Cohomology of Coxeter groups with group ring coefficients

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(joint work with Jan Dymara, Tadeusz Januszkiewicz, Boris Okun)

Suppose (W, S) is a Coxeter system. For $T \subset S$, W_T denotes the subgroup generated by T. T is *spherical* if W_T is finite. $\mathcal{S}$ denotes the set of spherical subsets of S.

Let X be a CW complex and $(X_s)_{s\in S}$ a family of subcomplexes. For each $x \in X$, put $S(x) := \{s \in S \mid x \in X_s\}$. Define $\mathcal{U}(W, X)$ $(= \mathcal{U})$ to be the quotient space $(W \times X)/\sim$, where $\sim$ is the equivalence relation defined by $(w, x) \sim (w', x')$ if and only if $x = x'$ and $wW_{S(x)} = w'W_{S(x)}$. W acts on $\mathcal{U}$ and X is a strict fundamental domain (i.e., $\mathcal{U}/W = X$). A W-action on a space is a *reflection group* if it is equivariantly homeomorphic to $\mathcal{U}(W, X)$ for some X. The action is proper and cocompact if and only if X is a finite complex and $S(x)$ is spherical for each $x \in X$. Henceforth, assume this.

For each $w \in W$, put

$$\mathrm{In}(w) := \{s \in S \mid l(ws) < l(w)\}$$
$$\mathrm{In}'(w) := \{s \in S \mid l(sw) < l(w)\},$$

where $l(\ )$ is word length. It is a basic fact that for any w, both $\mathrm{In}(w)$ and $\mathrm{In}'(w)$ $(= \mathrm{In}(w^{-1}))$ are spherical subsets of S. Let $A := \mathbf{Z}W$ be the group ring and $\{e_w\}_{w\in W}$ its standard basis. For each $T \in \mathcal{S}$, define elements a_T and h_T in A by

$$a_T := \sum_{w\in W_T} e_w \quad \text{and} \quad h_T := \sum_{w\in W_T} (-1)^{l(w)} e_w.$$

Let A^T denote the right ideal $a_T A$ and H^T the left ideal Ah_T. (If $T \notin \mathcal{S}$, set $A^T = H^T = 0$.) A_T is the set of finitely supported functions on W which are constant on each right coset in $W_T\backslash W$. Put

$$b'_w := a_{\mathrm{In}'(w)} e_w, \qquad b_w := e_w h_{\mathrm{In}(w)}.$$

Then $\{b'_w \mid \mathrm{In}'(w) \supset T\}$ is a basis for A^T and $\{b_w \mid \mathrm{In}(w) \supset T\}$ is a basis for H^T. So, if we define $\widehat{A}^T := \mathrm{Span}\{b'_w \mid \mathrm{In}(w) = T\}$ and $\widehat{H}^T := \mathrm{Span}\{b_w \mid \mathrm{In}(w) = T\}$, we have direct sum decompositions of abelian groups:

$$A^T = \bigoplus_{U\supset T} \widehat{A}^U \quad \text{and} \quad H^T = \bigoplus_{U\supset T} \widehat{H}^U.$$

Theorem.

$$H_*(\mathcal{U}) \cong \bigoplus_{T\in\mathcal{S}} H_*(X, X^T) \otimes \widehat{H}^T$$
$$H^*_c(\mathcal{U}) \cong \bigoplus_{T\in\mathcal{S}} H_*(X, X^{S-T}) \otimes \widehat{A}^T.$$

The first formula was originally proved in [1], the second in [2]. We give a different proof in [6] by using the identifications of these (co)homology groups with certain equivariant (co)homology groups: $H_*^W(\mathcal{U};\mathbf{Z}W) = H_*(\mathcal{U})$ and $H_W^*(\mathcal{U};\mathbf{Z}W) = H_c^*(\mathcal{U})$ and then using a direct sum decomposition of the coefficient system on X induced by $\mathbf{Z}W$. This point of view leads to a computation of the W-module structures on $H_*(\mathcal{U})$ and $H_c^*(\mathcal{U})$ in the following sense. We have a decreasing filtration of right W-modules $A = F_0 \supset \cdots \supset F_p$, where $F_p := \sum_{|T|\geq p} A^T$. This leads to a filtration of cohomology. (Similarly, there is a decreasing filtration of left W-modules for homology.) Put

$$A^{>T} := \sum_{U \supsetneq T} A^U \quad \text{and} \quad H^{>T} := \sum_{U \supsetneq T} H^U.$$

With this terminology, we can state the following result of [6].

Theorem. *In filtration degree p, the associated graded term in homology is the left W-module,*

$$\bigoplus_{|T|=p} H_*(X, X^T) \otimes H^T/H^{>T},$$

while in compactly supported cohomology it is the right W-module,

$$\bigoplus_{|T|=p} H_*(X, X^{S-T}) \otimes A^T/A^{>T}.$$

These formulas were suggested by our work in [5] on weighted L^2-cohomology of Coxeter groups (see [3] for an abstract).

If $\mathcal{U}$ is acyclic, then $H_c^*(\mathcal{U}) = H^*(W;\mathbf{Z}W)$. Moreover, there is a particularly nice choice of a contractible $\mathcal{U}$. We usually denote it Σ and its fundamental chamber K [2, 4, 6]. This leads to a formula for $H^*(W;\mathbf{Z}W)$ with each associated graded term a sum of terms of the form $H^*(K, K^{S-T}) \otimes A^T/A^{>T}$. A consequence is the following.

Corollary. *$H^*(W;\mathbf{Z}W)$ is always finitely generated as a W-module.*

Question. *Suppose a group Γ is virtually type FP. Is $H^*(\Gamma;\mathbf{Z}\Gamma)$ always a finitely generated Γ-module?*

References

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