$. Let $\theta = \chi \circ \eta$; then θ is a polynomial, and it is shown above that $\theta \not\equiv 0$. Since K is normal in L, for any $s \in L$ we have $\theta(n,ts) = \chi(s^{-1} t^{-1} g(n) a g(n)^{-1} ts) = \chi(t^{-1} g(n) a g(n)^{-1} t) = \theta(n,t)$ for all $t \in G^o$, $n \in \mathbb{Z}^m$. Thus, θ is defined on $\mathbb{Z}^m \times (G^o/L)$: there exists a polynomial θ' on

$\mathbb{Z}^m \times (G^o/L)$ such that $\theta(n,t) = \theta'(n,\tau(t))$, $t \in G^o$, $n \in \mathbb{Z}^m$. Let P be the restriction of θ' to $\mathbb{Z}^m \times Q'$. Now, $n \in S'$ iff $\gamma_n a \gamma_n^{-1} = t_n^{-1} g(n) a g(n)^{-1} t_n \in K$, iff $\theta(n,t_n) = 0$, iff $P(n,\tau(t_n)) = 0$.

Extend P to a polynomial on $\mathbb{R}^m \times Q'$. Write P in coordinates: $P(w,u) = \sum_{\alpha\in A} q_\alpha(w)u^\alpha$, where A is a set of multiindices and for each $\alpha \in A$, q_α is a polynomial on $\mathbb{R}^m$. Let $d = \max\bigl\{\deg q_\alpha,\ \alpha \in A\bigr\}$. For each $\alpha \in A$, let q_α^* be the homogeneous part of q_α of degree d. Let Σ be the sphere $\{\xi \in \mathbb{R}^m : |\xi| = 1\}$ and let $\Xi = \bigl\{\xi \in \Sigma : q_\alpha^*(\xi) \neq 0 \text{ for some } \alpha \in A\bigr\}$. For every $\xi \in \Sigma$ and $\alpha \in A$, $\lim_{s\to\infty} s^{-d} q_\alpha(s\xi) = q_\alpha^*(\xi)$, thus the polynomials $P_s(\xi,u) = s^{-d}P(s\xi,u)$ converge as $s \to \infty$ to the polynomial $p_\xi(u) = \sum_{\alpha\in A} q_\alpha^*(\xi)u^\alpha$ uniformly on $\Sigma \times Q'$. (Example: for $P\bigl((w_1,w_2),(u_1,u_2)\bigr) = (w_1^2+w_2)u_1^2 + w_2u_1u_2 + 2w_1w_2u_2$ we have $p_\xi(u_1,u_2) = w_1^2u_1^2 + 2w_1w_2u_2$, $\xi = (w_1,w_2) \in \Sigma$, and $\Xi = \{\xi \in \Sigma : p_\xi \neq 0\} = \{(w_1,w_2) \in \Sigma : w_1 \neq 0\}$.)

Fix $\varepsilon > 0$. For $\xi \in \Xi$, let $N_\xi = \bigl\{u \in Q' : p_\xi(u) = 0\bigr\}$ and let $\delta_\xi > 0$ be such that the set $N_{\xi,\delta_\xi} = \bigl\{u \in Q' : |p_\xi(u)| < \delta_\xi\bigr\}$ has measure $< \varepsilon$. Let $U_\xi \subset \Xi$ be an open neighborhood of ξ such that $|p_\zeta(u) - p_\xi(u)| < \delta_\xi/2$ for all $\zeta \in U_\xi$ and $u \in Q'$. Let $s_\xi > 0$ be such that $\bigl|s^{-d}P(s\zeta,u)(u) - p_\zeta(u)\bigr| < \delta_\xi/2$ for all $s > s_\xi$, $\zeta \in U_\xi$, and $u \in Q'$. Then for any $s > s_\xi$ and $\zeta \in U_\xi$, $\{u \in Q' : P(s\zeta,u) = 0\} \subseteq N_{\xi,\delta_\xi}$.

Since the sequence $u_n = \tau(t_n)$, $n \in \mathbb{Z}^m$, is well distributed in Q', for every $\xi \in \Xi$ there exists $M_\xi \in \mathbb{N}$ such that for any $M > M_\xi$ and any $v \in \mathbb{R}^m$, $\frac{1}{M^m}\bigl|\bigl\{n \in v + [1,M]^m : u_n \in N_{\xi,\delta_\xi}\bigr\}\bigr| < 2\varepsilon$. If $v \in \mathbb{R}^m$ and $M \in \mathbb{N}$ are such that $|v| > s_\xi + \sqrt{m}M$ and $v + [1,M]^m \subset \mathbb{R}_+U_\xi$, then for any $w \in v + [1,M]^m$ we have $\{u \in Q' : P(w,u) = 0\} \subseteq N_{\xi,\delta_\xi}$. Thus, for such v and M, $\frac{1}{M^m}\bigl|\bigl\{n \in v + [1,M]^m : P(n,u_n) = 0\bigr\}\bigr| < 2\varepsilon$, and hence, $\frac{1}{M^m}\bigl|S' \cap \bigl(v + [1,M]^m\bigr)\bigr| < 2\varepsilon$.

