Chapter 1
The Mathematics of Voting
The Paradoxes of Democracy

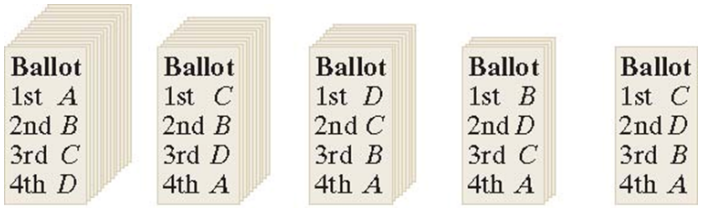
- Elections involving ballots ranking several candidates.
- Consider many examples of balloting in society.
- Understand several different methods for determining a winner.
- Understand several different fairness criteria.
- Arrows' impossibility theorem.

- **Preference ballots**
  A ballot in which the voters are asked to rank the candidates in order of preference.
- **Linear ballot**
  A ballot in which ties are not allowed.

Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot
1st *A*	1st *B*	1st *A*	1st *C*	1st *B*	1st *C*	1st *A*	1st *B*	1st *C*	1st *A*	1st *C*	1st *D*	1st *A*	1st *A*	1st *C*	1st *A*	1st *C*	1st *D*	
2nd *B*	2nd *D*	2nd *B*	2nd *B*	2nd *D*	2nd *B*	2nd *B*	2nd *D*	2nd *B*	2nd *B*	2nd *B*	2nd *C*	2nd *B*	2nd *B*	2nd *B*	2nd *B*	2nd *B*	2nd *C*	
3rd *C*	3rd *C*	3rd *C*	3rd *D*	3rd *C*	3rd *D*	3rd *C*	3rd *C*	3rd *D*	3rd *C*	3rd *D*	3rd *B*	3rd *C*	3rd *C*	3rd *D*	3rd *D*	3rd *B*		
4th *D*	4th *A*	4th *D*	4th *A*	4th *A*	4th *A*	4th *D*	4th *A*	4th *A*	4th *D*	4th *A*	4th *A*	4th *D*	4th *D*	4th *A*	4th *D*	4th *A*	4th *A*	

Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot
1st *C*	1st *A*	1st *D*	1st *D*	1st *C*	1st *C*	1st *D*	1st *A*	1st *D*	1st *C*	1st *A*	1st *D*	1st *B*	1st *A*	1st *C*	1st *A*	1st *A*	1st *D*	1st *A*	
2nd *B*	2nd *B*	2nd *C*	2nd *C*	2nd *B*	2nd *B*	2nd *C*	2nd *B*	2nd *C*	2nd *B*	2nd *B*	2nd *C*	2nd *D*	2nd *B*	2nd *D*	2nd *B*	2nd *B*	2nd *C*	2nd *B*	
3rd *D*	3rd *C*	3rd *B*	3rd *B*	3rd *D*	3rd *D*	3rd *B*	3rd *C*	3rd *B*	3rd *D*	3rd *C*	3rd *B*	3rd *C*	3rd *C*	3rd *B*	3rd *C*	3rd *C*	3rd *B*	3rd *C*	
4th *A*	4th *D*	4th *A*	4th *A*	4th *A*	4th *A*	4th *A*	4th *D*	4th *A*	4th *A*	4th *D*	4th *A*	4th *A*	4th *D*	4th *A*	4th *D*	4th *D*	4th *A*	4th *D*	

Ballot	Ballot	Ballot	Ballot	Ballot
1st *A*	1st *C*	1st *D*	1st *B*	1st *C*
2nd *B*	2nd *B*	2nd *C*	2nd *D*	2nd *D*
3rd *C*	3rd *D*	3rd *B*	3rd *C*	3rd *B*
4th *D*	4th *A*	4th *A*	4th *A*	4th *A*
14	10	8	4	1



Who wins the election?

Are most people happy about the result?

Assumptions about Ballots

- The first is that a voter's preference are **transitive**, i.e., that a voter who prefers candidate A over candidate B and prefers candidate B over candidate C automatically prefers candidate A over C (if B were not running).
- Secondly, that the relative preferences of a voter are not affected by the elimination of one or more of the candidates.

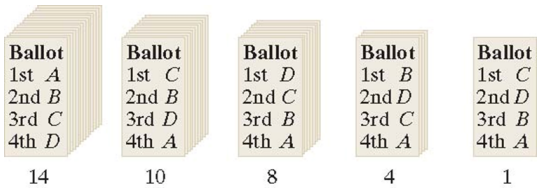
Methods of Choosing a Winner based on all the preference ballots

- **Plurality method**
  Election of 1st place votes
- **Plurality candidate**
  The Candidate with the most 1st place votes

- **Majority rule**
  The candidate with a more than half the votes should be the winner.
- **Majority candidate**
  The candidate with the majority of 1st place votes .

Criterion for selecting a Winner
(Things that "should" be true about an election)

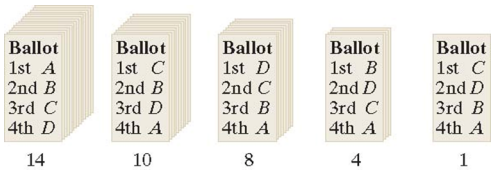
The Majority Criterion
If candidate X has a *majority* of the 1st place votes, then candidate X should be the *winner* of the election.



