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Concordance classes of regular U_n and Sp_n action on homotopy spheres

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Introduction

In this paper and a subsequent one [7], we shall apply the general theories of [4], [5], and [6] to some interesting and important cases. Let G_n be O_n, U_n or Sp_n, let ρ_n be the standard representation of G_n and let Σ be a smooth homotopy sphere. Consider a smooth action $G_n \times \Sigma \rightarrow \Sigma$ modeled on $k\rho_n$ with $n \geqq k$. This means that the orbit types and the normal representations of G_n on Σ occur among those of k times the standard representation of G_n. In other words, the orbits are Stiefel manifolds of the form G_n/G_{n-i} where $0 \leqq i \leqq k$. Following [12], [13], such actions will also be called 'regular G_n-actions'. There are two typical examples of these actions:

(A) The unit sphere of the representation $k\rho_n \oplus l\theta$ (where θ denotes the one dimensional trivial representation) is clearly such a G_n-sphere.

(B) Let $\Sigma^{2m+1}(p, q)$ be the Brieskorn variety (see [3], [9]) defined as the intersection of the unit sphere in $\mathbf{C}^{m+2}$ with the hypersurface $f^{-1}(\varepsilon)$ where

$$f(u, v, z_1, \cdots, z_m) = u^p + v^q + z_1^2 + \cdots + z_m^2 ,$$

and where ε is a sufficiently small positive number. For suitably chosen p and q, $\Sigma^{2m+1}(p, q)$ will be a homotopy sphere for $m > 1$. If $V^{2m+2}(p, q)$ is the intersection of $f^{-1}(\varepsilon)$ with the unit disc in $\mathbf{C}^{m+2}$, then $V^{2m+2}(p, q)$ is a parallelizable manifold with boundary equal to $\Sigma^{2m+1}(p, q)$. O_m acts linearly on $\mathbf{C}^{m+2}$ by operating on the last m coordinates and the invariant submanifold $V^{2m+2}(p, q)$ becomes an O_m-manifold modeled on $2\rho_m$. Write $m = dtn + l$, where $d = 1$, 2 or 4 as $G_n = O_n$, U_n or Sp_n, and consider the embedding $t\rho_n + l\theta\colon G_n \rightarrow O_m$. The restriction of the O_m-action to G_n gives $\Sigma^{2m+1}(p, q)$ the structure of a G_n-sphere modeled on $k\rho_n$ with $k = 2t$. As we shall see, these action on Brieskorn spheres can be distinguished from one another by the index or Kervaire invariant of $V^{2m+2}(p, q)$ which coincides with the index or Kervaire invariant of the fixed point set $V^{2l+2}(p, q)$. Thus, for any even

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k there are non-linear G_n-spheres modeled on $k\rho_n$.

It turns out, rather surprisingly, that for $G_n = U_n$ or Sp_n, these examples are essentially the only possibilities which can occur. More precisely, we shall show that if $G_n \times \Sigma \to \Sigma$ is modeled on $k\rho_n$ with $n \geqq k$ and with $G_n = U_n$ or Sp_n, then there is a regular G_n-action on $\Sigma \times [0, 1]$ such that the action on $\Sigma \times 0$ is the given action while on $\Sigma \times 1$ it is equivalent to either the linear action or one of the above actions on a Brieskorn sphere[3]—we shall say that the G_n-sphere $\Sigma \times 0$ is *concordant* to one of the typical examples. For $k = 2$, this result was essentially proved by Bredon in [2]. Two immediate consequences of our result are particularly worthwhile mentioning:

(A) If the regular G_n-sphere Σ modeled on $k\rho_n$ is fixed point free, then Σ is concordant to the linear action.

(B) If the regular G_n-sphere is modeled on $k\rho_n$ ($k \leqq n$) for k odd and if the dimension of the fixed point set is not equal to 3, then Σ is concordant to the linear action.

In particular, in either of the above cases, the underlying differentiable structure on Σ is the standard one. The case of $G_n = O_n$ is dealt with in the next paper [7].

Let us now give a rough sketch of our proof. We shall work entirely in the category of smooth G-manifolds and (equivariant) 'stratified maps' and in the analogous category of 'local G-orbit space' and stratified maps (see Sections II.4–II.6 in [5]). For now, it should suffice to mention that a G-manifold is stratified by the orbit types as is its orbit space and that we only wish to consider smooth maps which are 'stratified' in the sense that they preserve the strata and that they map the normal bundle of each stratum transversely.

For a G_n-manifold M which is modeled on $k\rho_n$, the strata can be indexed by the integers between 0 and k. Thus, M_i is the invariant submanifold consisting of the orbits of type G_n/G_{n-i}.

The first part of our program is carried out in [6] where it is shown that we have the following set-up. Let D denote the unit disc in the representation $k\rho_n + m\theta$ and let $S = \partial D$ be the unit sphere. Here $m\theta$ is the trivial m-dimensional representation, where $m = \dim \Sigma_0 + 1$ (Σ_0 is the fixed point set). If the homotopy sphere Σ admits a G_n-action modeled on $k\rho_n$, then Σ equivariantly bounds a parallelizable G_n manifold V, also modeled on $k\rho_n$, and there is a stratified map $F: (V, \Sigma) \to (D, S)$ which is a homotopy equi-

[3] We have some mild condition on the dimension of the fixed point set and the number k to avoid the usual low dimensional surgery difficulties. See Theorem 2 for the precise statment.

valence on the boundary. In this paper, we investigate the question of when we can choose such a V to be a disc.

Let A, B, K and L denote the orbit spaces of V, Σ, D and S, respectively. Then, F induces a stratified map $f\colon (A, B) \rightarrow (K, L)$. It is also shown in [6] how we can faithfully translate our problem to the orbit space level. For $G_n = U_n$ or Sp_n, the condition that F is a homotopy equivalence on the boundary is precisely that $\pi_1(B) = 0$ and the restriction of f to the i-stratum B_i induces an isomorphism in (integral) homology for each i. (In the case $G_n = O_n$, the failure of this to be true is one of the major difficulties [7].) Also, V is a disc if and only if $f \mid A_i$ induces an isomorphism in (integral) homology and $\pi_1(A) = 0$.

