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The geometry and topology of Coxeter groups, by Michael W. Davis, London Mathematical Society Monographs Series, 32, Princeton University Press, Princeton, NJ, 2008, xvi+584 pp., ISBN 978-0-691-13138-2

Topological methods in group theory, by Ross Geoghegan, Graduate Texts in Mathematics, 243, Springer, New York, 2008, xiv+473 pp., ISBN 978-0-387-74611-1

The exploration of mathematics that lies at the intersection of geometry, topology and group theory has been going on for least a century, with the work of Max Dehn in the 1910s being frequently put forward as an early example of the geometric approach to the study of groups (see Stillwell's compilation and translation of selected works [5]). In 1936 Hurewicz published his proof that the homotopy type of an aspherical CW complex—a cell complex whose universal cover is contractible—is determined by its fundamental group [6]. Starting from this result, one can ascribe topological invariants such as Euler numbers to groups that are fundamental groups of compact, aspherical cell complexes. In the last 30 years this exploration of geometric and topological aspects of group theory has lead to a number of illuminating examples and deep insights. Below I discuss two such results, which are directly related to the books under review.

1. Davis manifolds

In 1934, as part of an incorrect proof of the Poincaré Conjecture, Whitehead claimed that any open, contractible 3-manifold must be homeomorphic to $\mathbb{R}^3$. In the following year he constructed the *Whitehead contractible 3-manifold*, which is the complement of a rather complicated subset of the 3-sphere. While this manifold is a contractible open 3-manifold, it is not homeomorphic to $\mathbb{R}^3$.

Keeping in mind Whitehead's counterexample as a source for caution, it was still natural to wonder if Whitehead's original instinct contained a kernel of truth. Whitehead's contractible 3-manifold is not the universal cover of a closed aspherical manifold. On the other hand, if M is a closed Riemannian manifold with nonpositive sectional curvature, then the nonpositive curvature ensures that M is aspherical and, in this situation, the universal cover of M is homeomorphic to Euclidean space, $\widetilde{M} \approx \mathbb{R}^n$. The argument points out that the exponential map from the tangent space at a fixed point to M is a covering map. Thus it was quite natural to ask

Question 1.1. Is it always the case that the universal cover of a closed aspherical manifold is homeomorphic to Euclidean space?

In [4], Mike Davis showed that the answer is "no". The construction of his counterexample begins with a contractible 4-manifold, K, which admits a finite simplicial structure. Let L be the boundary of K. By Lefschetz duality, the boundary manifold L has the same homology as a 3-sphere, but there are examples where L has nontrivial fundamental group. From now on, we assume that $\pi_1(L)$ is nontrivial, and so we are working with what is often called a *Mazur manifold* [7].

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We can construct a Davis manifold using K and a naturally associated Coxeter group. Coxeter groups are groups generated by reflections. The symmetry groups of the Platonic solids are finite Coxeter groups, and elementary examples of infinite Coxeter groups are associated with some of the standard tilings of $\mathbb{R}^n$ and $\mathbb{H}^n$. For example, $\mathbb{R}^n$ can be tiled by n-cubes. The associated Coxeter group is generated by the $2n$ reflections whose reflecting hyperplanes pass through the codimension-1 faces of a given fixed cube. In this case the corresponding Coxeter group would be D_∞^n, the product of n copies of the infinite dihedral group.

To construct a Davis manifold from the Mazur manifold K, first define a Coxeter group W, whose generating reflections correspond to the vertices in $L = \partial K$. The defining relations state that two generating reflections commute if and only if their associated vertices bound an edge in L. Just as one could repeatedly reflect a single n-cube to generate a tiling of $\mathbb{R}^n$, one can use W to repeatedly "reflect" K across (the barycentric subdivision of) its boundary L to form a manifold $\mathcal{U}$. Davis showed that this space $\mathcal{U}$ has a number of nice properties:

(1) The space $\mathcal{U}$ is a contractible 4-manifold.
(2) $\mathcal{U}$ is the universal cover of a closed aspherical 4-manifold.
(3) $\mathcal{U}$ is not homeomorphic to $\mathbb{R}^4$.

To establish claim (3), one first notes that $\mathbb{R}^4$ is simply connected at infinity. A contractible, locally finite cell complex is *simply connected at infinity* if given a nested, exhaustive sequence of finite subcomplexes

$$K_0 \subset K_1 \subset K_2 \cdots \text{ where } \bigcup_{i=0}^{\infty} K_i = K\,,$$

the inverse limit of the induced sequence of fundamental groups

$$\pi_1(X \setminus K_0) \leftarrow \pi_1(X \setminus K_1) \leftarrow \pi_1(X \setminus K_2) \leftarrow \cdots$$

is trivial. Work of Stallings and Freedman shows that in dimensions ≥ 4, being simply connected at infinity is both necessary and sufficient for establishing that a contractible n-manifold is homeomorphic to $\mathbb{R}^n$. In contrast to $\mathbb{R}^4$, the manifold $\mathcal{U}$ is not simply connected at infinity. In fact, a judicious choice for the exhaustive sequence that is informed by an understanding of the structure of Coxeter groups combined with an iterated use of van Kampen's theorem shows that $\pi_1^\infty(\mathcal{U})$ is the free product of infinitely many copies of $\pi_1(L)$.

