REFLECTION GROUPS AND CAT(0) COMPLEXES WITH EXOTIC LOCAL STRUCTURES

MICHAEL W. DAVIS *

Department of Mathematics
The Ohio State University
231 W. 18th Ave
Columbus, OH 43210, USA
E-mail: mdavis@math.ohio-state.edu

JOHN MEIER†

Department of Mathematics
Lafayette College
Easton, PA 18042, USA
E-mail: meierj@lafayette.edu

We show that, in contrast to the situation for the standard complex on which a right angled Coxeter group W acts, there are cocompact W-actions on CAT(0) complexes such that the local topology of the complex is distinctly different from the end topology of W.

1. Introduction

If $\widetilde{X}$ is a contractible n-manifold or homology n-manifold, then, since it satisfies Poincaré duality, its cohomology with compact supports, $H_c^*(\widetilde{X})$, is concentrated in dimension n and is isomorphic to $\mathbb{Z}$ in that dimension. It follows that the homology at infinity of $\widetilde{X}$ is concentrated in dimension $n-1$ and is isomorphic to $\mathbb{Z}$ in that dimension (see [5] for definitions and references for homology at infinity). More particularly, recall that a simplicial complex is a homology n-manifold if and only if the link of each vertex is a "generalized homology $(n-1)$-sphere" (i.e., a homology $(n-1)$-manifold with the same homology as S^{n-1}). So, the above argument shows that if the link of each vertex in an aspherical simplicial complex X is

*Davis was partially supported by NSF grant DMS-0104026.
†Meier thanks The Ohio State University for hosting him while on sabbatical.

a generalized homology $(n-1)$-sphere, then the homology at infinity of its universal cover $\widetilde{X}$ is concentrated in dimension $n-1$ and hence, $\widetilde{X}$ is $(n-2)$-acyclic at infinity. This shows that hypotheses concerning the local topology of an aspherical space can have implications for the end topology of its universal cover.

There are other local-to-asymptotic results for nonpositively curved complexes. If L is a simplicial complex, we let $\mathcal{S}(L)$ denote the set of all closed simplices of L, including the empty simplex. Let $\widetilde{X}$ be a (locally finite) CAT(0) cubical complex, and for each vertex $x \in \widetilde{X}$, let L_x denote its link. If for each vertex x and for each closed simplex $\sigma \in \mathcal{S}(L_x)$, $L_x - \sigma$ is m-connected (resp., m-acyclic), then $\widetilde{X}$ is m-connected (resp., m-acyclic) at infinity (see [2] and the references cited there). We call the complexes $L_x - \sigma$ the *punctured links* of $\widetilde{X}$.

In recent work [5], we have shown there is a close connection between local topology and end topology of the standard complexes on which Coxeter groups act. The *nerve* of a Coxeter system (W, S) is the simplicial complex L with one vertex for each element of the generating set S and one simplex for each subset of S which generates a finite subgroup of W. As explained in [3, 4 or 5], associated to (W, S) there is natural cell complex, here denoted $|W|$, such that $|W|$ is a model for $\underline{E}W$ and such that the link of each of its vertices is isomorphic to L (see [7] for the definition of $\underline{E}G$). This implies, for example, that if L is a triangulation of an $(n-1)$-sphere, then $|W|$ is a contractible n-manifold.

If S is finite (which we shall henceforth always assume), then L is a finite complex and the quotient space $|W|/W$ is compact. It is proved in [6] and [8] that the natural piecewise Euclidean metric on $|W|$ is CAT(0). In [5] the authors established a direct correspondence between the topological properties of L and the asymptotic topological properties of $|W|$ (or of any locally finite building with associated Coxeter system (W, S)). For example:

Theorem 1.1. (See 4.1, 4.2 and 4.3 in [5]) Let W be a finitely generated Coxeter group with associated nerve L. Then

(1) W is simply connected at infinity if and only if $L - \sigma$ is simply connected for each $\sigma \in \mathcal{S}(L)$.

(2) W is m-acyclic at infinity if and only if $L - \sigma$ is m-acyclic for each $\sigma \in \mathcal{S}(L)$.

(3) W is m-connected at infinity if and only if $L - \sigma$ is m-connected for each $\sigma \in \mathcal{S}(L)$

Thus, not only do the connectivity properties of the punctured links $L - \sigma$ determine the connectivity at infinity of W (i.e., of $|W|$), the converse is also true. This leads to speculation that, in the general context of nonpositive curvature, similar asymptotic-to-local results might hold. For example, one might speculate that if X is a nonpositively curved, finite Poincaré complex with, say, extendable geodesics, then the links of vertices in X are forced to be generalized homology spheres (and hence, X is a homology manifold). Similarly, it could be speculated that if X is a nonpositively curved cubical complex (with extendable geodesics) and if $\widetilde{X}$ is m-connected (resp., m-acyclic) at infinity, then the punctured links of vertices must be m-connected (resp., m-acyclic). The purpose of this note is to give some examples which set such speculations to rest: there are no general results of this nature. (For further examples illuminating the difficulty of getting asymptotic-to-local results, see [2].)