Therefore, we try to do 'surgery' on F rel B to get a new orbit space A' together with a map $f'\colon A' \rightarrow K'$ in such a way that for each i, $f' \mid A'_i$ will induce an isomorphism on homology. If we succeed, our new G_n-manifold V' (obtained by the pullback construction of [5]) will be contractible. From such a V', it is easy to produce a concordance of Σ to the linear action. It turns out that modulo the usual low dimensional difficulties *the only obstruction to doing this surgery is the index or Kervaire invariant of the fixed point set* $V_0 = A_0$. Furthermore, *if k is odd, the surgery obstruction must automatically be zero*. From this, we deduce the results.

By 'surgery', we essentially mean surgery on a stratified space (compare [4]). This type of surgery is a generalization of surgery on a manifold with boundary (which has two strata). The way in which surgery obstructions are computed can also be illustrated by considering this simple example. So suppose that we are given a normal map $\varphi\colon (M, \partial M) \rightarrow (N, \partial N)$ where N and ∂N are simply connected, and that we are trying to do surgery on φ to a homotopy equivalence of pairs. First, we try to do surgery on $\varphi \mid \partial M$. There is no obstruction (i.e., no index or Kervaire invariant), since $\varphi \mid \partial M$ is the boundary of a surgery problem, namely φ. Hence surgery is always possible. In this argument, we are "looking up one stratum." To continue, one tries to do surgery rel the boundary on $M \cup X$, where X is the trace of the surgery on ∂M. We might meet an obstruction. If so, one simply changes the cobordism X by adding the negative of this obstruction, and so surgery will again be possible. In this argument, we are "going down one stratum and changing the cobordism."

The problem of doing surgery on an orbit space is analogous. In our case, we must solve a sequence of (ordinary) surgery problems indexed by $\{0, 1, \cdots, k\}$. The definition of the surgery problem on the i-stratum depends on the choice of the solutions of problems on the lower strata. Thus, as in the case of a manifold with boundary, we can sometimes change the obstruction by going down one (or more) stratum and changing the cobordism. Furthermore, the possible surgery obstructions for the problem on the i-stratum are not arbitrary. In particular, part of the boundary of the $(i+1)$-stratum is a fibre bundle over the i-stratum with fibre CP^m (or QP^m). When m is even, by using a slight generalization of the product formula, we can sometimes look up one stratum and conclude that the original problem must have the vanishing surgery obstruction. Using one or the other argument one sees that there is no obstruction except possibly on the 0-stratum. In this case, for k even, we would like to go down one stratum and change the cobordism; however, there is no lower stratum, so we are left with an obstruction (which is realized by the Brieskorn examples).

There is a close analogy between this program and classical obstruction theory. "Looking up one stratum" to eliminate some possible surgery obstructions corresponds to observing that a primary obstruction must be annihilated by some cohomology operation if the map is to extend to the next higher skeleton (of the domain, viewed as a cell complex), while "going down one stratum" corresponds to observing that certain nonzero candidates for obstructions are indeterminacy tied to the next lower skeleton.

It should be pointed out that here we are doing surgery in a slightly different context from that proposed by Browder and Quinn [4]. Their treatment deals with a stratified map $F:(M, \partial M) \to (M', \partial M')$ which is required to be an *isovariant homotopy equivalence* on the boundary. (In other words, the homotopy inverse of $F|\partial M$ is required to be equivariant and strata preserving as are the homotopies to the identity.) In the general situation, this hypothesis is necessary to insure that $F|\partial M$ will induce a homotopy equivalence on each stratum of the orbit space of ∂M. However, it is more natural for us to assume that F is an isovariant map which only induces a homotopy (homology) equivalence on the boundary (but not necessarily an isovariant homotopy equivalence on the boundary). Even with this weaker hypothesis, for regular U_n, Sp_n manifolds, we can still conclude that $F|\partial M$ induces on each stratum an isomorphism in integral homology (and $Z_{(2)}$-homology for the corresponding O_n case). The reason for this is that we can use Smith theory, since for regular U_n, Sp_n actions the conjugacy classes of isotropy groups have distinct ranks [6], [10] (see Theorem B in § 1).

In Sections 1 and 2, we summarize the results of [5], [6] necessary for our argument. In Section 3, we state the main result and deduce some consequences from it. In 4, we prove the main theorem.

1. Preliminaries

In this section, we review some general definitions from [5] concerning the stratification of a G-manifold and the existence of pullbacks.

Suppose that a compact Lie group G acts smoothly on a manifold M (smooth $= C^\infty$). For $x \in M$, G_x denotes the isotropy subgroup at x and $G(x)$ is the orbit passing through x. The *slice representation* (at x) is the G_x-module

$$S_x = T_x(M)/T_x(G(x)) .$$

The famous (Differentiable) Slice Theorem asserts that $G(x)$ has an invariant tubular neighborhood of the form $G \times_{G_x} S_x$. Thus, the local structure of M is completely determined by the slice representation.

A slice representation can be decomposed as $S_x = F_x \oplus V_x$, where F_x is the subspace on which G_x acts trivially and $V_x = S_x/F_x$. V_x is called the *normal representation* at x. The (conjugate) equivalence class of the G_x-module V_x is called the *normal orbit type* of x. A *stratum* of M is the set of points of a given normal orbit type. It follows from the Slice Theorem, that a stratum is a smooth invariant submanifold. Notice that a fibre at x of the normal bundle of a stratum is V_x. A smooth equivariant map of G-manifolds $\varphi\colon M' \to M$ is *stratified* if $G_x = G_{\varphi(x)}$ and if the differential of φ induces an isomorphism $V_x \cong V_{\varphi(x)}$.

