Compactly supported cohomology of buildings

Michael Davis

(joint work with Jan Dymara, Tadeusz Januszkiewicz, Boris Okun)

Suppose (W, S) is a Coxeter system. A subset $T \subset S$ is *spherical* if it generates a finite subgroup of W. $\mathcal{S}$ denotes the poset of spherical subsets of S. Let Φ be a building in the sense of [6] (i.e., Φ is a set of chambers, equipped with a family of adjacency relations indexed by S and a W-valued distance function, $\Phi \times \Phi \to W$).

Let A denote the set of finitely supported $\mathbf{Z}$-valued functions on Φ (i.e., A is the free abelian group on Φ). For each $T \in \mathcal{S}$, A^T denotes the set of $f \in A$ which are constant on all residues of type T. If $U \supset T$, then $A^U \subset T$. We show that the quotient, $D^T := A^T / \sum_{s \in S-T} A^{T \cup \{s\}}$, is free abelian. Let $\hat{A}^T$ be a summand of A^T which projects isomorphically onto D^T.

Decomposition Theorem. *For each $T \in \mathcal{S}$,*

$$A^T = \bigoplus_{U \supset T} \hat{A}^U.$$

Suppose X is a CW complex and that $\{X_s\}_{s \in S}$ is a mirror structure over S on X (defined in [2, p.63]). For a cell c of X or point $x \in X$, put

$$S(c) := \{s \in S \mid c \subset X_s\}, \quad S(x) := \{s \in S \mid x \in X_s\}.$$

The X-*realization* of Φ is the quotient space $\mathcal{U}(\Phi, X) := (\Phi \times X)/\sim$, where $\sim$ is the equivalence relation defined by $(\phi, x) \sim (\phi', x')$ if and only if $x = x'$ and ϕ, ϕ' belong to the same $S(x)$-residue. (Here Φ has the discrete topology.) For each $T \subset S$, put

$$X_T := \bigcap_{s \in T} X_s \quad \text{and} \quad X^T := \bigcup_{s \in T} X_s.$$

We are primarily interested in the case where $X = K$, the geometric realization of the poset $\mathcal{S}$ and where K_s is the geometric realization of $\mathcal{S}_{\geq \{s\}}$ (cf. [2, Chap.7]).

For any subgroup B of A and $T \subset S$, put $B^T := B \cap A^T$. We have a "coefficient system" $\mathcal{I}(B)$ on X, giving a cochain complex

$$\mathcal{C}^i(X; \mathcal{I}(B)) := \bigoplus_{c \in X^{(i)}} B^{S(c)},$$

where $X^{(i)}$ denotes the set of i-cells in X. Let $\mathcal{H}^*(X; \mathcal{I}(B))$ denote the cohomology groups of this cochain complex. The Decomposition Theorem gives us a direct sum decomposition of coefficient systems

$$\mathcal{I}(A) = \bigoplus_{T \in \mathcal{S}} \mathcal{I}(\hat{A}^T),$$

leading to the following.

Theorem.

$$\mathcal{H}^*(X; \mathcal{I}(A)) = \bigoplus_{T \in \mathcal{S}} \mathcal{H}^*(X; \mathcal{I}(\hat{A}^T)) = \bigoplus_{T \in \mathcal{S}} H^*(X, X^{S-T}) \otimes \hat{A}^T.$$

If X is compact and if, for each cell $c \subset X$, $S(c)$ is spherical, then $\mathcal{H}^*(X; \mathcal{I}(A)) = H_c^*(\mathcal{U}(\Phi, X))$. This gives our main result, the following corollary.

Corollary. (cf. [3, 4]).

$$H_c^*(\mathcal{U}(\Phi, K)) = \bigoplus_{T \in \mathcal{S}} H^*(K, K^{S-T}) \otimes \hat{A}^T.$$

When Φ is an irreducible affine building, this corollary is the classical computation of Borel-Serre [1]. When $\Phi = W$ (the thin building) or when Φ is right-angled, proofs can be found in [3]. A version of the general result is claimed in [5]; however, there is a mistake in the proof.

Our proof of the Decomposition Theorem is modeled on a homological argument from [4] for a similar result. The key to the proof is a calculation for the standard realization of Φ where X is the simplex Δ of dimension $\mathrm{Card}(S) - 1$ with its codimension-one faces indexed by S. The Decomposition Theorem follows from the next result (and some similar statements).

Theorem. $\mathcal{H}^*(\Delta; \mathcal{I}(A))$ *is concentrated in the top degree* ($= \mathrm{Card}(S) - 1$) *and is a free abelian group.*

We also need versions of this which assert the concentration in the top degree of $\mathcal{H}^*(\sigma, \sigma^U; \mathcal{I}(A))$, where σ ranges over the spherical faces of Δ and U over the subsets of S.

References

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