2. The Construction

The construction of our examples is essentially the same as the construction of [1]. We will show that by making minor modifications in the construction of $|W|$ one gets a model for $\underline{E}W$, $\mathfrak{W}$, with a CAT(0) cubical structure so that the connectivity properties at infinity do not descend to connectivity properties of links. In fact, in all of our examples $|W|$ will be a manifold while $\mathfrak{W}$ will not even be a homology manifold. We show that $|W|$ and $\mathfrak{W}$ are equivariantly proper homotopy equivalent, so if Γ is any torsion-free subgroup of finite index in W, then $\mathfrak{W}/\Gamma$ is a nonpositively curved Poincaré complex that is not a homology manifold. For simplicity, we restrict our construction to right angled Coxeter groups, which we briefly review below.

Right Angled Coxeter Groups. A simplicial complex L is a *flag complex* if any complete graph in the 1-skeleton of L is actually the 1-skeleton of a simplex in L. The barycentric subdivision of any cell complex is a flag complex; hence, the condition of being a flag complex imposes no restriction on the topology of L — it can be any polyhedron. The importance of flag complexes in CAT(0) geometry stems from the result of Gromov that the natural piecewise Euclidean metric on a cubical complex is nonpositively curved (= locally CAT(0)) if and only if the link of each vertex is a flag complex [6, p. 122].

Suppose L is a finite flag complex. For each integer $k \geq 0$, let $L^{(k)}$ denote the set of k-simplices in L and as before let $\mathcal{S}(L)$ denote the poset of all simplices in L (including the empty simplex).

Associated to L there is a group W defined as follows. For each $i \in L^{(0)}$ introduce a symbol s_i and set $S = \{s_i\}_{i \in L^{(0)}}$. W is defined by the presentation:

$$W = \langle S \mid s_i^2 = 1, s_i s_j = s_j s_i \text{ when } \{i, j\} \in L^{(1)} \rangle \ .$$

(W, S) is called a *right angled Coxeter system*. Its nerve is L.

The Cubical Complex |W|. For each $\sigma \in \mathcal{S}(L)$, let W_σ denote the subgroup generated by the elements of S which correspond to vertices of σ. Then $W_\sigma \simeq (\mathbb{Z}_2)^{\dim(\sigma)+1}$. Set

$$W\mathcal{S}(L) = \coprod_{\sigma \in \mathcal{S}(L)} W/W_\sigma \ .$$

$W\mathcal{S}(L)$ is called the *poset of spherical cosets* (the partial order is given by inclusion). The complex $|W|$ is defined to be the geometric realization of $W\mathcal{S}(L)$. There is an obvious left W action on $|W|$. The cubical structure on $|W|$ is defined as follows. There is one vertex of $|W|$ for each element of W $(= W/W_\emptyset)$. For each spherical coset wW_σ, we then fill in a Euclidean cube of dimension $\dim(\sigma) + 1$ with vertices corresponding to the elements of wW_σ. (Note that the elements of W_σ can naturally be identified with the vertices of a cube of dimension $\dim(\sigma) + 1$.) The poset of cubes in $|W|$ is $W\mathcal{S}(L)$ and the link of each vertex is L.

The geometric realization of the poset $\mathcal{S}(L)$ is denoted K. The inclusion $\mathcal{S}(L) \hookrightarrow W\mathcal{S}(L)$ defined by $\sigma \mapsto W_\sigma$ induces an inclusion $K \hookrightarrow |W_L|$ and we identify K with its image in $|W|$. Similarly, the orbit projection $W\mathcal{S}(L) \twoheadrightarrow \mathcal{S}(L)$ defined by $wW_\sigma \mapsto \sigma$ induces a projection $|W| \twoheadrightarrow K$ which factors through a homeomorphism $|W|/\mathrm{W} \to K$. Thus, K is a fundamental domain for the W-action on $|W|$ and the orbit projection $|W| \to K$ restricts to the identity on K.

The geometric realization of $\mathcal{S}(L)_{>\emptyset}$ can be identified with the barycentric subdivision L' of L. Thus, K is the cone on L' (the empty set provides the cone point). For each $i \in L^{(0)}$, let K_i denote the geometric realization of $\mathcal{S}(L)_{\geq \{i\}}$, i.e., K_i is the closed star of i in L'. We call K_i the *mirror* of K of type i.