Let $\pi\colon M \to B$ be the projection of M onto its orbit space. There is a natural induced 'smooth' structure on B. Essentially, this is obtained by defining a function $\varphi\colon B \to \mathbf{R}$ to be *smooth* if $\varphi\circ\pi$ is smooth (see [1]). In view of the Slice Theorem, B is locally isomorphic to $(G \times_{G_x} S_x)/G \cong S_x/G_x$. If M_α is a stratum of M, then its image $B_\alpha = \pi(M_\alpha)$, also a smooth manifold, is called a *stratum* of B.

In what follows, a theorem of [14] plays an important technical role. It says that the orbit space of an H-module (e.g., S_x) is smoothly isomorphic to a certain semi-algebraic subset of euclidean space.

We may define a 'local G-orbit space' as a space equipped with a stratification and local charts to the orbit space of an H-module (H is a closed subgroup of G). Everything we shall say about orbit spaces is also true for local G-orbit spaces.

Once we are given the 'smooth' functions on B, we can define for any $b \in B$ the tangent space T_bB in the usual fashion. It is a finite dimensional vector space of constant dimension along each stratum. It follows that $T(B)\,|\,B_\alpha$ is a smooth vector bundle and that the ordinary tangent bundle of a stratum $T(B_\alpha)$ is a subbundle. So it makes sense to define the normal bundle of B_α in B by $N(B_\alpha) = (T(B)\,|\,B_\alpha)/T((B_\alpha)$. A map $f\colon B' \to B$ of (local) G-orbit spaces is *stratified* if it preserves the smooth structure and the stratification and if for each stratum the induced map $N(B'_\alpha) \to N(B_\alpha)$ is an isomorphism when restricted to each fibre. It is not difficult to see that a stratified map of G-manifolds induces a stratified map of the orbit spaces. (In [5], there is a further condition in the definition of a stratified map; however, it is unnecessary in this paper.) The following theorem was suggested by the proof of a special case in [2]. It is proved in Section III. 1 of [5].

THEOREM. *Suppose that M is a smooth G-manifold over B and that B' is another local G-orbit space. If $f\colon B' \to B$ is a stratified map, then the pullback*

$$f^*(M) = \{(x, y) \in M \times B' \,|\, \pi(x) = f(y)\}$$

is a smooth G-manifold over B'. Moreover, the natural map $f^*(M) \to M$ is stratified and

$$\begin{array}{ccc} f^*(M) & \longrightarrow & M \\ \downarrow & & \downarrow \\ B' & \longrightarrow & B \end{array}$$

is a Cartesian square.

So, for example, we can produce an equivariant cobordism of a stratified map of G-manifolds by producing a stratified cobordism of the induced map of orbit spaces.

Remark. It is necessary to take some care in formulating the definition of a stratified map $f\colon B' \to B$ if the above theorem is to be true. For example, if we had only required that f preserve the stratification, then it would not, in general, be true that the space f^*M is a manifold.

2. Regular G_n-manifolds

In this section, we set up some notation and review the results of [6] which we need.

First we consider the linear model. As $d = 1, 2$ or 4, let $\mathbf{F}(d)$ be the field of real, complex or quaternion numbers, respectively; and let G_n^d stand for, respectively, O_n, U_n or Sp_n. Let $M^d(n, k)$ be the vector space of $n \times k$ matrices with entries in $\mathbf{F}(d)$ and let $H_+^d(k)$ be the set of $k \times k$ positive semidefinite $\mathbf{F}(d)$-hermitian matrices. The representation $k\rho_n^d$ can be defined as the action of G_n^d on $M^d(n, k)$ given by matrix multiplication. For $n \geqq k$, the orbit space can be identified with $H_+^d(k)$ and the orbit map with $\pi(x) = x^*x$

(x^* is the conjugate transpose of x).

If $x \in M^d(n, k)$ is a matrix of rank i (over $\mathbf{F}(d)$), then the isotropy subgroup at x is conjugate to G^d_{n-i} and the normal representation at x is equivalent to $M^d(n-i, k-i)$. Thus, a stratum of $M^d(n, k)$ is the union of all x of a given rank. Similarly, $H^d_+(k)$ is stratified by rank.

A G^d_n-manifold M is *modeled on* $k\rho^d_n$ if its normal orbit types occur among those of G^d_n on $M^d(n, k)$. This means that the orbits are Stiefel manifolds of the form G^d_n/G^d_{n-i} $(0 \leqq i \leqq k)$, and that the normal representation at an orbit of type G^d_n/G^d_{n-i} is equivalent to $(k-i)\rho^d_{n-i}$. Throughout this paper we will assume that $n \geqq k$. Also, we will suppress the d's in our notation when there is no ambiguity.

The strata of M can be indexed by integers $0, 1, \cdots, k$. Thus, the union of orbits of type G_n/G_{n-i} is the *i-stratum* of M and denoted by M_i. Similarly, if B is the orbit space of M, we have the i-stratum, $B_i = \pi(M_i)$ of B.

Let D^{dkn+m} denote the unit disc in $M^d(n, k) \times \mathbf{R}^m$, where G^d_n acts trivially on $\mathbf{R}^m$. Let $S^{dkn+m-1}$ be the unit sphere. The notations $D = D^{dkn+m}$ and $S = S^{dkn+m-1}$ are used when there is no ambiguity. Then both D and S are G_n-manifolds modeled on $k\rho_n$. Let $K = D/G_n = \pi(D)$, $L = S/G_n = \pi(S)$. The following theorem is the beginning of our program.

THEOREM A. *Let Σ be a G^d_n-manifold modeled on $k\rho^d_n$. Suppose further that Σ is an integral homology sphere of dimension $dnk + m - 1$. Then, Σ equivariantly bounds a parallelizable G^d_n-manifold V, also modeled on $k\rho^d_n$, and there is a stratified map of pairs*

$$F:(V, \Sigma) \longrightarrow (D, S) ,$$

where $D = D^{dkn+m}$, $S = S^{dkn+m-1}$. Furthermore, except for the case where $d = 1$, $m = 0$, k is even and n is odd, F can be chosen to be of degree 1.