2. Bestvina–Brady groups

A group G is of *type* F_n if there is an aspherical cell complex whose fundamental group is G, that is, there is a $K(G, 1)$, where the n-skeleton is finite. This property generalizes two standard notions of finiteness for groups: a group is of type F_1 if and only if it is finitely generated; it is of type F_2 if and only if it is finitely presented. If H is a finite index subgroup of G, and G is of type F_n, then H must be of type F_n as well. Proof. Take the appropriate cover of a $K(G, 1)$ to create a $K(H, 1)$. However, if H is of infinite index in G, almost anything can happen. The following example is due to Stallings ($n = 3$) and Bieri ($n \geq 4$). Define G_n to be the direct product of n copies of the free group of rank two:

$$G_n = \mathbb{F}_2 \times \cdots \times \mathbb{F}_2\,.$$

Let $\phi : G_n \to \mathbb{Z}$ be the diagonal map, taking each of the standard generators of G_n to the generator $1 \in \mathbb{Z}$. Then the kernel of ϕ is of type F_{n-1}, but it is not of type F_n.

There is an alternative theory of finiteness properties that arises in a more algebraic approach to group cohomology. Fix a group G and look at resolutions of $\mathbb{Z}$ viewed as a trivial $\mathbb{Z}G$-module. Then G is of *type* FP_n if there is a projective resolution of $\mathbb{Z}$ as a trivial $\mathbb{Z}G$-module that is finitely generated through dimension n.

If G is of type F_n you can use the action of G on the universal cover of a $K(G, 1)$ to get a resolution of $\mathbb{Z}$ as a $\mathbb{Z}G$ module that is finitely generated through dimension n. So $F_n \Rightarrow FP_n$. Being of type FP_1 is equivalent to being finitely generated, so $F_1 \Leftrightarrow FP_1$. If G is finitely presented, then type FP_n implies type F_n for all n. Thus the natural question to ask was

Question 2.1. Does a group have type F_n if and only if it has type FP_n?

In [1] Mladen Bestvina and Noel Brady showed that the answer is "no" and their approach uses Artin groups. Let K be a finite flag complex, that is, a simplicial complex where every complete graph in the 1-skeleton corresponds to a simplex in K. Define the associated *right-angled Artin group* to be

$$A_K = \langle V(K) \mid vw = wv \text{ whenever } \{v, w\} \in E(K)\rangle .$$

In other words, A_K is generated by elements associated with the vertices of K, with relations saying two generators commute when their associated vertices share an edge. This should sound quite similar to the construction of Coxeter groups used in creating Davis manifolds.

The groups A_K do not have exotic finiteness properties. In fact, they admit finite classifying spaces that are subcomplexes of tori. However, they do have exotic subgroups of infinite index. Let BB_K denote the kernel of the diagonal map $\phi : A_K \to \mathbb{Z}$ that sends each standard generator of A_K to the generator $1 \in \mathbb{Z}$. Then the kernel BB_K is finitely generated if and only if the complex K is connected; BB_K is finitely presented if and only if K is simply connected. And in general, the presence of a finiteness property for the kernel BB_K is directly tied to the triviality of a homology or homotopy group of K. Bestvina and Brady's proof of this result uses a variation on classical Morse theory, and this technique has had a continued impact on the field.

To answer the question above, let K be a finite acyclic flag complex, with nontrivial fundamental group. Thus K has trivial homology, but $\pi_1(K)$ is not trivial. Then the kernel BB_K of the diagonal map $\phi : A_K \to \mathbb{Z}$ is not finitely presentable, yet there is a finite projective (in fact free) resolution of $\mathbb{Z}$ as a trivial $\mathbb{Z}G$-module. That is to say, BB_K is not of type F_2 but it is of type FP_n for all n.

3. The books

I hope that readers of the two stories above suspect that I have left out a number of important details. They are right, and moreover, not only have I skipped important details, I have skipped a number of *interesting* details. My only defense is to say that there are now two books that do a wonderful job of presenting both the group theory and the geometry/topology that underlies these and many other results.

Ross Geoghegan's *Topological methods in group theory* starts at an elementary level (discussing cell complexes, cellular homology, and fundamental groups) but it moves at a brisk pace. Section 1.2 defines CW complexes; section 3.1 introduces the fundamental group; and section 4.5 presents a nice, geometric argument for Hurewicz's Theorem: $\pi_n(X) \approx H_n(X)$ when X is simply connected and $(n-1)$-acyclic. All of that is accomplished in the first 123 pages.

