CHAPTER X

Finite Group Actions on Homotopy 3-Spheres

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The Smith conjecture states that certain types of smooth actions of a cyclic group on a homotopy 3-sphere are essentially linear. Our working hypothesis is that every smooth action of a finite group on a homotopy 3-sphere is essentially linear. By making use of the methods presented earlier in this volume, we have established this hypothesis in a substantial number of cases. Our results are summarized by the following theorem.

THEOREM A. *Let $\tilde{\Sigma}$ be a homotopy 3-sphere and let $G \times \tilde{\Sigma} \to \tilde{\Sigma}$ be an action of a finite group of orientation-preserving diffeomorphisms. If every isotropy group is cyclic and if at least one isotropy group has order greater than 5, then the action is essentially linear.*

It is an immediate consequence of the Schönflies theorem and the definition of "essentially linear" that any essentially linear action on S^3 is equivariantly diffeomorphic to a linear action. (The definition of essentially linear is given in Chapter 1.) Thus we have the following corollary of the above theorem.

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THEOREM B. *Let $G \times S^3 \to S^3$ be an action of a finite group of orientation-preserving diffeomorphisms. If every isotropy group is cyclic and if at least one isotropy group has order greater than 5, then the given action is linear.*[1]

This chapter is divided into eight sections. The first introduces the notion of an orbifold. This is a term coined by Thurston to describe spaces that are locally the quotients of finite group actions. In Sections 2 and 3 we discuss two- and three-dimensional examples of these spaces in some detail. We find the language of orbifolds a convenient framework for dealing with quotient spaces of properly discontinuous group actions. Using them, one can treat all properly discontinuous actions the way one normally deals with free, properly discontinuous actions—by working with the quotient space.

We establish the connection between finite group actions and linear actions through the intermediary of Seifert-fibered orbifolds. Essentially, a Seifert-fibered orbifold comes about when one has a group acting on a Seifert-fibered manifold preserving the Seifert-fibered structure. The basic properties of these orbifolds are developed in Section 4. In Section 5 we prove an analogue of a theorem of Seifert and Threlfall. We show that any three-dimensional, Seifert-fibered orbifold with finite fundamental group is diffeomorphic to S^3/G for some finite group G in SO(4) that normalizes the standard S^1 in SO(4). This reduces the problem of showing a finite group action is linear to the problem of showing that its quotient orbifold can be given the structure of a Seifert fibration.

We prove Theorem A by showing that if $G \times \tilde{\Sigma} \to \tilde{\Sigma}$ is a finite group action on a homotopy 3-sphere, then the quotient $\tilde{\Sigma}/G$ is isomorphic to $X_1 \# \Sigma$, where X_1 is Seifert-fibered and Σ is a homotopy 3-sphere. Since X_1 is equivalent to the quotient of a linear action, it follows easily that the action we began with is essentially linear.

In Section 6 we reformulate the result along these lines. In Section 7 we study an important special case. In the special case we study a knot K in a homotopy sphere Σ with the property that the cyclic n-sheeted branched cover of Σ, branched over K, has finite fundamental group for some $n > 5$. We show that this implies that K is unknotted in Σ. This is very similar to the situation of the Smith conjecture. The difference is that in the Smith conjecture the cyclic branched cover is assumed to be simply connected (also, one doesn't restrict to $n > 5$). The general plan of attack that resolves the Smith conjecture, bolstered by some old results of Dickson's concerning subgroups

[1] By linear we mean that the action is equivariantly diffeomorphic to an action arising from a representation $G \hookrightarrow SO(4)$.

of $PSL_2(F)$, for F a finite field, is used in Section 7 to deal with the special case.

In Section 8 we show how to reduce the general case to the special case. The basic idea is to start with an action and restrict it to a certain type of normal subgroup. We show that if the quotient by the subgroup is Seifert-fibered, then so is the quotient by the full group. The main result needed to establish this is that a prime three-dimensional orbifold with boundary is Seifert-fibered if and only if its fundamental group contains a normal, infinite, cyclic subgroup. Using this reduction result, one deduces Theorem A from the special case considered in Section 7.

Lastly, there is an appendix that gives Dickson's classification of subgroups of $PSL_2(F)$ that contain an element whose order equals the characteristic of F.

The results of the first three sections of this chapter are not original. They are properly called folklore. We have adopted Thurston's terminology and point of view on the subject. The heart of the chapter is Section 7, where we generalize the argument proving the Smith conjecture to deal with the special case considered there.

1. Orbifolds

Suppose that H is a discrete group acting smoothly, effectively, and properly discontinuously on a manifold M^n. We wish to analyze the local structure of the orbit space M^n/H. For each $x \in M$ the isotropy group H_x is finite. Furthermore, each $x \in M$ has an H_x-invariant open neighborhood U_x such that $h(U_x) \cap U_x = \emptyset$ for all $h \in H - H_x$. It follows that the image of HU_x in M^n/H is naturally isomorphic to U_x/H_x.

Since the action is smooth, its differential induces a linear action $H_x \times T_xM \to T_xM$. Choose an H_x-invariant metric on M and an isometry $T_xM \cong \mathbf{R}^n$. This yields a faithful representation $\mu: H_x \hookrightarrow O(n)$, well defined up to conjugation. Let $[x]$ denote the image of x in M^n/H. The conjugacy class of $\mu(H_x) \subset O(n)$ is called the *local group type at* $[x]$. It is independent of all choices. If $D^n \subset \mathbf{R}^n$ is a disk about the origin of sufficiently small radius, then the exponential map $\mathbf{R}^n \cong TM_x \to M$ can be used to define an H_x-invariant, smooth embedding $\varphi: D^n \hookrightarrow M^n$, taking 0 to x. Clearly, we can arrange that the image of φ is contained in U_x. Assuming this, it induces an embedding $\bar{\varphi}: D^n/\mu(H_x) \hookrightarrow M^n/H$ taking $[0]$ to $[x]$. The map $\bar{\varphi}$ is called a *smooth orbifold chart for M^n/H centered at* $[x]$.