To exploit Theorem A, we shall need the following theorem.

THEOREM B. *Let $G_n = U_n$ or Sp_n. Suppose that M and M' are G_n-manifolds modeled on $k\rho_n$ and that $\Phi: M \to M'$ is a stratified map. Let $\varphi: B \to B'$ be the induced map of orbit spaces and let $\varphi_i = \varphi | B_i$. Then*

$$\Phi_*: H_*(M; \mathbf{Z}) \longrightarrow H_*(M'; \mathbf{Z})$$

is an isomorphism if and only if for each i,

$$(\varphi_i)_* = (\varphi | B_i)_*: H_*(B_i; \mathbf{Z}) \longrightarrow H_*(B'_i; \mathbf{Z})$$

is an isomorphism.

The proofs of Theorems A and B will appear in [6].

Theorem B tells us the following. With the hypothesis of Theorem A, suppose that $(A, B) = (V/G_n, \Sigma/G_n)$ and that $f: (A, B) \to (K, L)$ is the map of orbit spaces induced by F. Then, since $F|\Sigma$ is of degree 1, it induces a homology isomorphism; hence, for each i $(f \mid B_i)_*$ is an isomorphism. Also, we see that F will induce an isomorphism on homology if and only if each $(f \mid A_i)_*$ is an isomorphism.

Remarks. (1) Of course, it is implicit in the statement of Theorem A that if Σ is any homology sphere which admits an action modeled on $k\rho_n^d$, then the dimension of Σ is $\geqq dkn - 1$.

(2) The proof of Theorem B is a relatively straightforward argument using Smith's theory and Mayer-Vietoris sequences. (A similar application of these arguments can be found, for example, in Section 4 of [8].) As stated, Theorem B is not valid for $G_n = O_n$. Essentially, the reason for this is that O_{2r} and O_{2r+1} have the same maximal torus and this prevents us from using Smith's theory with integral coefficients in the same manner as for U_n or Sp_n. However, a slightly more complicated version of Theorem B is still true in this case. (See [6].)

(3) The proof of Theorem A contains most of the main ideas in [6] (which is a revised version of the first author's thesis). Since this work has not yet appeared, we will sketch the line of thought in the proof.

First, it is shown that if M is any G_n-manifold modeled on $k\rho_n$ with $n \geqq k$, and if the bundle of principal orbits is a trivial fibre bundle, then M is the pullback of the linear model $M(n, k)$ via a stratified map $f: B \to H_+(k)$ (where B is the orbit space of M). Next, it is shown that if M is a homology sphere, then the bundle of principal orbits is trivial so that the above result applies (this is proved by showing that the base space B_k is acyclic). Next, it is shown that if M is a pullback of $M(n, k)$, then it equivariantly bounds a V which is also a pullback of $M(n, k)$. This is proved by constructing the orbit space A of V and an extension of f to A. In this construction it so happens that if M is a π-manifold, then so is V (more will be said about this below). By a slight modification of our original argument, Theorem A is then proved by showing that (V, Σ) is a pullback of (D, S).

A few words concerning the tangential structure of B are in order. For any $y \in H_+^d(k)$, it is easy to see that $T_y(H_+^d(k)) = H^d(k)$ where $H^d(k)$ is the vector space of all $k \times k$ hermitian matrices. It follows that the union of all the tangent spaces has the structure of a bundle, the *tangent bundle*. The same is true for B, since it is locally modeled on $H_+(k)$. Thus TB is a well-defined vector bundle over B. In the proof that V is parallelizable, the following observation of Bredon [2] is essential (see also [6]).

THEOREM C. *Suppose that M is a pullback of $M(n, k)$ with $n \geqq k$. Then M is a π-manifold if and only if TB is trivial.*

In fact, a trivialization of TB induces a stable trivialization of the equivariant tangent bundle TM.

In what follows it is also important to understand the normal bundles of the strata. The normal bundle of M_i in M, denoted by $\nu(M_i)$, is a G_n-vector bundle over M_i with fibre $M(n-i, k-i)$. Its orbit space $\nu(M_i)/G_n$ is a bundle over B_i with fibre $M(n-i, k-i)/G_{n-i} = H_+(k-i)$. We shall also wish to consider the unit disc bundle $\bar{\nu}(M_i)$ and the unit sphere bundle $\partial\bar{\nu}(M_i)$. Since for $x \in M(n, k)$, $\|x\|^2 = \text{trace } x^*x$, we see that the image of the unit disc in $M(n, k)$ in $H_+(k)$ is just the space $\bar{H}_+(k)$ consisting of matrices of trace less than or equal to one. Similarly, the image of the unit sphere is $W_+(k)$, the set of all matrices in $H_+(k)$ of trace 1. Let $C(B_i)$ denote the fibre bundle $\bar{\nu}(M_i)/G_n \to B_i$. It has the fibre $\bar{H}_+(k-i)$ and the structure group G_{k-i} (or actually G_{k-i}/center), which acts on $H_+(k-i)$ by conjugation. Similarly, let $S(B_i)$ be the bundle $\partial\bar{\nu}(M_i)/G_n$ with fibre $W_+(k-i)$. Notice that the normal bundle $N(B_i)$ of B_i in B can be identified with the restriction to B_i of the tangent bundle along the fibres of $C(B_i)$. Thus $N(B_i)$ is a vector bundle over B_i with fibre $H(k-i)$ and the structure group G_{k-i}. The crucial fact is that the i-stratum $H_+^d(k)_i$ is a 'fat' Grassmannian, i.e., it is of the homotopy type of $G_k^d/G_i^d \times G_{k-i}^d$. (This can be seen for example by considering the transitive $\mathrm{GL}^d(k)$ action on $H_+^d(k)_i$ and computing the isotropy subgroup.) It can also easily be seen that K_i and L_i are homotopy equivalent to the same Grassmannian (except for $L_0 = S^{m-1}$). In particular, for $G_n = U_n$ or Sp_n, K_i and L_i are both simply connected (except for $L_0 = S^1$). Essentially the same observation shows that $W_+(k)_1 = G_k/G_1 \times G_{k-1}$ (since G_k acts by conjugation transitively on $W_+(k)_1$ with isotropy subgroup $G_1 \times G_{k-1}$). In other words,

$$W_+^d(k)_1 = \mathbf{F}P^{k-1} \quad \text{where } \mathbf{F} = \mathbf{F}(d) .$$

Let $S(B_i)_{i+1}$ be the $(i+1)$-stratum of $S(B_i)$. Then $S(B_i)_{i+1} \to B_i$ is a fibre bundle with fibre $W_+(k-i)_1$. Thus we have the following lemma.