While much of this foundational material will be known to potential readers, it should not be skipped. Not only is Geoghegan's exposition spare yet enlightening, he presents this material in a manner that serves as an appropriate foundation for the mathematics that follows. As one example, Geoghegan's presentation of the Hurewicz Theorem leads nicely into a proper-homotopy version of this result (Theorem 17.1.6), and essentially all that is necessary to prove the proper-homotopy version is to simply point back to the earlier argument. Another place where Geoghegan's style and insights are particularly notable is in his discussion of duality in Chapter 15. In this thirteen-page chapter he presents a geometric treatment of Poincaré Duality for manifolds and quickly builds from there a treatment of duality properties in the context of groups. While this material is presented quite efficiently, an attentive reader will gain a good deal of intuition and understanding by working through these ideas.

The choice of topics in this book is deeply influenced by shape theory, which Geoghegan points out explicitly in his introduction. This becomes most apparent starting in Chapter 10, where proper maps and proper homotopy theory is developed, and it continues through the remainder of the text. Because of this, this book is an invaluable resource for anyone wanting a deep understanding of topics related to the ends of groups. In addition to presenting properties like being simply connected at infinity, this book also develops material on filtered or relative ends of spaces, the fundamental group at infinity, and compactifiability at infinity. For much of this material I can think of no place in the literature where it is presented as well as it is here. In fact, there is a good deal of material in this book that does not appear anywhere else in the literature.

Mike Davis's *The geometry and topology of Coxeter groups* develops many themes in the study of geometric group theory in the context of Coxeter groups. Coxeter groups appear in many branches of mathematics, which is evidenced by the fact that there are over a dozen books on Coxeter groups. Many of these other texts focus on the theory of finite Coxeter groups, and/or they focus on connections between Coxeter groups and combinatorics. None of the other books on Coxeter groups provides the same insights into the geometry of infinite Coxeter groups that is available in Davis's book.

The first sixty pages of this book provide the foundation for the study of infinite Coxeter groups, which begins in earnest in Chapter 5. Here Davis describes how to construct spaces $\mathcal{U}$ upon which Coxeter groups act. The classic example of this construction yields what is commonly called the Davis Complex, which thankfully Mike Davis chooses not to rename in his book.

Davis presents the foundational results about the topology of the spaces $\mathcal{U}$ in Chapters 7–9. This includes acyclicity conditions and cohomology with compact supports, both of which are key to establishing topological properties at infinity for the associated Coxeter groups.

Infinite Coxeter groups are of great interest in geometric group theory because of the flexibility of their construction combined with the fact that their associated Davis complexes support nonpositively curved metric structures. The two stories told above suggest what I mean by flexibility, where one can use topological properties present in finite cell complexes to establish the existence of topological properties in certain groups. Another example that should be mentioned is the reflection group trick, described in Chapter 11 of Davis's book. This is a method for converting aspherical complexes into closed aspherical manifolds, which retract onto the original aspherical complexes. This technique can be used to establish that there are closed, aspherical manifolds whose fundamental groups are not residually finite, and there are closed, aspherical manifolds whose fundamental groups have unsolvable word problems. Within the realm of manifold theory, this approach can also be used to establish that in each dimension ≥ 13 there are closed, aspherical manifolds that are not homotopy equivalent to smooth manifolds.

Mike Davis's book concludes its discussion of the geometry and topology of Coxeter groups by surveying recent work on weighted L^2-cohomology of the spaces $\mathcal{U}$ and more generally, buildings. There are approximately 150 pages of appendices describing topics such as complexes of groups, the Novikov and Borel conjectures, and CAT(0) geometry.

The construction of Davis manifolds and the results of Bestvina and Brady can be found in both of these texts. Davis manifolds appear in section 16.6 in Geoghegan's book, and they are presented in section 10.5 in Davis's book. The examples of Bestvina–Brady show up in section 8.3 in Geoghegan's book and section 11.6 in Davis's book. While these two books do cover some of the same ground, they are a great complement to each other, not competing expositions.

The field of geometric group theory is too large for any single volume to attempt to present all of the material that is now viewed as foundational. Such is the nature of a field with rich connections to well-established branches of mathematics. As one example, neither of the books being reviewed here would serve as good resources for someone wanting to learn about hyperbolic groups. The topic is touched upon in both texts, but is not a focus of either. A reader who is interested in focusing on the cohomology of groups is well served by reading Ken Brown's text [3]; a reader wanting a rich understanding of CAT(0) geometry should work through Bridson and Haefliger's book [2]. The books of Davis and Geoghegan are excellent introductions to other, important aspects of the study of geometric and topological approaches to group theory. Davis's exposition gives a delightful treatment of infinite Coxeter groups that illustrates their continued utility to the field; Geoghegan's book provides a well-presented, concrete development of geometric group theory focused on a topological approach.

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