Here is another description of $|W|$. For each point $x \in K$ let $\sigma(x)$ be the simplex spanned by $\{i \in L^{(0)} \mid x \in K_i\}$. Then

$$|W| = (W \times K)/\sim$$

where the equivalence relation $\sim$ is defined by $(w, x) \sim (w', x')$ if and only if $x = x'$ and $w^{-1}w' \in W_{\sigma(x)}$.

For a flag complex L that can be decomposed as $L = L_1 \cup L_2$, we will construct a different CAT(0) cubical complex $\mathfrak{W}$ on which the associated right angled Coxeter group W acts as a cocompact reflection group. The complexes $\mathfrak{W}$ and $|W|$ will have the same pro-homotopy type. However, the topology of the links of vertices in $\mathfrak{W}$ can differ dramatically from that of the links of $|W|$.

The construction of $\mathfrak{W}$. Suppose that a finite flag complex L can be decomposed as the union of two full subcomplexes: $L = L_1 \cup L_2$. Set $L_0 = L_1 \cap L_2$. Since L_0, L_1 and L_2 are full subcomplexes of L each of them is a flag complex.

For any simplicial complex L and a point z not in L, let $C_z L$ be the simplicial complex defined by taking the cone on L with cone point z.

Let x_1, x_2 and v be points that are not in L and define new simplicial complexes:

$$\begin{aligned}
\widehat{L}_1 &= L_1 \cup C_v L_0 \\
\widehat{L}_2 &= L_2 \cup C_v L_0 \\
\widehat{K}_1 &= C_{x_1} \widehat{L}_1 \\
\widehat{K}_2 &= C_{x_2} \widehat{L}_2
\end{aligned}$$

Let $\widehat{K}$ denote the result of gluing $\widehat{K}_1$ to $\widehat{K}_2$ along $C_v L_0$.

In Figure 1 we show a simple example that highlights the difference between K and $\widehat{K}$. The original simplicial complex L is a circuit of length 8, and K is the cone on this octagon. We let $L_0 \simeq S^0$ be two antipodal vertices (indicated by dots in the figure on the right), and let L_1 and L_2 be the two simplicial arcs in L which are separated by L_0.

Returning to the case where L is an arbitrary finite flag complex, we note that L is a subcomplex of $\widehat{K}$ (we think of it as the boundary of $\widehat{K}$). Also, $\widehat{K}$ is contractible (it is the union of two contractible pieces glued along a contractible subcomplex). The space $\mathfrak{W}$ is defined by hollowing out each copy of K in $|W|$ and replacing it with a copy of $\widehat{K}$. Since K and $\widehat{K}$ are both contractible, $|W|$ and $\mathfrak{W}$ are proper homotopy equivalent; hence, $\mathfrak{W}$ is also contractible.

Here is a more precise description of $\mathfrak{W}$. Recalling that for $i \in L^{(0)}$, K_i is the closed star of i in the barycentric subdivision of L (which is a subspace of $\widehat{K}$), we see that K_i is identified with a subspace of $\widehat{K}$. So, define $\widehat{K}_i$ to be K_i. We then proceed as before. For each point $x \in \widehat{K}$, let

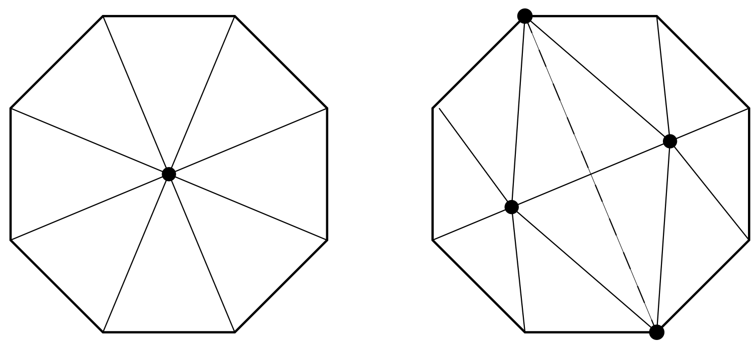

Figure 1. The difference between K (left) and $\widehat{K}$ (right)

$\sigma(x)$ be the simplex spanned by $\{i \in L^{(0)} \mid x \in \widehat{K}_i\}$ and let

$$\mathfrak{W} = (W \times \widehat{K})/\sim$$

where the equivalence relation $\sim$ is defined as before.