LEMMA D. *$S(B_i)_{i+1} \to B_i$ is a fibre bundle with fibre $\mathbf{F}P^{k-i-1}$ (where $\mathbf{F} = \mathbf{F}(d)$).*

3. Statement of results

For the remainder of this paper let $G_n = U_n$ or Sp_n. Let Σ be a homology sphere and let $G_n \times \Sigma \to \Sigma$ be an action modeled on $k\rho_n$. By Theorem A, Σ bounds a parallelizable G_n-manifold V and there is a diagram

$$\begin{array}{ccc} (V, \Sigma) & \xrightarrow{F} & (D, S) \\ \downarrow & & \downarrow \\ (A, B) & \xrightarrow{f} & (K, L) \end{array}$$

where both F and f are stratified and F is of degree 1. By Theorem C, we can choose a framing $TA \to \mathbf{R}^N$ of the tangent bundle of A. Since f is stratified, it induces an isomorphism

$$N(A_i) \cong f_i^*(N(K_i)) .$$

Thus, Φ induces a framing

$$\psi_i\colon f_i^*(N(K_i)) \oplus TA_i \longrightarrow \mathbf{R}^N .$$

Since $f_0\colon (A_0, B_0) \to (K_0, L_0) = (D^m, S^{m-1})$ is a map of degree 1, the data (f_0, ψ_0) form a surgery problem in the sense of [15]. Since $f_0 | B_0$ is a homology isomorphism by Theorem B, there is an obstruction to doing surgery rel B_0 on f_0 to a homology isomorphism. Since K_0, L_0 are simply connected ($m \neq 2$), this obstruction is the same as the ordinary surgery obstruction (where $f_0 | B_0$ is required to be a homotopy equivalence). Hence for $m \neq 4$, the obstruction $\sigma_0 \in L_m(1)$ is defined by

$$\sigma_0 = \begin{cases} \dfrac{1}{8} \text{ index of } (A_0, B_0) & \text{if } m \equiv 0 \ (4), \\ \text{the Kervaire inveriant of } (f_0, \psi_0) & \text{if } m \equiv 2 \ (4), \\ 0 & \text{if } m \equiv 1 \ (2). \end{cases}$$

Our main result is the following theorem.

THEOREM 1. *Let Σ be a homology sphere and let $G_n \times \Sigma \to \Sigma$ be an action modeled on $k\rho_n$. Suppose that the fixed point set B_0 is of dimension $m - 1$ with $m \neq 4$. Moreover, if $m \leqq 3$, suppose that $k \geqq 3$ for U_n and $k \geqq 2$ for* Sp_n. *(These assumptions are made to avoid the usual low dimensional surgery difficulties.) Then, Σ equivariantly bounds a contractible G_n-manifold V modeled on $k\rho_n$ if and only if $\sigma_0 = 0$. If k is even, σ_0 can assume any possible value. On the other hand, if k is odd, σ_0 must always vanish.*

Suppose that Σ and Σ' are G_n-manifolds modeled on $k\rho_n$ and that the underlying manifolds are homotopy spheres. Σ and Σ' are said to be *concordant*, if there is an action of $\Sigma \times I$ such that its restriction to $\Sigma \times \{0\}$ is equivalent to the action on Σ and its restriction to $\Sigma \times \{1\}$ is equivalent to the action on Σ'. Let $\theta^d(k, n, m)$ denote the set of concordance classes of G_n^d-actions on homotopy $(dkn + m - 1)$-spheres which are modeled on $k\rho_n^d$. For $m > 0$, $\theta^d(k, n, m)$ has a group structure induced by taking the equivariant connected sum along the fixed point sets. Let $P_m = L_m(1)$. It will follow from the proof of Theorem 1 that the map

$$\sigma_0\colon \theta^d(k, n, m) \longrightarrow P_m$$

which sends Σ to the surgery obstruction of (f_0, ψ_0) is a well-defined homomorphism. Therefore, as a corollary to Theorem 1 we have the following theorem.

THEOREM 2. *Assume that* $m \neq 4$ *and if* $m \leqq 3$ *then* $k \geqq 3$ *for* U_n *and* $k \geqq 2$ *for* Sp_n. *If* k *is odd or if* m *is zero, then* $\theta^d(k, n, m)$ *is the trivial group. If* k *is even and* $m > 0$, *then*

$$\sigma_0\colon \theta^d(k, n, m) \longrightarrow P_m$$

is an isomorphism.

Let $F\colon \theta^d(k, n, m) \to \theta_{dkn+m-1}$ be the forgetful homomorphism, i.e., $F(\Sigma)$ is the underlying homotopy sphere. According to Theorem A, the image of F is contained in bP_{dkn+m}, the subgroup consisting of those homotopy spheres which bound π-manifolds. Then, as a further corollary, we have the following theorem.

THEOREM 3. *For* $m \geqq 6$, *the following diagram commutes*

$$\begin{tikzcd} \theta^d(k, n, m) \arrow[r, "F"] \arrow[dr, "\sigma_0"'] & bP_{dkn+m} \\ & P_m = P_{dkn+m} \arrow[u, "b"'] \end{tikzcd}$$

where b *is the canonical map.*

Proof. It follows from Theorem 1 that it suffices to check this for the Brieskorn examples for which it is well-known.