Suppose that G_1 and G_2 are conjugate subgroups of $O(n)$ and that $b: D^n/G_1 \to D^n/G_2$ is a map taking $[0]$ to $[0]$. The map b is called a *smooth* *isomorphism* if it can be lifted to a diffeomorphism $\tilde{b}: D^n \to D^n$ so that the following diagram commutes:

$$\begin{array}{ccc} D^n & \xrightarrow{\tilde{b}} & D^n \\ \downarrow & & \downarrow \\ D^n/G_1 & \xrightarrow{b} & D^n/G_2 \end{array}$$

The map $\tilde{b}$ is called a *lifting* of b. If $\tilde{b}'$ is another lifting of b, then there is an element $g \in G_1$ so that $\tilde{b}' = \tilde{b} \cdot g$. A map of orbit spaces $f: M/H \to M'/H'$ is a *smooth isomorphism* (or a *diffeomorphism*) if it is a homeomorphism and if for each $[x] \in M/H$ there are smooth orbifold charts centered at $[x]$ and $f[x]$ so that in these coordinates the map is a smooth isomorphism. Notice that if $f: M/H \to M'/H'$ is a smooth isomorphism, then it preserves the local group types. (However, it is not necessary that $M \cong M'$ or that $H \cong H'$ for M/H to be diffeomorphic to M'/H'.)

Now we generalize these concepts from spaces that are globally quotients of properly discontinuous group actions to those that locally have such descriptions. Such a notion was first suggested by Satake [8] under the name of *V-manifolds*. Recently, Thurston has made use of these spaces and has introduced the word *orbifolds*. We follow Thurston's point of view and terminology.

By a (*smooth*) *n-dimensional orbifold* we shall mean a paracompact Hausdorff space X together with a maximal atlas of local charts of the form $D^n/G \hookrightarrow X$, where G ranges over the conjugacy classes of finite subgroups of $O(n)$ and where the overlap maps are diffeomorphisms in the sense defined above. The notions of the *local group at a point* and of a *diffeomorphism of orbifolds* then have unambiguous meanings. A point in X is called a *manifold point* if its local group is trivial; otherwise, it is *exceptional*. An orbifold is a *manifold* if it has empty exceptional set.

It is clear that if M^n is a manifold and $H \times M^n \to M^n$ is an effective, properly discontinuous, smooth action, then M^m/H receives naturally the structure of an orbifold.

An *orientation* for D^n/G is an orientation for D^n in which G acts as a group of orientation-preserving maps. An *orientation* for an orbifold is a compatible system of orientations for the charts in the atlas defining the orbifold structure.

There is a similar notion of *orbifold with boundary*. Here one uses, in addition, charts of the form $(D^n)_+/G$, where $(D^n)_+ = \{(x_1, \ldots, x_n) | \Sigma x_i^2 \leq r$ and $x_n \geq 0\}$ and $G \subset O(n)$ leaves the half-space $\{x_n \geq 0\}$ invariant. The boundary of an n-dimensional orbifold is an $(n-1)$-dimensional orbifold without boundary.

An orbifold is *compact* if its underlying topological space is compact. An orbifold is *closed* if it is compact and has empty boundary. (N.B.: The underlying topological manifold can have a boundary even if the orbifold is closed.)

If X is an orbifold whose underlying space is Q and if $H \times Q \to Q$ is a group action, then the action is said to be a (*smooth*), *properly discontinuous action on the orbifold* X if

(1) $H \times Q \to Q$ is properly discontinuous, and
(2) each $h \in H$ acts via a smooth isomorphism on X.

Suppose that $H \times X \to X$ is a properly discontinuous action on the orbifold X. We shall give the quotient space the structure of an orbifold. Let Q be the underlying space of X, and let x be a point in Q/H. Choose $\tilde{x} \in Q$ a point that projects to x. Choose an orbifold chart for X centered at $\tilde{x}$, $\varphi: D^n/G \hookrightarrow X$, so that $h(\varphi(D^n/G)) \cap \varphi(D^n/G) = \emptyset$ for all $h \in H - H_{\tilde{x}}$. Choose an open set $U \subset D^n/G$ with $[0] \in U$ and with $h(\varphi(U)) \subset \varphi(D^n/G)$ for all $h \in H_x$. Then $\varphi^{-1} \circ h \circ \varphi: U \hookrightarrow D^n/G$ is a diffeomorphism onto its image for all $h \in H_x$. Thus there is a lifting $(\varphi^{-1} \circ h \circ \varphi)^{\sim}: \tilde{U} \hookrightarrow D^n$, where $\tilde{U}$ is the preimage of U in D^n. This lifting can be varied by any element of G. Let $\tilde{H}_{\tilde{x}}$ be the group of germs at $0 \in D^n$ of all liftings of all $\varphi^{-1} \circ h \circ \varphi$ for $h \in H_{\tilde{x}}$. There is an exact sequence

$$1 \to G \to \tilde{H}_{\tilde{x}} \to H_{\tilde{x}} \to 1$$

In particular, $\tilde{H}_{\tilde{x}}$ is a finite group of germs of diffeomorphisms. Each element $\alpha \in \tilde{H}_{\tilde{x}}$ is represented by a smooth embedding $\varphi_\alpha: \tilde{U} \hookrightarrow D^n$. Pull back the standard metric on D^n under each of the φ_α. Average the resulting finite collection of metrics on $\tilde{U}$. This produces a riemannian metric on $\tilde{U}$, in which the elements of $\tilde{H}_{\tilde{x}}$ are germs of isometries. It follows that there is a disk D' of radius ε centered at $0 \in \tilde{U}$ so that $D'/\tilde{H}_{\tilde{x}}$ is embedded in $(D^n/G)/H_{\tilde{x}}$, which, in turn, is embedded by φ in Q/H. The composite is an orbifold chart for Q/H centered at x. The collection of all charts constructed in this manner defines the orbifold structure which we denote X/H. It is called the *quotient* of the action of H on X.

Local models for the map $X \to X/H$ are $\pi: D^n/G \to D^n/G'$, where $G \subset G' \subset O(n)$ and π is the natural projection.

