Factoring Quadratic Polynomials

In this handout we will factor quadratic polynomials of the form $ax^2 + bx + c$, where a , b , and c are integers and $a \neq 0$. Recall that factoring is actually the reverse process of multiplying, for example:

Example 1. (a) Multiply $3(3x+1)(2x-1)$.

Solution: $3(3x+1)(2x-1) = 3(6x^2 - 3x + 2x - 1) = 3(6x^2 - x - 1) = 18x^2 - 3x - 3$.

(b) Factor $18x^2 - 3x - 3$.

Solution: $18x^2 - 3x - 3 = 3(6x^2 - x - 1) = 3(6x^2 - 3x + 2x - 1) = 3(3x+1)(2x-1)$.

Example 2. (a) Multiply $(5x+1)(5x-1)$.

Solution: $(5x+1)(5x-1) = 25x^2 - 5x + 5x - 1 = 25x^2 - 1$.

(b) Factor $25x^2 - 1$.

Solution: $25x^2 - 1 = 25x^2 - 5x + 5x - 1 = (5x+1)(5x-1)$.

Notice that the last step in factoring was simply factoring by (re)grouping:

In Example **1**: $6x^2 - 3x + 2x - 1 = 3x(2x-1) + 1(2x-1) = (3x+1)(2x-1)$; and

in Example **2**: $25x^2 - 5x + 5x - 1 = 5x(5x-1) + 1(5x-1) = (5x+1)(5x-1)$.

It remains to explain how to *regroup* or *split* the middle term bx ; in **1.b** we replaced $-x$ by $-3x + 2x$, and in **2.b** we replaced $0x$ by $-5x + 5x$. How do we determine those coefficients in more complicated cases? Let us consider the general case.

Suppose that it is possible to write the given polynomial $ax^2 + bx + c$ as a product of the form $(mx+n)(px+q)$ where m , n , p , and q are integers to be determined. Then we have

$$(mx+n)(px+q) = mpx^2 + mqx + npx + nq = mpx^2 + (mq+np)x + nq = ax^2 + bx + c ,$$

where $mp = a$, $mq + np = b$, $nq = c$. Written in reverse, this becomes an exercise in factoring; the only problem is to replace b by the sum $mq + np$. We already know that the sum of mq and np must be equal to b ; their product is $(mq)(np) = (mp)(nq) = a \cdot c$. In other words, we are looking for two integers whose sum is b and whose product is ac . The factoring method for polynomials of the form $ax^2 + bx + c$ can now be stated as follows:

(1) Calculate ac . Find two integers whose product is ac and whose sum is b .

(2) Replace the middle term, bx , by the sum of two terms whose coefficients are integers found in (1).

(3) Factor by (re)grouping.

Notice that if $ac \neq 0$ there are only finitely many ways to represent ac as a product of two integers, so that the method is guaranteed to produce a factoring or to show that no factoring (over the integers) is possible. If $ac = 0$ just factor directly or use $b = 0 + b$.

Example 3. Factor $4x^2 + 3x - 7$.

Solution: Here $ac = 4(-7) = -28$, $b = 3$, so we are looking for two numbers whose product is -28 and whose sum is 3 . It is convenient to put the possible factors in a table:

OVER

First factor:	1	2	4	7	14	28
Second factor:	–28	–14	–7	–4	–2	–1
Their sum:	–27	–12	–3	3	12	27

The fourth column tells us to try $3x = 7x - 4x$. We have $4x^2 + 3x - 7 = 4x^2 + 7x - 4x - 7$ $= x(4x+7) - 1(4x+7) = (4x+7)(x-1)$.

Example 4. Factor $49x^2 - 1$.

Solution: Here $ac = 49(-1) = -49$, $b = 0$

First factor:	–1	–7 . . .
Second factor:	49	7 . . .
Their sum:	48	0 . . .

The second column is the good one (the sum is zero as required), so we *omit* the rest of the table. This means $0x = -7x + 7x$ should work. We have $49x^2 - 1 = 49x^2 - 7x + 7x - 1 = 7x(7x-1)$ $+ (7x-1) = (7x-1)(7x+1)$.

Example 5. Factor $4x^2 - 4x + 1$.

Solution: Here $ac = 4 \cdot 1 = 4$, $b = -4$. The (relevant part of the) table is now

First factor:	−1	−2 . . .
Second factor:	−4	−2 . . .
Their sum:	−5	−4 . . .

Replacing $-4x$ by $-2x - 2x$ we have $4x^2 - 4x + 1 = 4x^2 - 2x - 2x + 1 = 2x(2x-1) - (2x-1)$ $= (2x-1)(2x-1) = (2x-1)^2$.

Example 6. Factor $5x^2 + 7x + 3$.

Solution: Here $ac = 5 \cdot 3 = 15$, $b = 7$. The table is

First factor:	–1	–3	1	3
Second factor:	–15	–5	15	5
Their sum:	–16	–8	16	8

In this example the sum is never equal to 7 . Since we have tried *every* possible factoring of 15 , the conclusion is that $5x^2 + 7x + 3$ is *not* factorable over the integers.

Example 7. Factor $6x^2 + 5xy + y^2$.

Solution: Although this trinomial contains *two* variables, the same method still works. Here we have $ac = 6 \cdot 1 = 6$, $b = 5$ and the table is

First factor:	–1	–2	1	2
Second factor:	–6	–3	6	3
Their sum:	–7	–5	7	5

Using $5xy = 2xy + 3xy$ (from the last column) we have $6x^2 + 5xy + y^2 = 6x^2 + 2xy + 3xy + y^2$ $= 2x(3x+y) + y(3x+y) = (3x+y)(2x+y)$.

Your friendly Math Instructor Miroslav