$E = \Sigma \setminus \Xi$ is a proper algebraic subvariety of Σ, therefore there exists a compact set $D \subset \Xi$ such that $d(\mathbb{R}_+D \cap \mathbb{Z}^m) > 1 - \varepsilon$. (Indeed, E can be represented as a finite union of smooth submanifolds of Σ of dimension $\leq m-2$, thus it can be covered by a finite union $\mathcal{E}$ of open balls with $\sigma(\mathcal{E}) < \varepsilon\sigma(\Sigma)$, where σ is the standard $(m-1)$-dimensional volume on Σ. For such a set $\mathcal{E}$ we have $d(\mathbb{R}_+\mathcal{E} \cap \mathbb{Z}^m) = \sigma(\mathcal{E})/\sigma(\Sigma) < \varepsilon$, and for $D = \Sigma \setminus \mathcal{E}$ we have $d(\mathbb{R}_+D \cap \mathbb{Z}^m) > 1 - \varepsilon$.) Let $\xi_1,\ldots,\xi_l$ be such that $\bigcup_{j=1}^l U_{\xi_j} \supseteq D$ and let $s = \max_{1\leq j\leq l} s_{\xi_j}$, $M = \max_{1\leq j\leq l} M_{\xi_j}$. Let $r > s + \sqrt{m}M$ be such that for any cube $C = v + [1,M]^m \subset \mathbb{R}_+D$ with $|v| > r$ we have $C \subset \mathbb{R}_+U_{\xi_j}$ for some j. Then for any such cube C we have $\frac{1}{|C|}|S' \cap C| < 2\varepsilon$. Thus, $d(S') < 3\varepsilon$. Hence, $d(S') = 0$. ∎

Remark. The proof of Proposition 4.1 gives more information about the set $S = \bigl\{n \in \mathbb{Z}^m : \bigl|\int_{g(n)Y} f - \int_{g(n)Z} f\bigr| > \varepsilon\bigr\}$ than just the fact that S has zero density. Actually, the *uniform* density of S is zero, – if we ignore a small set $\mathcal{E}$ of "bad" directions in $\mathbb{R}^m$; indeed, S has uniform density 0 in $\mathbb{R}_+(\Sigma \setminus \mathcal{E}) \cap \mathbb{Z}^m$, whereas $\sigma(\mathcal{E}) < \varepsilon\sigma(\Sigma)$.

We say that a mapping $\psi\colon \mathbb{Z}^m \longrightarrow \mathbb{C}$ is a *basic polynomial d-step m-parameter nilsequence of degree $\leq s$* if there exist a d-step nilmanifold $M = J/\Lambda$, a polynomial mapping $g\colon \mathbb{Z}^m \longrightarrow J$ of naive degree $\leq s$, a function $h \in C(M)$, and a point $w \in M$ such that $\psi(n) = h(g(n)w)$, $n \in \mathbb{Z}^m$, and we will say that an m-parameter numerical sequence is a *polynomial d-step nilsequence of degree $\leq s$* if it is a uniform limit of basic polynomial d-step m-parameter nilsequences of degree $\leq s$. The definitions and facts related to one-parameter polynomial sequences and nilsequences are translated almost literally to the multiparameter case; one only has to use results from [L2] and [L3] instead of the corresponding results from [L1] and [HK2]. (In particular, any (basic) polynomial m-parameter nilsequence is a (basic) m-parameter nilsequence; see the proof of Theorem B* in [L2].) In the same way as we got Theorems 2.5 and 3.1, we now obtain:

Theorem 4.2. *Let $X = G/\Gamma$ be a d-step nilmanifold, let Y be a subnilmanifold of X, let $g\colon \mathbb{Z}^m \longrightarrow G$ be a polynomials sequence of naive degree $\leq s$, let $f \in C(X)$, let $\varphi(n) = \int_{g(n)Y} f\,d\mu_{g(n)Y}$, $n \in \mathbb{Z}^m$. There exists a basic polynomial d-step m-parameter nilsequence ψ of degree $\leq s$ such that $\varphi(n) - \psi(n) \longrightarrow 0$ in density.*