We turn now to the question of covering spaces for orbifolds. Let $\{U_\alpha\}_{\alpha \in I}$ be a family of orbifolds. A mapping $p: \amalg_{\alpha \in I} U_\alpha \to D^n/G$ *evenly covers* D^n/G if for each $\alpha \in I$ there is an isomorphism $\varphi: D^n/G_\alpha \cong U_\alpha$, so that $G_\alpha \subset G$ and $p \circ \varphi: D^n/G_\alpha \to D^n/G$ is the natural projection.

If $p: Y \to X$ is a continuous mapping between the topological spaces underlying two orbifolds, then p is said to be a *covering projection* if every point

$x \in X$ has an orbifold chart $\varphi\colon D^n/G \hookrightarrow X$ that is evenly covered by $p|p^{-1}(\varphi(D^n/G))$. In this case we also say that Y is a *covering orbifold* of X.

If $p\colon Y \to X$ is a covering projection, then the group of covering transformations $G_X(Y)$ acts properly discontinuously on the orbifold Y. The quotient orbifold $Y/G_X(Y)$ covers X via the map induced by p. Also, the quotient mapping $Y \to Y/G_X(Y)$ is a covering projection.

Conversely, if $H \times Y \to Y$ is a properly discontinuous action on the orbifold Y, then $Y \to Y/H$ is a covering projection whose group of covering transformations is H.

A covering of an orbifold $Y \to X$ is said to be *regular* if the induced covering $Y/G_X(Y) \to X$ is an isomorphism, i.e., if X is naturally isomorphic to the quotient orbifold of the group of covering transformations acting on Y.

The usual proof of the existence of universal coverings can be adapted to show that any orbifold X has a universal covering, $\tilde{X} \to X$. This is a regular covering, and the group of covering transformations is called the *orbifold fundamental group* of X, $\pi_1^{orb}(X)$. All connected coverings of X come, up to isomorphism, by dividing $\tilde{X}$ by a subgroup of $\pi_1^{orb}(X)$. A connected cover of X is regular if and only if it is isomorphic to the quotient of $\tilde{X}$ by a normal subgroup of $\pi_1^{orb}(X)$.

Suppose that X is an orbifold whose underlying space is Q. Let $p\colon P \to Q$ be a topological covering with group of covering transformations G. Since p is a local homeomorphism, we can induce the orbifold structure X on Q up to an orbifold structure Y on P. The projection $p\colon Y \to X$ becomes a covering of orbifolds. Clearly, G acts as the group of covering transformations for this covering of orbifolds. Thus if $P \to Q$ is a regular covering, then so is $Y \to X$, and the group of covering transformations is the same. Applying this to the universal topological covering of Q we see that X has a regular covering whose group of covering transformations is $\pi_1(Q)$. This proves Lemma 1.1.

LEMMA 1.1. *If Q is the space underlying an orbifold X, then there is a natural surjection $\pi_1^{orb}(X) \to \pi_1(Q) \to 1$.*

An orbifold is said to be *good* if its universal cover is a manifold (as an orbifold). There is a way to formulate this concept in terms of local groups. First, note that $\pi_1^{orb}(D^n/G) \cong G$. Thus if $\varphi\colon D^n/G \hookrightarrow X$ is an orbifold chart centered at x, then $\pi_1^{orb}(D^n/G)$ is identified with the local group at x. The map φ induces $\varphi_*\colon \pi_1^{orb}(D^n/G) \to \pi_1^{orb}(X)$.

The orbifold X is good if and only if for each chart $\varphi\colon D^n/G \hookrightarrow X$ the homomorphism $\varphi_*\colon \pi_1^{orb}(D^n/G) \to \pi_1^{orb}(X)$ is an injection. We denote this by saying that "the local groups inject."

This general discussion of orbifolds impinges on the problem of classifying actions of finite groups on homotopy 3-spheres. This is brought out clearly by the next theorem.

THEOREM 1.2. *Classifying finite group actions on homotopy 3-spheres, up to equivariant diffeomorphism, is the same as classifying closed, good, smooth, three-dimensional orbifolds with finite fundamental group, up to diffeomorphism.*

Proof. The correspondence between group actions and orbifolds is given as

$$\left\{\begin{array}{l}\text{finite group actions}\\ \text{on homotopy 3-spheres}\end{array}\right\} \underset{\text{universal cover}}{\overset{\text{quotient}}{\rightleftarrows}} \left\{\begin{array}{l}\text{orbifolds with}\\ \text{finite fundamental}\\ \text{groups}\end{array}\right\}.$$

■

There is also a notion of locally smooth, topological orbifolds. One uses the same charts but requires that the overlap functions lift to homeomorphisms instead of diffeomorphisms. It turns out that this concept is the same as the underlying topological space with its stratification by local group type and with the local group type associated to each stratum. We shall describe in more detail these stratified spaces in dimensions two and three in the next two sections.

2. Two-Dimensional Orbifolds

The finite subgroups of O(2), up to conjugacy, are (1) cyclic subgroups of SO(2), (2) O(1), and (3) dihedral groups $D_{2n} \subset O(2)$, $n \geq 2$. The resulting quotients of the 2-disk by these groups are shown in Fig. 2.1. From this it follows that if X is a two-dimensional orbifold without boundary, then its underlying topological space, Q, is a 2-manifold (possibly with boundary). There is a discrete set of points in the interior with nontrivial local groups.

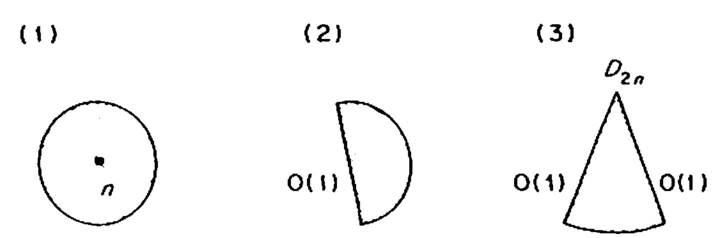


Figure 2.1

We label these points by the orders of their local groups. There is a discrete set of points on the boundary with dihedral local groups. The rest of the boundary consists of points with local groups O(1).