For $k = 2$, these theorems are essentially all due to Bredon [2]. For $k = 1$, they are implicit in [11].

The concordance relation is introduced to take care of difficulties with the fundamental groups of the strata. Suppose that all the strata of Σ are simply connected. By taking connected sum with a Brieskorn sphere (which also has 1-connected strata), we may assume that $\sigma_0 = 0$. Then, modulo the usual 3 and 4 dimensional difficulties, we can actually do surgery so that $f \mid A_i$ will be a homotopy equivalence for each i. The concordance produced in this manner will be equivariantly diffeomorphic to the linear action on $S \times I$. We therefore have the following theorem.

THEOREM 4. *With the hypothesis of Theorem* 1, *suppose also that for each* i, Σ_i *is* 1-*connected. Then,* Σ *is equivariantly diffeomorphic to the linear action on* $S^{dkn+m-1}$ *or an action on a Brieskorn sphere* (*depending on whether or not* $\sigma_0 = 0$).

On the other hand, it is easy to construct examples where the fundamental groups of the strata are not 1-connected. One way to do this is to alter any stratum of the linear orbit space L by taking the connected sum with a homology sphere and altering the higher strata appropriately.

4. Proof of Theorem 1

Before beginning the proof, we need to make one technical digression. A stratum of a compact G-manifold or its orbit space is in general an open manifold. Usually, one replaces such a stratum by a compact manifold with corners (called the *closed stratum*), the interior of which is the original stratum. A closed stratum is essentially a stratum of the G-manifold (or its orbit space) minus open tubular neighborhoods of the lower strata. It does not matter very much how we remove these tubular neighborhoods, although in [5[, [13] it is shown how to remove them in a 'canonical' way. So, for example, if B is the orbit space of a regular G_n-manifold, from now on we will use the notation B_i to denote the manifold with corners

$$B_i = \overline{\left(B - \bigcup_{j<i} C(B_j)\right)}$$

where $C(B_j)$ is a fibre bundle neighborhood of B_j in B. B_j is a 'manifold with faces' (see [13]). We have

$$\partial B_i = \partial_0 B_i \cup \cdots \cup \partial_{i-1} B_i \ ,$$

where $\partial_j B_i = \left(S(B_j)\right)_i$. Thus, $\partial_j B_i$ is a fibre bundle over B_j with fibre $W_+(k-j)_i$. Moreover, if we remove the neighborhoods in the canonical way, a stratified map $f: B \to C$ will induce a bundle map $f \mid \partial_j B_i: \partial_j B_i \to \partial_j C_i$ covering f_j (see section III. 1-2 in [5]).

Now, we begin the proof. Recall that we are given a map $f: (A, B) \to (K, L)$ and a framing $\Phi: TA \to \mathbf{R}^N$. We want to construct a 'stratified' normal cobordism rel B to a new map $f': (A', B) \to (K, L)$ which is a homology isomorphism. To do this, we will inductively construct for each i a normal cobordism rel B to

$$f(i): \left(A(i), B\right) \longrightarrow (K, L) \ ,$$
$$\Phi(i): T\left(A(i)\right) \longrightarrow \mathbf{R}^N$$

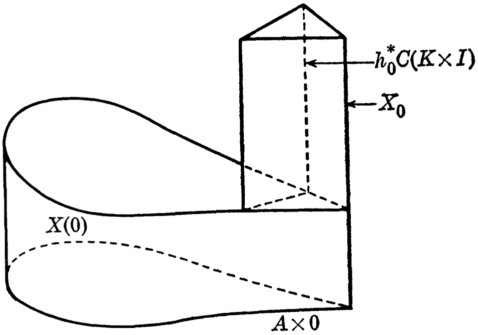
so that $f(i)$ will induce a homology isomorphism on the j-stratum for $0 \leqq j \leqq i$. To start with, we have the surgery problem

$$f_0: (A_0, B_0) \longrightarrow (K_0, L_0) \ ,$$
$$\psi_0: TA_0 \oplus f_0^* N(K_0) \longrightarrow \mathbf{R}^N \ .$$

Let us assume that the surgery obstruction $\sigma_0 = 0$ so that we can begin our induction. Then, there is a normal cobordism rel B_0 to $f_0': (A_0', B_0) \to (K_0, L_0)$ where f_0' induces an isomorphism on homology. Denote this cobordism by

$$h_0\colon (X_0, A_0, A_0') \longrightarrow (K_0 \times I, K_0 \times 0, K_0 \times 1)\ ,$$
$$\tilde{\psi}_0\colon TX_0 \oplus h_0^*(K_0 \times I) \longrightarrow \mathbf{R}^{N+1}\ .$$

We use this to construct a cobordism of f. Let $X(0) = A \times I \cup h_0^*(C(K_0 \times I))$ where the union is via the identification $f_0^*C(K) \cong C(A_0) \times 1$ given by the differential of f (using the fact that f is stratified).



$X(0)$ is a cobordism from A to

$$A(0) = (A - C(A_0)) \cup h_0^*S(K_0 \times I) \cup f_0'^*C(K_0)\ .$$

(See § 2 for definitions of S and C.) The map h_0 induces a cobordism

$$h(0)\colon (X(0); A, A(0)) \longrightarrow (K \times I; K \times 0, K \times 1)$$

from f to $f(0)\colon (A(0), B) \to (K, L)$ in an obvious fashion. Also, since

$$T(h_0^*C(K_0 \times I)) = p^*(TX_0 \oplus h_0^*N(K_0 \times I))\ ,$$

where $p\colon h_0^*C(K_0 \times I) \to X_0$ is the projection map, we have that the framing $\tilde{\psi}_0$ induces a framing

$$\tilde{\Phi}(0)\colon TX(0) \longrightarrow \mathbf{R}^{N+1}\ .$$

By construction,

$$f(0)_0 = f_0'\colon (A_0', B_0) \longrightarrow (K_0, L_0)$$

is a homology isomorphism of pairs.