Theorem 4.3. *Let $(W, \mathcal{B}, \mu, T)$ be an ergodic invertible measure preserving system with $\mu(W) < \infty$, let $h_1, \dots, h_k \in L^\infty(W)$, let $p_1, \dots, p_k$ be polynomials $\mathbb{Z}^m \longrightarrow \mathbb{Z}$, and let $\varphi(n) = \int_W T^{p_1(n)} h_1 \cdot \ldots \cdot T^{p_k(n)} h_k d\mu$, $n \in \mathbb{Z}^m$. Let the complexity of $\{p_1, \dots, p_k\}$ be c and let $s = \max_i(\deg p_i)$; then there exists a $(c+1)$-step m-parameter polynomial nilsequence ψ of degree $\leq s$ such that $\varphi(n) - \psi(n) \longrightarrow 0$ in density.*

5. Appendix

We will show here how Proposition 1.1 can be derived from Green-Tao's result in [GT].

We first need to introduce some terminology from [GT]. Let G be a connected nilpotent Lie group with a discrete cocompact subgroup Γ, and let $X = G/\Gamma$.

A filtration $G_\bullet$ on G is a finite decreasing sequence of subgroups $G = G_1 \supseteq G_2 \supseteq \ldots \supseteq G_d \supseteq G_{d+1} = \{\mathbf{1}_G\}$ with the property that $[G_i, G_j] \subseteq G_{i+j}$ for all i, j.

For a sequence $g\colon \mathbb{Z} \longrightarrow G$, "the derivative" ∂g is defined by $(\partial g)(n) = g(n)^{-1} g(n+1)$, $n \in \mathbb{Z}$. Given a filtration $G_\bullet = (G_1 \supseteq G_2 \supseteq \ldots \supseteq G_d)$ on G, $\mathrm{poly}(\mathbb{Z}, G_\bullet)$ denotes the group of polynomial sequences g in G with the property that, for each $i = 1, \dots, d$, $\partial^i g$ takes values in G_i,

Given a filtration $G_\bullet = (G_1 \supseteq G_2 \supseteq \ldots \supseteq G_d)$ on G, a Malcev basis $\mathcal{M}$ adapted to this filtration can be constructed (which means that for any i, $\mathcal{M} \cap G_i$ is a basis in G_i), and this basis naturally defines a locally Euclidean metric ρ on X.

A (horizontal) *character* on X is a mapping $\chi\colon X \longrightarrow \mathbb{R}/\mathbb{Z}$ induced by a character on the torus $T = [G, G]\backslash X$ (or equivalently, by a continuous homomorphism $G \longrightarrow \mathbb{R}/\mathbb{Z}$ trivial on Γ). A Malcev basis in G defines coordinates $(t_1, \dots, t_l)$ on T, and in these coordinates any character χ on X has the form $m_1 t_1 + \ldots + m_l t_l$, $(t_1, \dots, t_l) \in T$, with $m_1, \dots, m_l \in \mathbb{Z}$; the modulus $|\chi|$ of χ is defined by $|\chi| = |m_1| + \ldots + |m_l|$.

Given $\delta > 0$, a finite sequence $(x_1, \dots, x_N)$ is said to be *δ-equidistributed* in X if $\bigl|\frac{1}{N}\sum_{n=1}^N f(x_n) - \int_X f\,d\mu_X\bigr| < \delta\|f\|_{\mathrm{Lip}}$ for any Lipschitz function f on X, where $\|f\|_{\mathrm{Lip}} = \sup|f| + \sup_{x \neq y} \frac{\rho(f(x), f(y))}{\rho(x,y)}$.

The following theorem was obtained in [GT]:

Theorem 5.1. ([GT] Theorem 1.16) *Let $G_\bullet$ be a filtration on G and let $g \in \mathrm{poly}(\mathbb{Z}, G_\bullet)$. There exist constants C and c, which only depend on X, such that for any $\delta > 0$ small enough and any $N \in \mathbb{N}$, either the sequence $(g(n))_{n=1}^N$ is δ-equidistributed in X, or there is a nontrivial character χ on X with $|\chi| < C\delta^{-c}$ such that $\bigl|\chi(g(n)) - \chi(g(n-1)\bigr| < C\delta^{-c}/N$*

for all $n \in \{1,\dots,N\}$.

(In this theorem and below, the "either ... or ..." expression should be understood in the "inclusive" sense, that is, that both possibilities may also occur simultaneously.)
(We skipped some details; in particular, there is also a condition on the Malcev basis chosen in G and so, on the metric on X; this condition is satisfied if δ is small enough.)