If we allow X to be an orbifold with boundary, then the situation is more complicated. There is a disjoint union of circles and closed intervals in the boundary of the underlying space which forms the space underlying ∂X. This space is unlabeled except for the end points of the intervals, which are labeled by O(1).

To each two-dimensional orbifold X we can associate its *underlying triple* (Q, K, ρ), where Q is the underlying space, $K \subset Q$ is the exceptional set, and ρ is the function on Q that assigns to each point its local group type. Given any such triple (where Q is a 2-manifold and K and ρ are as described above), there is an orbifold structure on Q that gives rise to this triple. There are many such. Any two are diffeomorphic by a diffeomorphism arbitrarily close to the identity.

One can check easily that there are four types of compact two-dimensional orbifolds that are not good:

(a) S^2 with one point labeled,
(b) S^2 with two points labeled by different integers,
(c) D^2 with one boundary point labeled by a dihedral group and the rest of ∂D^2 labeled by O(1), and
(d) D^2 with two boundary points labeled by dihedral groups of different order and the rest of ∂D^2 labeled by O(1).

Notice that all these orbifolds are closed.

All other compact two-dimensional orbifolds are good. The good orbifolds with finite fundamental group are

S(i) S^2, $\mathbf{R}P^2$; S^2 with 2 points labeled n, and $\mathbf{R}P^2$ with one point labeled.
S(ii) S^2 with 3 points labeled p, q, and r, where $1/p + 1/q + 1/r > 1$,
S(iii) D^2 with ∂D^2 labeled by O(1), D^2 with two boundary points labeled by D_{2n} and the rest of ∂D^2 labeled by O(1),
S(iv) D^2 with three boundary points labeled by dihedral groups of order $2p$, $2q$, and $2r$ with $1/p + 1/q + 1/r > 1$ and the rest of ∂D^2 labeled O(1);
S(v) D^2 with an interior point labeled by p and a boundary point labeled by a dihedral group of order $2n$, where $(2/p) + (1/n) > 1$ and the rest of ∂D^2 labeled by O(1); and
D orbifolds isomorphic to D^2/G for $G \subset$ O(2).

The orbifolds S(i)–S(v) are *spherical orbifolds* in the sense that they are diffeomorphic to S^2/G for some $G \subset$ O(3). Those of type D are called 2-*disk orbifolds*.

All other compact two-dimensional smooth orbifolds have infinite fundamental group. It turns out that all of these are diffeomorphic to either flat or hyperbolic two-dimensional orbifolds. A *flat two-dimensional orbifold* is A/G, where $A \subset \mathbf{R}^2$ is a region bounded by straight lines and G is a discrete group of euclidean motions leaving A invariant. A *hyperbolic orbifold* is K/Γ, where $K \subset \mathbb{H}^2$ is a region bounded by geodesics (here $\mathbb{H}^2$ is the hyperbolic plane) and Γ is a discrete group of hyperbolic isometries which acts leaving K invariant.

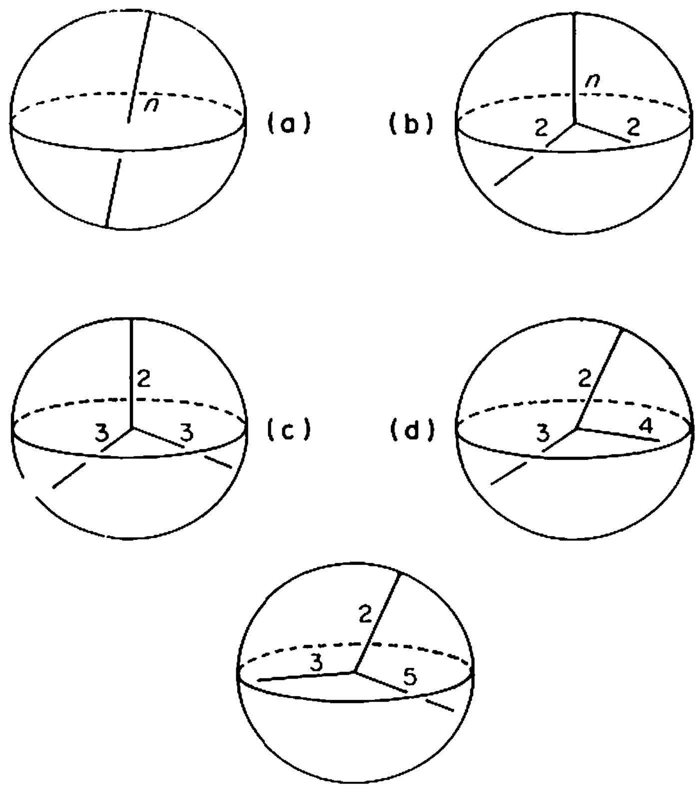
Let X be a two-dimensional orbifold whose underlying space is Q and whose exceptional set is $K \subset Q$. Suppose that all local group of X are cyclic subgroups of SO(2). Then K is a discrete set of points in Q, and we can view ρ as a function from K to the natural numbers greater than one—ρ assigns to each $k \in K$ the order of the local group at k. An orbifold covering $Y \to X$ induces a ramified covering of Q, ramified over K with index of ramification over $k \in K$ dividing $\rho(k)$. Conversely, any ramified covering of this type corresponds to an orbifold cover of X.

3. Three-Dimensional Orbifolds

We shall consider three-dimensional orbifolds whose local groups are contained in SO(3). These are called *locally orientable* three-dimensional orbifolds. The finite subgroups of SO(3) are (1) the cyclic groups, (2) the dihedral groups, (3) the tetrahedral group, (4) the octahedral group, and (5) the icosahedral group. The quotients of D^3 by these groups are shown in Fig. 3.1. In each case the orbit space is homeomorphic to D^3. In case (a) the exceptional set is a line segment; the label n means that the local group is cyclic of order n. In cases (b)–(e), the exceptional set is a cone on three points. The central vertex is the image of the origin; it is the only point with noncyclic local group type.