Criterion for selecting a Winner
(Things that "should" be true about an election)

The Condorcet Criterion

If candidate X is preferred by the voters over each of the other candidates in a head-to-head comparison, then candidate X should be the winner of the election.



http://en.wikipedia.org/wiki/Marquis_de_Condorcet

2008 Heisman Trophy Results

Player	School	First	Second	Third	Total points
Sam Bradford	Oklahoma	300	315	196	1,726
Colt McCoy	Texas	266	288	230	1,604
Tim Tebow	Florida	309	207	234	1,575
Graham Harrell	Texas Tech	13	44	86	213
Michael Crabtree	Texas Tech	3	27	53	116
Shonn Greene	Iowa	5	9	32	65
Patrick White	West Virginia	3	1	8	19
Nate Davis	Ball State	0	1	8	10
Rey Maualuga	USC	2	1	1	9
Javon Ringer	Michigan State	1	0	5	8

2000 Presidential Election

Candidate	Party	Votes	Percent	Electoral College Votes
George W. Bush	Republican	50,456,002	47.87%	271
Al Gore	Democratic	50,999,897	48.38%	266
Ralph Nader	Green	2,882,955	2.7%	0
Pat Buchanan	Reform	448,895	0.4%	0
Harry Browne	Libertarian	384,431	0.4%	0
Howard Phillips	Constitution	98,020	0.1%	0
John Hagelin	Natural Law	83,714	0.1%	0

Another examples where the plurality method fails to satisfy the Condorcet Criterion

Number of voters	49	48	3
1st choice	*R*	*H*	*F*
2nd choice	*H*	*S*	*H*
3rd choice	*F*	*O*	*S*
4th choice	*O*	*F*	*O*
5th choice	*S*	*R*	*R*

Nice Web Page to Compare Several Types of Voting Methods

Insincere Voting (or Strategic Voting)

Three voters decide not to "waste" their vote on F and swing the election over to H in doing so.

The Real Preferences			
Number of voters	49	48	3
1st choice	*R*	*H*	*F*
2nd choice	*H*	*S*	*H*
3rd choice	*F*	*O*	*S*
4th choice	*O*	*F*	*O*
5th choice	*S*	*R*	*R*

The Actual Votes			
Number of voters	49	48	3
1st choice	*R*	*H*	*H*
2nd choice	*H*	*S*	*F*
3rd choice	*F*	*O*	*S*
4th choice	*O*	*F*	*O*
5th choice	*S*	*R*	*R*

The "Election Spoiler" Controversy

The extremely close race between the Democratic and Republican presidential candidates, Al Gore and George W. Bush, helped to create some additional controversy around the 2000 campaign. Many Democrats claimed that Nader had no realistic chance of winning in the close election. They felt that those who supported Nader should have instead voted for Gore, and that a victory for Gore would have been preferable to a victory for George W. Bush [3]. Many prominent liberal politicians, activists, and celebrities campaigned for Nader [4]; others made the argument of prominent Democrats to voters in swing states, sometimes using the catch phrase "a vote for Nader is a vote for Bush." The Republican Leadership Council ran pro-Nader ads in a few states in a likely effort to split the "left" vote, a tactic from which the Nader campaign disassociated itself.[20] Nader and many of his supporters, including filmmaker Michael Moore, responded with their own catch phrase: "a vote for Gore is a vote for Bush."[21][22] The Nader campaign proclaimed that while Gore was perhaps marginally preferable to Bush, the differences between the two were not great enough to merit support of Gore.[citation needed]

Results of Bush, Gore, Nader Presidential Vote in 2000

In the **Borda Count Method** each place on a ballot is assigned points. In an election with *N* candidates we give 1 point for *last* place, 2 points for *second from last place*, and so on.

At the top of the ballot, a *first-place* vote is worth *N* points. The points are tallied for each candidate separately, and the candidate with the highest total is the winner.

We call such a candidate the *Borda winner*.

http://en.wikipedia.org/wiki/Jean-Charles_de_Borda

Number of voters	14	10	8	4	1
1st choice: 4 points	*A*	*C*	*D*	*B*	*C*
2nd choice: 3 points	*B*	*B*	*C*	*D*	*D*
3rd choice: 2 points	*C*	*D*	*B*	*C*	*B*
4th choice: 1 point	*D*	*A*	*A*	*A*	*A*

A gets 56 + 10 + 8 + 4 + 1 = 81 points
B gets 42 + 30 + 16 + 16 + 2 = 106 points
C gets 28 + 40 + 24 + 8 + 4 = 104 points
D gets 14 + 20 + 32 + 12 + 3 = 81 points

What if the point system is different?

Number of voters	14	10	8	4	1
1st choice: 4 points 10	*A*	*C*	*D*	*B*	*C*
2nd choice: 3 points 5	*B*	*B*	*C*	*D*	*D*
3rd choice: 2 points 2	*C*	*D*	*B*	*C*	*B*
4th choice: 1 point 1	*D*	*A*	*A*	*A*	*A*

A gets 140 + 10 + 8 + 4 + 1 = 163 points
B gets 70 + 50 + 16 + 40 + 2 = 178 points
C gets 28 + 100 + 40 + 8 + 10 = 188 points
D gets 14 + 20 + 80 + 20 + 5 = 139 points

http://en.wikipedia.org/wiki/Voting_system

Attachments

- Heisman Trophy Winner Selection
- Alternate Voting Methods for Presidential Primaries
- Results of Bush, Gore, Nader Presidential Vote in 2000
- Wikipedia Article on Voting Methods and Criteria
- Monotonicity Criterion
- Wikipedia Voting Systems Page
- wikipedia Arrows Impossibility Theorem
- Wikipedia Page on Kenneth Aarow
- Nice Web Page to Compare Several Types of Voting Methods