Now, suppose by induction that we have constructed a normal cobordism rel B from (f, Φ) to

$$f(i-1)\colon (A(i-1), B) \longrightarrow (K, L)\ ,$$
$$\Phi(i-1)\colon TA(i-1) \longrightarrow \mathbf{R}^{N+1}$$

in such a way that

$$f(i-1)_j\colon (A(i-1)_j, B_j) \longrightarrow (K_j, L_j)$$

induces an isomorphism on homology for each j, $0 \leqq j \leqq i-1$. Let

$$h(i-1)\colon (X(i-1), A, A(i-1)) \longrightarrow (K \times I, K \times 0, K \times 1)$$

denote the cobordism. Let X_j denote the cobordism from $A(j-1)_j$ to $A(j)_j$. We consider the map $f(i-1)$ on the i-stratum. To simplify notation, let $Y_i = A(i-1)_i$ and let

$$r_i = f(i-1)_i\colon (Y_i, \partial Y_i) \longrightarrow (K_i, \partial K_i)\ .$$

Notice that

$$\partial Y_i = B_i \cup \partial_0 A(i-1)_i \cup \cdots \cup \partial_{i-1} A(i-1)_i\ .$$

Since $\partial_j A(i-1)_i$ is a fibre bundle over $A(i-1)_j$ and since $r_i \mid \partial_j A(i-1)_i$ is a bundle map, it follows from the induction hypothesis that $r_i \mid \partial_j A(i-1)_i$ is a homology isomorphism. Since $r_i \mid B_i$ is also a homology isomorphism, a simple argument using Mayer-Vietoris sequences show that $r_i \mid \partial Y_i$ is a homology isomorphism. Also, we have a framing

$$\psi_i\colon T(Y_i) \oplus r_i^* N(K_i) \longrightarrow \mathbf{R}^{N+1}$$

induced by $\Phi(i-1)$. Thus, we have a surgery problem (r_i, ψ_i) where we want to do surgery rel ∂Y_i to a homology isomorphism. Since K_i and ∂K_i are simply connected, this obstruction lies in the ordinary surgery group. Thus, let $\sigma_i \in L_p(1)$ $(p = \dim\ Y_i)$ be this obstruction.

LEMMA 1. *For $i > 0$, with a possible alternation of the choice of X_{i-1}, we can always make $\sigma_i = 0$. Furthermore, if k is odd, $\sigma_0 = 0$.*

Let us defer the proof of the above lemma and continue our proof of Theorem 1. Since $\sigma_i = 0$, we can de surgery rel ∂Y_i on r_i to a homology isomorphism

$$f'_i\colon (A'_i, \partial Y_i, B_i) \longrightarrow (K_i, \partial K_i, L_i)\ .$$

Let X_i be the trace of the completed surgery on r_i. Let $h_i\colon X_i \to K_i \times I$ and $\tilde{\psi}_i\colon TX \oplus h_i^* N(K_i \times I) \to \mathbf{R}^{N+1}$ be the normal cobordism. As before, define $X(i) = X(i-1) \cup h_i^* C(K_i \times I)$ where the union is via the identification $r_i^* C(K_i) \cong C\big(A(i-1)_i\big)$. Use h_i to extend $h(i-1)$ to $h(i)\colon X(i) \to K \times I$, and $\tilde{\psi}_i$ to extend $\Phi(i-1)$ to $\Phi(i)\colon TX(i) \to \mathbf{R}^{N+1}$. This completes our induction. Hence, setting $A' = A(k)$ and $f' = f(k)$, we have our required map $f'\colon (A', B) \to (K, L)$, which induces a homology isomorphism on each stratum. Using the pullback construction, we obtain a G_n-manifold $V' = f^* D$. By Theorem B, V' is acyclic. If A'_k is simply connected, then by the fibration sequence so is V'_k; and hence, by a general position argument, we have that $\pi_1(V') = 0$. Conversely, according to Corollary II. 6.3 on page 91 of [1], if V' is simply connected then so is A'_k. Therefore, V' is simply connected (and hence, contractible) if and only if $\pi_1(A'_k) = 0$. But when we do surgery on r_i we may clearly assume that the resulting manifold A'_i is simply connected.

Thus, $\sigma_0 = 0$ implies that Σ equivariantly bounds a contractible G_n-manifold V'.

We must also show that if Σ equivariantly bounds a contractible manifold, then $\sigma_0 = 0$. So, suppose that V, V' are parallelizable G_n-manifolds modeled on $k\rho_n$ and that the bundles of principal orbits are trivial. Suppose further that $\partial V = \partial V' = \Sigma$. Let $A = V/G_n$ and $A' = V'/G_n$. Choose framings $\Phi: TA \to \mathbf{R}^N$, $\Phi': TA' \to \mathbf{R}^N$. Since B is acyclic, Φ and Φ' are homotopic on B. Thus, we may assume that Φ, Φ' agree on B and there is a global framing $\theta: T(A \cup A') \to \mathbf{R}^N$. This together with our classifying map provides a framing of the normal bundle of the closed manifold $M_0 = A_0 \cup A_0'$. Since M_0 is a closed framed manifold, its index or Kervaire invariant is zero. Now, suppose V' is contractible, then it follows from Theorem B that A_0' is acyclic. Thus, $\sigma(A_0') = 0$ and hence

$$\sigma_0 = \sigma(A_0) = \sigma(M_0) - \sigma(A_0') = 0 .$$

Modulo Lemma 1, this completes the proof of Theorem 1.

We need the following lemma for the proof of Lemma 1.

LEMMA 2. *Let M^s be an s-dim compact (simply connected) manifold with boundary, and let $p: E \to M$ be a fibre bundle with fibre F^t a closed simply connected manifold of* dim t. *Suppose that $f: (M', \partial M') \to (M, \partial M)$ is a normal map which induces a homology isomorphism on the boundary (we suppress the normal data). The natural map*

$$\bar{f}: (f^*E, \partial f^*E) \longrightarrow (E, \partial E)$$

is also a normal map and induces a homology isomorphism on the boundary. Then

$$I(\bar{f}) = I(F^t)\cdot I(f) ,$$
$$C(\bar{f}) = \chi(F^t)\cdot C(f)$$

where $I(F^t)$, $\chi(F^t)$ denote the index and the Euler characteristic of F^t respectively, and $I(f)$, $I(\bar{f})$, $C(f)$, $C(\bar{f})$ are the index, and the Kervaire invariant of $f, \bar{f}$ respectively.