If X is a locally orientable, three-dimensional orbifold, then we can extract the *underlying triple* (Q, K, ρ). The first element, Q, is the underlying space; the second, $K \subset Q$, is the exceptional set; and the third, ρ, is the function which assigns local group type to each point. In this case we can view ρ as a function from Q-{vertices of K} to the positive integers. Each point $q \in Q$ that is not a vertex has cyclic local group. We think of ρ as associating to that point the order of the local group. From this function one can recover the local group at each point of Q.

If (Q, K, ρ) is a triple, which locally near each point of Q is of one of the five types in Fig. 3.1, then there is an orbifold structure on Q whose underlying triple is (Q, K, ρ). There are many such structures. Any two are isomorphic by an isomorphism that is arbitrarily close to the identity on Q.



(e)
Figure 3.1

A three-dimensional orbifold is of *cyclic type* if all its local group types are represented by cyclic subgroups of SO(3). This simply means that the exceptional set K is a disjoint union of circles. An orbifold is of *dihedral type* if each of its vertices is dihedral, or equivalently, if each local group type is represented either by a cyclic or dihedral subgroup of SO(3).

Suppose that X is a three-dimensional, locally orientable orbifold whose underlying triple is (Q, K, ρ). Covering spaces of X are exactly ramified coverings of Q (with the total space being a topological manifold) which are ramified over K, so that above any $q \in$ (K-vertices) the index of ramification divides ρ (q). This means that any covering $Y \to X$ yields such a ramified covering on the underlying triples and, conversely, any ramified covering $P \to Q$ of the type specified above yields a covering of orbifolds $Y \to X$.

The universal cover of X corresponds to the universal ramified covering of (Q, K, ρ). The orbifold X is good if in the universal ramified covering the indices of ramification over any $q \in$ (K-vertices) do not merely divide $\rho(q)$ but are equal to $\rho(q)$.

There is an explicit description of the fundamental group of the orbifold X in terms of the underlying triple (Q, K, ρ). Let $\Gamma = \pi_1(Q - K)$. Number the edges and circles of K as $e_1, \ldots, e_T$. Let $\mu_i \in \Gamma$ be the class of the *meridian* around e_i. It is well-defined up to conjugation and taking inverses in Γ. The fundamental group $\pi_1^{\mathrm{orb}}(X)$ is then $\Gamma/\{(\mu_1)^{\rho(e_1)}, \ldots, (\mu_T)^{\rho(e_T)}\}$. The kernel of $\Gamma \to \pi_1^{\mathrm{orb}}(X) \to 1$ corresponds to an unramified cover of $Q - K$. This covering can be completed in only one way to form a ramified covering $\tilde{Q}_{\mathrm{ram}} \to Q$, which is the universal covering of the orbifold.

REMARK 3.1. The order of μ_i in $\pi_1^{\mathrm{orb}}(X)$ divides $\rho(e_i)$. The orbifold X is good if and only if the order of each μ_i is equal to $\rho(e_i)$. If X has finite fundamental group and is of dihedral type, then it is automatically good. Basically this follows from the fact that Q is a rational homology 3-sphere.

Let Y be a two-dimensional orbifold and X a three-dimensional orbifold. An embedding of Y in X is a mapping $f: Y \to X$ that is a homeomorphism onto its image such that for each $x \in X$ there is an orbifold chart $\varphi: D^3/G \hookrightarrow X$ centered at x so that either.

(1) $\varphi(D^3/G) \cap f(Y) = \emptyset$, or

(2) $G \subset \mathrm{O}(2) \subset \mathrm{O}(3)$; $f(Y) \cap \varphi(D^3/G) = \varphi((D^2 \times \{0\})/G)$; and $f^{-1} \circ \varphi: D^2/G \hookrightarrow Y$ is an orbifold chart (in the structure of Y) centered at $f^{-1}(x)$.

If $Y^2 \to X^3$ is an embedding, then we can cut X^3 open along Y. The result is a (smooth) orbifold with boundary. (If Y meets the boundary of X, then it is necessary to "round the corners" at $\partial Y = \partial X \cap Y$.)

If Y^2 is an orientable spherical orbifold, then $Y^2 \cong S^2/G$ for some $G \subset \mathrm{O}(3)$. Thus Y is the boundary of D^3/G. A three-dimensional orbifold is *prime* if whenever $Y^2 \hookrightarrow X^3$ is an embedding of a spherical orbifold which locally separates X, then Y bounds an orbifold isomorphic to D^3/G in X. If $Y^2 \hookrightarrow X^3$ is an embedding of a spherical orbifold in X that separates, then we can write $X = X_1 \cup_Y X_2$, where X_1 and X_2 are orbifolds with Y as a boundary component. If $Y \cong S^2/G$, then we let $\hat{X}_i = X_i \cup_Y D^3/G$. We say that X is the *connected sum* of $\hat{X}_1$ and $\hat{X}_2$; $X \cong \hat{X}_1 \# \hat{X}_2$. If Y separates, but one side, say X_2, is isomorphic to D^3/G, then $\hat{X}_2 \cong S^3/G$ for some $G \subset \mathrm{SO}(3) \subset \mathrm{SO}(4)$. Otherwise, X is said to be a *nontrivial* connected sum.

PROPOSITION 3.2. *Let X be a closed three-dimensional orbifold of cyclic type with finite fundamental group. Then $X \cong X_1 \# \Sigma$, where X_1 is prime and Σ is a simply connected manifold.*

Proof. If X is a 3-manifold, then the result follows immediately from the existence of a prime decomposition and the Seifert–van Kampen theorem. Let us assume that X is not a manifold. Let Q be the underlying space for X and $K \subset Q$ the exceptional set. We know that Q is a 3-manifold. The prime decomposition theorem for manifolds allows us to write $Q \cong Q_1 \# \Sigma$, where

$K \subset (Q_1 - \text{(3-ball)}) \subset Q_1 \# \Sigma$ and where $Q_1 - K$ is prime. This induces a connected sum decomposition $X \cong X_1 \# \Sigma$, where Σ is a manifold.

We claim that Σ is simply connected. Since X is good and X is not a manifold, X_1 is good and X_1 is not a manifold. Thus, $\pi_1^{\text{orb}}(X_1) \neq \{e\}$. Since $\pi_1^{\text{orb}}(X) = \pi_1^{\text{orb}}(X_1) * \pi_1(\Sigma)$ is finite, $\pi_1(\Sigma) = \{e\}$.