We do not need much from this very strong "quantitative" theorem. Let X be connected but G not necessarily connected; represent X as $X = G^o/(\Gamma\cap G^o)$. Define the filtrations $G_\bullet = \{G_1 \supseteq G_2 \supseteq \dots\}$ on G and $G^o_\bullet = \{G^o_1 \supseteq G^o_2 \supseteq \dots\}$ on G^o by $G_1 = G$, $G_i = [G_{i-1}, G]$ for $i \geq 2$, and $G^o_i = G_i \cap G^o$, $i \in \mathbb{N}$. Let $f \in C(X)$, and let $\varepsilon > 0$. Choose a Lipschitz function h on X with $|h-f| < \varepsilon/3$. Choose $\delta > 0$ small enough to satisfy Theorem 5.1 and such that $\delta\|f\|_{\mathrm{Lip}} < \varepsilon/3$. Let $\chi_1,\dots,\chi_r$ be the nontrivial characters on X satisfying $|\chi_i| < C\delta^{-c}$. Then for any $g \in \mathrm{poly}(\mathbb{Z}, G^o_\bullet)$ and $N \in \mathbb{N}$, either there exists i such that $\bigl|\chi_i(g(n)\mathbf{1}_X) - \chi_i(g(n-1)\mathbf{1}_X)\bigr| < C\delta^{-c}/N$ for all $n = 1,\dots,N$, or $\bigl|\frac{1}{N}\sum_{n=1}^N h(g(n)\mathbf{1}_X) - \int_X h\,d\mu_X\bigr| < \delta\|f\|_{\mathrm{Lip}}$, and then $\bigl|\frac{1}{N}\sum_{n=1}^N f(g(n)\mathbf{1}_X) - \int_X f\,d\mu_X\bigr| < \varepsilon$. Sending N to infinity, we get that either $\chi_i(g(n)\mathbf{1}_X) \equiv 1$ for some i, or $\limsup_{N\to\infty}\bigl|\frac{1}{N}\sum_{n=1}^N f(g(n)\mathbf{1}_X) - \int_X f\,d\mu_X\bigr| \leq \varepsilon$.

Now let Y be a connected subnilmanifold of X with $\mathbf{1}_X \in Y$. Choose an element $a \in G$ such that the sequence $(a^n\mathbf{1}_X)_{n\in\mathbb{N}}$ is dense in Y. Choose $\gamma \in \Gamma$ such that $a\gamma^{-1} \in G^o$. (Such γ exists since $X = G/\Gamma$ is connected.) Put $g(n) = a^n\gamma^{-n}$, $n \in \mathbb{N}$; then $g(n)\mathbf{1}_X = a^n\mathbf{1}_X$ for all n, and since $g \in \mathrm{poly}(\mathbb{Z}, G_\bullet)$ and $g(n) \in G^o$ for all n, we have $g \in \mathrm{poly}(\mathbb{Z}, G^o_\bullet)$. Let $\chi_1,\dots,\chi_r$ be as above, let $V'_i = \{x \in X : \chi_i(x) = 0\}$, $i = 1,\dots,r$, and for each i, let V_i be the connected component of the nilmanifold V'_i that contains $\mathbf{1}_X$. We have that either $\chi_i(a^n\mathbf{1}_X) \equiv 1$ for some i, or $\limsup_{N\to\infty}\bigl|\frac{1}{N}\sum_{n=1}^N f(a^n\mathbf{1}_X) - \int_X f\,d\mu_X\bigr| \leq \varepsilon$. In the first case, $Y \subseteq V'_i$, and so, $Y \subseteq V_i$; in the second case, since $\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^N f(a^n\mathbf{1}_X) = \int_Y f\,d\mu_Y$ by [L1] (or by one more application of Theorem 5.1), we get that $\bigl|\int_Y f\,d\mu_Y - \int_X f\,d\mu_X\bigr| \leq \varepsilon$. We obtain

Corollary (Proposition 1.1). *Let X be a connected nilmanifold. For any $f \in C(X)$ and any $\varepsilon > 0$ there are subnilmanifolds $V_1,\dots,V_r$ of X, connected, of codimension 1, and containing $\mathbf{1}_X$, such that for any connected subnilmanifold Y of X with $\mathbf{1}_X \in Y$, either $Y \in V_i$ for some $i \in \{1,\dots,r\}$, or $\bigl|\int_Y f\,d\mu_Y - \int_X f\,d\mu_X\bigr| < \varepsilon$.*

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