It is not difficult to define the cubical structure on $\mathfrak{W}$ and to see that it is CAT(0). The vertex set is $Wx_1 \coprod Wx_2$. For $\alpha = 1, 2$ and for a spherical coset $wW_\sigma \in W\mathcal{S}(L_\alpha)$, the vertices $wW_\sigma x_\alpha$ span a cube of dimension $\dim(\sigma)+1$. Also, for each spherical coset $wW_\sigma \in W\mathcal{S}(L_0)$, we have a cube spanned by $wW_\sigma x_1 \coprod wW_\sigma x_2$. Its dimension is $\dim(\sigma)+2$. (In particular, corresponding to the case where σ is empty, we have an edge from wx_1 to wx_2.) For $\alpha = 1, 2$, the link of x_α in $\mathfrak{W}$ is $\widehat{L}_\alpha$. Since $\widehat{L}_\alpha$ is a flag complex, the cubical structure is CAT(0). We also note that the punctured link $\widehat{L}_\alpha - v$ is homotopy equivalent to L_α.

Remark. In [1] the above construction was used only in the case where L is a homology sphere and $L_0 \subset L$ is a homology sphere embedded in codimension one.

In the following examples we will always choose L to be a triangulation of an n-sphere and L_0 to be a codimension one submanifold triangulated as a full subcomplex. L_1 and L_2 will then be n-manifolds with boundary. (However, L_0, L_1 and L_2 need not be connected.) Since $L \simeq S^n$, $|W|$ is a contractible $(n+1)$-manifold; however, $\mathfrak{W}$ need not be a manifold.

Example 2.1. Suppose that L is a 2-sphere, that L_1 is an annulus (a collared neighborhood of the equator) and that L_2 is the disjoint union of the two 2-disks (neighborhoods of the north and south poles). Then $L_0 = L_1 \cap L_2$ is the disjoint union of two circles, and $\widehat{L}_1$ is an annulus with its boundary coned off. So, $\widehat{L}_1$ is homeomorphic to a 2-sphere with two points identified. In particular, the link $\widehat{L}_1$ is not simply connected ($\pi_1(\widehat{L}_1) \simeq \mathbb{Z}$). Similarly, $\widehat{L}_2$ is the wedge of two 2-spheres. The punctured links L_1 and L_2 are also not simply connected. Nevertheless, the theorem quoted at the beginning implies that $|W|$ (and hence $\mathfrak{W}$) is simply connected at infinity, since the nerve L is a 2-sphere.

Example 2.2. Suppose L is an n-sphere and L_0 is a codimension one submanifold separating L into two pieces L_1 and L_2. Then $\mathfrak{W}$ is $(n-1)$-connected at infinity by the theorem quoted at the beginning. On the other hand, $\widetilde{H}_*(\widehat{L}_1) \simeq H_*(L_1, L_0)$ can be nonzero in any dimension $< n$. Similarly, the homology of the punctured link L_1 is fairly arbitrary.

A true optimist might believe that these examples occur because there are two W-orbits of vertices, and that if W acts transitively on the 0-skeleton, then such examples disappear. The following modified version of our construction shows that this speculation is also false.

A construction with only one vertex orbit. Suppose L_0 is a subcomplex of L_1 and that t is a simplicial involution on L_0. Let L denote the result of gluing together two copies of L_1 along L_0 via the map t. Call the two copies L_1 and L_2. Then t extends to an involution on L (also denoted t) that interchanges L_1 and L_2. Let W be the right angled Coxeter group associated to L. Let G denote the semidirect product, $G = W \rtimes \mathbb{Z}_2$. Here $\mathbb{Z}_2$ acts on the vertex set of L (the generating set of W) via t. The W-action on $\mathfrak{W}$ extends to a G-action. Now there is only one G-orbit of vertices.

Example 2.3. Suppose that L_1 is the solid torus, $L_1 = D^2 \times S^1$ and that L_0 is its boundary, $L_0 = S^1 \times S^1$. Let $t : S^1 \times S^1 \to S^1 \times S^1$ be the involution which switches the factors. Then $L = S^3$ and $\mathfrak{W}$ is 2-connected at infinity. The link of each vertex is isomorphic to $\widehat{L}_1$. However, $H_2(\widehat{L}_1) \simeq \mathbb{Z}$ and although $\pi_1(\widehat{L}_1) \simeq 0$, for the punctured link, L_1, we have $\pi_1(L_1) \simeq \mathbb{Z}$.

Remark. We note that there is a simple method of altering the local topology of $|W|$ so that the connectivity of the links does not coincide with the connectivity at infinity: Form $|W|'$ by attaching a copy of $[0,1]$ (or $[0,\infty)$) to each vertex of $|W|$. If one attaches unit intervals, then the resulting complex does not have extendable geodesics; if one attaches half lines, then the resulting complex is not cocompact. Further, while the complex $|W|'$ deformation retracts onto $|W|$, $|W|$ does not sit as a retract inside $\mathfrak{W}$.

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