Finally, it remains to show that X_1 is prime. If $Y^2 \hookrightarrow X_1$ is an embedded spherical orbifold which locally separates, then Y^2 is of cyclic type. We shall show that Y^2 is the boundary in X of an orbifold isomorphic to D^3/G for some $G \subset \text{SO}(3)$. The underlying space of Y is a closed 2-manifold. Since $\pi_1^{\text{orb}}(X_1)$, and hence $\pi_1(Q_1)$, is finite, this 2-manifold must separate Q_1. Thus Y^2 meets every component of K in an even number of points. If $Y^2 \cap K = \emptyset$, then Y^2 is diffeomorphic to a 2-sphere. Since $Q_1 - K$ is prime, Y^2 bounds a 3-ball in X_1. If $Y^2 \cap K \neq \emptyset$, then Y^2 must be a 2-sphere with two points labeled n. We have a decomposition $X_1 = X_1' \cup_Y X_2''$. Since X_1 is good, $\pi_1^{\text{orb}}(Y) \to \pi_1^{\text{orb}}(X)$ injects. Thus we have a free product with amalgamation decomposition

$$\pi_1^{\text{orb}}(X_1) = \pi_1^{\text{orb}}(X_1') *_{\mathbf{Z}/n\mathbf{Z}} \pi_1^{\text{orb}}(X_1'').$$

Since $\pi_1^{\text{orb}}(X_1)$ is finite, this decomposition must be trivial; i.e., either $Y \hookrightarrow X_1'$ or $Y \hookrightarrow X_2'$ must induce an isomorphism on orbifold fundamental groups. Suppose that it is $Y \hookrightarrow X_1'$. Let $\tilde{X}_1'$ be the universal cover of X_1'. It is a simply connected 3-manifold whose boundary is S^2. Thus, by the solution to the Smith conjecture, $X_1' \cong (D^3/G) \# \Sigma$, where $G \subset \text{O}(3)$ and Σ is a homotopy 3-sphere. Since $Q_1 - K$ is prime, this homotopy 3-sphere must be standard. Thus $X_1' \cong D^3/G$. This proves that X_1 is prime. ■

4. Seifert-Fibered Orbifolds

In this section we shall introduce the notion of a Seifert fiber structure for a locally orientable three-dimensional orbifold. This will generalize the classical notion for 3-manifolds. The basic reason for introducing Seifert orbifolds is that they provide a bridge to quotients of linear group actions in S^3.

Let $S^1 \subset \text{SO}(4)$ be the subgroup of matrices

$$\begin{pmatrix} R(\zeta) & 0 \\ 0 & R(\zeta) \end{pmatrix}, \quad \zeta \in S^1,$$

where $R(\zeta)$ is the matrix for rotation by angle ζ in the plane. If we take the natural action of SO(4) on S^3 and restrict it to this circle, then the result is a free action. The projection mapping to the quotient space, $p: S^3 \to S^2$, is the Hopf fibration. Let $N_{\text{SO}(4)}(S^1)$ be the normalizer of this circle in SO(4).

It is an extension of the unitary group $U(2)$, by $\mathbf{Z}/2\mathbf{Z}$. There is also an extension

$$1 \to S^1 \to N_{SO(4)}(S^1) \xrightarrow{p} O(3) \to 1.$$

In this sequence the action of O(3) on S^1 is given by $g\zeta g^{-1} = \zeta^{\det(p(g))}$ for all $g \in N_{SO(4)}(S^1)$.

Suppose that $G \subset N_{SO(4)}(S^1)$ is a finite subgroup. Since G normalizes the S^1, its action on S^3 sends fibers of the Hopf fibration to fibers. The induced action on the fibers is either complex linear or complex antilinear. There is induced on the quotient orbifold S^3/G a decomposition into one-dimensional sets; namely, the images of the fibers of the Hopf fibration. Most of these images are circles, but it is possible for an image to be an interval. This happens when there is an element $g \in G$ which leaves a fiber invariant but acts on this fiber in an orientation-reversing manner.

It is exactly these one-dimensional decompositions of three-dimensional linear orbifolds which serve as models for Seifert-fibered orbifolds.

DEFINITION 4.1. Let X be a locally orientable, three-dimensional orbifold. Let $\mathscr{F}$ be a decomposition of X into intervals and circles. We say that $\mathscr{F}$ *is a (smooth) Seifert fibration of* X if for each element $T \in \mathscr{F}$ there are

(1) an open set $U_T \subset X$, containing T, that is a union of elements of $\mathscr{F}$;
(2) a finite finite subgroup $G_T \subset N_{SO(4)}(S^1)$;
(3) a G_T-invariant open set $V_T \subset S^3$, which is a union of Hopf fibers; and
(4) a smooth isomorphism $\varphi_T\colon V_T/G_T \to U_T$ so that φ_T carries the decomposition of V_T/G_T by images of Hopf fibers to the decomposition which $\mathscr{F}$ induces on U_T.

We say that X is *Seifert-fibered* if it admits a Seifert fibration structure.

It is clear from this definition that if $G \subset N_{SO(4)}(S^1)$ is a finite subgroup, then the orbifold S^3/G has a natural Seifert fibration induced by the Hopf fibration in S^3. The base space of this natural Seifert fibration on S^3/G is the orbifold $S^2/p(G)$. The Hopf fibration induces a continuous map $\pi\colon S^3/G \to S^2/p(G)$, which is called the *projection* of the Seifert fibration. The fibers of π form the decomposition. Thus $\pi\colon S^3/G \to S^2/p(G)$, on the level of underlying spaces, is just the quotient mapping of the decomposition.

In general, if X is an orbifold with a smooth Seifert fibration $\mathscr{F}$, then there is a quotient space A for the decomposition and a continuous map $\pi\colon Q \to A$, where Q is the underlying space of X. Since X is locally isomorphic to S^3/G, A is locally isomorphic to $S^2/p(G)$. These local isomorphisms define on A the structure of a smooth two-dimensional orbifold. This orbifold is denoted B and is called the *base of the Seifert fibration*. The map $\pi\colon X \to B$ is called the *projection*.