In particular, if $F = \mathbf{C}P^j$ or $\mathbf{Q}P^j$, then the correspondence $\sigma(f) \to \sigma(\bar{f})$ defines a homomorphism $L_p(1) \to L_{p+t}(1)$ of the surgery groups where $\sigma(f)$, $\sigma(\bar{f})$ denote the surgery obstruction of $f, \bar{f}$ respectively. This homomorphism is an isomorphism if j is even and the zero map if j is odd.

Proof. If $s > 4$, let us perform surgery on f rel ∂M to make it highly connected. We may assume that M' is a boundary connected sum $M'' \# M'''$ such that $f \mid M''$ induces a homology isomorphism and M''' carries the homology kernel. Since the induced bundle over M''' is trivial, the lemma follows from the standard product formula for surgery. If $s < 4$, we can reduce it back to $s > 4$ by taking a product of our surgery problem with $\mathbf{C}P^{2l}$.

Proof of Lemma 1. First suppose that $k - i - 1$ is even (in which case we 'look up one stratum'). Recall that σ_i is the obstruction to completing the surgery $r_i: (Y_i, \partial Y_i) \to (K_i, \partial K_i)$ where $Y_i = A(i-1)_i$. Consider the surgery problem

$$s_{i+1}: (W_{i+1}, \partial_i W_{i+1}) \longrightarrow (K_{i+1}, \partial K_{i+1})$$

where $W_{i+1} = A(i-1)_{i+1}$, $\partial_i W_{i+1} = \partial_i A(i-1)_{i+1}$, and $s_{i+1} = f(i-1)_{i+1}$. Let $s'_{i+1} = s_{i+1} | \partial_i W_{i+1}$. By the lemma at the end of Section 2, $\partial_i W_{i+1} \to Y_i$ is a fibre bundle with fibre $\mathbf{F}P^{k-i-1}$ $(= \mathbf{C}P^{k-i-1}$ or $\mathbf{Q}P^{k-i-1})$. Since $f(i-1)$ is stratified, s'_{i+1} is a bundle map covering r_i. In fact, s'_{i+1} is just the normal map induced by r_i. Thus, by Lemma 2, the obstruction to doing surgery on s'_{i+1} rel boundary is equal to σ_i. But s'_{i+1} is the boundary of a surgery problem (namely s_{i+1}) hence the index or Kervaire invariant of s'_{i+1} is zero, i.e., $\sigma_i = 0$. In particular, if k is odd, this argument shows that $\sigma_0 = 0$.

Now, suppose that $k - i - 1$ is odd, so that $k - i$ is even. In this case we 'go down one stratum and change the cobordism'. Suppose that $\sigma_i \neq 0$. It follows from the construction that $Y_i = A(i-1)_i = Q' \cup Q$ where $Q' = A(i-2)_i - C(A(i-2)_{i-1})$ and where Q is an $\mathbf{F}P^{k-1}$-bundle over X_{i-1}. Furthermore, $r_i | Q$ is a bundle map covering $h_{i-1}: X_{i-1} \to K_{i-1} \times I$. Recall that X_{i-1} is a cobordism rel B_{i-1} from $A(i-2)_{i-1}$ to A'_{i-1}. If $\dim X_{i-1} = t$, then $t = \dim Y_i - \dim \mathbf{F}P^{k-i}$ and therefore $t \equiv \dim Y_i(4)$. We may choose a parallelizable manifold U^t with ∂U^t a homotopy sphere and a normal map $\lambda: (U^t, \partial U^t) \to (D^t, S^{t-1})$ such that $\sigma(\lambda) = -\sigma_i$. (Here we use the fact that $\dim U = t \equiv \dim Y_i(4)$ to identify the surgery group in the usual fashion.) Let $X'_{i-1} = X_{i-1} \# U$, $h'_{i-1} = h_{i-1} \# \lambda$, where $\#$ means the boundary connected sum along A'_{i-1}. Thus A'_{i-1} is changed to $A'_{i-1} \# \partial U$ and Y_i is changed to

$$Y'_i = \big(Y_i - (D^t \times \mathbf{F}P^j)\big) \cup \big((U - D^t) \times \mathbf{F}P^j\big)$$

where D^t is a small disc in X_{i-1} and $j = k - 1$. By Lemma 2 again, $\sigma(\lambda \times \mathrm{id}) = -\sigma_i$. It therefore follows from the addition formula that the surgery obstruction σ'_i of $r'_i: (Y'_i, \partial Y'_i) \to (K_i, \partial K_i)$ is equal to $\sigma_i + \sigma(\lambda \times \mathrm{id}) = \sigma_i - \sigma_i = 0$. The only time this argument fails to work is when $i = 0$ (and so $X_{i-1} = \varnothing$); therefore, the only possible nontrivial contribution to the surgery obstruction occurs for $\sigma_i = \sigma_0$, and this happens only if $k = k - i$ is even. This proves Lemma 1 and thereby Theorem 1.

Remark. The same argument calculates the 'isovariant surgery group' ${}_GL_*(M, \partial M)$ of [4]. Thus let M be a compact G_n-manifold with boundary which is modeled on $k\rho_n$ with $n \geqq k$. Suppose further that $\pi_1(M_i) = \pi_1(\partial M_i) = 0$ and the fixed point set $M_0 \neq \emptyset$. Then, we have the following theorem.

THEOREM 5. *For k odd, ${}_GL_p(M, \partial M) = 0$; while for k even, ${}_GL_p(M, \partial M) = L_m(M_0, \partial M_0) = P_m$ where $m = p - dkn$.*

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