Suppose that $\pi: X \to B$ is the projection of a Seifert fibration of an orbifold and that $\mu: C \to B$ is a covering of two-dimensional orbifolds. Form the "fiber product" of X and C over B, and call the result Y,

$$\begin{array}{ccc} Y & \xrightarrow{\pi'} & C \\ {\scriptstyle \mu'}\downarrow & & \downarrow{\scriptstyle \mu}. \\ X & \xrightarrow{\pi} & B \end{array}$$

(Some care is needed in defining this "fiber product;" it is *not* the set theoretic fiber product.) The orbifold structure on X induces one on Y so that $\mu': Y \to X$ is a covering of orbifolds. The fibers of $\pi': Y \to C$ give a Seifert fibration on Y with base C. If $\mu: C \to B$ is a regular covering with group of covering transformations G, then the G-action on C induces a G-action on Y. The quotient orbifold is X. If C is a manifold, i.e., if the exceptional set is empty, then $\pi': Y \to C$ is a smooth circle bundle.

In the next two lemmas we shall consider two consequences of this fiber product construction.

LEMMA 4.2. *Suppose that $\pi: X \to B$ is the projection of a Seifert fibration on an orbifold. Let $b \in B$ be a point with local group $G_b \subset \mathrm{O}(2)$. Then there is a neighborhood N of $\pi^{-1}(b) \subset X$ of the form $(S^1 \times D^2)/G_b$, where G_b acts orthogonally on both factors and where $\pi | N$ is induced by projection onto the second factor.*

Proof. The point b has a neighborhood in B of the form D^2/G_b, where $G_b \subset \mathrm{O}(2)$. Let N be the preimage of this neighborhood in X. Form the fiber product of N and D^2 over D^2/G_b. Call the result $\tilde{N}$. Then $\tilde{N}$ is a smooth circle bundle over D^2. Moreover, G_b acts on it by bundle maps. It follows that there is a product structure $\tilde{N} = S^1 \times D^2$, where G_b acts orthogonally on both factors. Hence $N \cong \tilde{N}/G_b \cong (S^1 \times D^2)/G_b$, and $\pi | N$ is induced by projection onto the second factor. ■

REMARK. One consequence of Lemma 4.2 is that the orbifold X must be of dihedral type.

LEMMA 4.3. *Suppose that $\pi: X \to B$ is the projection of a Seifert fibration on an orbifold. Then π induces a surjection $\pi_1^{\mathrm{orb}}(X) \to \pi_1^{\mathrm{orb}}(B) \to 1$. The kernel of this homomorphism is a cyclic normal subgroup generated by the class of a generic fiber in the Seifert fibration. This kernel is nontrivial if $\partial X \neq \varnothing$.*

Proof. Form the fiber product of X and the universal cover $\tilde{B} \to B$. Call the result Y. Then $\pi_1^{\mathrm{orb}}(B)$ is the group of covering transformations of Y and the surjection $\pi_1^{orb}(X) \to \pi_1^{orb}(B) \to 1$ is induced by π. If B is good, then $\tilde{B}$ is either $\mathbf{R}^2$, S^2, or D^2, and $Y \to \tilde{B}$ is a smooth circle bundle. Hence $\pi_1^{orb}(Y)$ is cyclic and generated by the class of the fiber. If $\pi_1^{orb}(Y)$ is trivial and B is good, then $\tilde{B} = S^2$ and X is closed.

If $\tilde{B}$ is bad, then $\tilde{B}$ is either S^2 with one exceptional point or S^2 with two exceptional points with local groups having relatively prime order. It is easy to see that once again $\pi_1^{orb}(Y)$ is generated by the class of the generic fiber. Also, Y, and hence X, is closed in this case. ■

Definition. If each local group of B is cyclic, then the Seifert fibration $\pi: X \to B$ is said to be of *restricted type*.

We next prove a technical proposition about Seifert fibrations of restricted type that will be useful later.

Proposition 4.4. *Let X^3 be an orbifold of cyclic type and let Q be the underlying topological space. Suppose that Q is Seifert-fibered (in the classical sense) so that the exceptional set $K \subset Q$ is a union of fibers. Then X is a smooth Seifert-fibered orbifold.*

First we consider a simple relative version of the proposition.

Lemma 4.5. *Let X be an orbifold with underlying topological space $S^1 \times D^2$ and exceptional set $S^1 \times \{0\}$. Suppose that there is given a Seifert fibration on ∂X that extends to a Seifert fibration (in the classical sense) on $S^1 \times D^2$. Then the Seifert fibration on ∂X extends to one for the orbifold X.*

Proof of Lemma 4.5. Suppose that $\rho(S^1 \times \{0\}) = n$. Let $Y \to X$ be the cyclic n-sheeted cover (branched along $S^1 \times \{0\}$). Then Y is a manifold. The fibration on ∂X lifts to one on ∂Y which is invariant under the $\mathbf{Z}/n\mathbf{Z}$-action. It is easy to extend this to a Seifert fibration of Y which is invariant under $\mathbf{Z}/n\mathbf{Z}$. Taking the quotient yields the Seifert fibration of X. ■

Proof of Proposition 4.4. Let $\nu(K)$ be a disk bundle neighborhood of K in Q. By hypothesis, there is a smooth Seifert fibration of $Q - \nu(K)$. Use Lemma 4.5 on each component of $\nu(K)$ to extend this to a Seifert fibration for X. ■

Finally, notice that if $\pi: X \to B$ is the projection of a Seifert fibration and $x \in X$ has local group G_x, then G_x is a subgroup of the local group $G_{\pi(x)}$. Thus if $K \subset X$ is a circle of points with local group cyclic of order >2, then $\pi(K)$ consists of points with local group of order larger than 2. This implies that $\pi(K)$ is a single point. This proves the following lemma.

LEMMA 4.6. *If $K \subset X$ is a circle of points whose local group has order greater than 2 and if $\mathscr{F}$ is a Seifert fibration of X, then K is a fiber of $\mathscr{F}$.*

5. Seifert-Fibered Orbifolds and Linear Actions

In [11], Seifert and Threlfall identified the Seifert fiber spaces with finite fundamental group with the orbit spaces of finite subgroups of SO(4) acting freely on S^3. In this section we shall generalize this to Seifert orbifolds and nonfree actions. Thus our goal is to identify Seifert orbifolds with finite fundamental group with linear orbifolds coming from the action of finite subgroups of $N_{SO(4)}(S^1)$ on S^3. As we saw in the last section, all such linear orbifolds are Seifert-fibered. Here we prove the converse.

THEOREM 5.1. *Let X^3 be a good, orientable three-dimensional orbifold with finite fundamental group. Suppose that we are given a Seifert fibration for X. There is a subgroup $G \subset N_{SO(4)}(S^1)$ so that X and S^3/G are isomorphic as orbifolds.*

By Remark 3.2, X is good, and thus $\tilde{X}$ is a simply connected 3-manifold. Since it is Seifert-fibered, it is diffeomorphic to S^3. Let $G = \pi_1^{orb}(X)$. Let $\mathscr{D}$ be the group of orientation-preserving diffeomorphisms of S^3. Identifying $\tilde{X}$ with S^3 gives a representation $G \hookrightarrow \mathscr{D}$. We wish to show that this representation is conjugate to one whose image is contained in $N_{SO(4)}(S^1)$.

Let B be the base of the given Seifert fiber structure on X, and let $\pi\colon X \to B$ be the projection. The argument is divided into two cases, depending on whether B is a good orbifold. Let $\tilde{B}$ denote the universal cover of B.

Case I. B is not good. Let $N = (O(2) \times O(2)) \cap SO(4)$. It is our plan to construct an effective action of N on $\tilde{X}$ so that the group of covering transformations G acts as a subgroup of N. Since it is easy to see that any such N-action on $\tilde{X}$ is equivariantly diffeomorphic to the standard linear N-action on S^3, this will prove that $X = \tilde{X}/G$ is isomorphic to $S^3/\rho(G)$, where ρ is the linear representation $G \hookrightarrow N \subset N_{SO(4)}(S^1)$.

According to the list in Section 2, if B is not good, then either it is S^2 with at most two exceptional points or it is D^2 with at most two dihedral points on the boundary (See Fig. 5.1). Hence B decomposes as

$$B = B_1 \bigcup_{\partial B_1} (\partial B_1 \times I) \bigcup_{\partial B_2} B_2,$$

where $B_i \cong D^2/G_i$ for some $G_i \subset O(2)$. (Actually, the argument in this case will work for any such B.) Let $X_i = \pi^{-1}(B_i)$ and let $Y = \pi^{-1}(\partial B_1 \times I)$. Also let, $\partial_i Y = Y \cap X_i$. Clearly,

$$X = X_1 \bigcup_{\partial_1 Y} Y \bigcup_{\partial_2 Y} X_2.$$

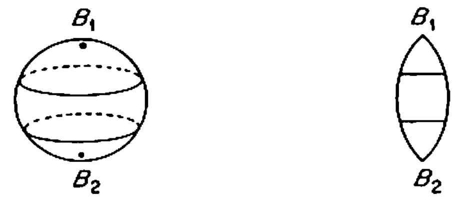


Figure 5.1

Let $\tilde{B}_i \subset \tilde{B}$ be the preimage of B_i. Each $\tilde{B}_i$ is connected. Let $\tilde{X}_i$ and $\tilde{Y}$ be the preimages of X_i and Y, respectively, in $\tilde{X}$. They are also connected. The group G acts on $\tilde{X}_i$, $\tilde{Y}$, and $\partial_i \tilde{Y}$ with quotients X_i, Y, and $\partial_i Y$.

There is a natural action of N on $S^1 \times D^2$ given by the standard linear action of O(2) on each factor. This action has principal isotropy group $\mathbf{Z}/2\mathbf{Z}$ and has one singular orbit—the core. By Lemma 4.2, X_i is isomorphic to $(S^1 \times D^2)/G_i$, where G_i acts linearly on each factor. Such an identification induces one of $\tilde{X}_i$ with $S^1 \times D^2$, so that G acts as a subgroup of N. ($\tilde{X}_i$ is a manifold since X is good, and $\tilde{X}_i$ is a finite cover of X_i since G is finite.) Choose such identifications and let $\psi_i: N \times \tilde{X}_i \to \tilde{X}_i$ be the resulting actions. Also, let $\lambda_i: \partial_i Y \to T^2/G$ be the resulting identifications.

Since Y is Seifert-fibered with base $\partial B_1 \times I$, it is diffeomorphic to $\partial_1 Y \times I$. Choose an identification $\lambda: Y \to (T^2/G) \times I$ so that $\lambda | \partial_1 Y = \lambda_1$. Then λ defines an N-action on $\tilde{Y}$, so that $G \subset N$, and so that, when restricted to $\partial_1 \tilde{Y}$, this action agrees with ψ_1 restricted to $\partial \tilde{X}_1$. The problem is that, in general, the restriction of this action to $\partial_2 \tilde{Y}$ will differ from the action of ψ_2 restricted to $\partial \tilde{X}_2$. However, we shall show that after changing the action $\psi_2: N \times \tilde{X}_2 \to \tilde{X}_2$ by an automorphism of N, these actions become equivalent. This will allow us to amalgamate these actions into an N-action on $\tilde{X}$.

The group N is a semidirect product $T^2 \times \mathbf{Z}/2\mathbf{Z}$. We denote the elements of N by pairs (t, ω^i), where $t \in T^2$ and ω is the generator of $\mathbf{Z}/2\mathbf{Z}$. The multiplication is defined by $(t, \omega^i)\,(t', \omega^j) = (t + (-1)^i t', \omega^{i+j})$. Consider the natural action on N of $S^1 \times S^1 \subset S^1 \times D^2$. A map $\alpha: S^1 \times S^1 \to S^1 \times S^1$ will be called an *affine map* if it is the composition of a map induced by $A \in \mathrm{GL}(2, \mathbf{Z})$ acting linearly on $\mathbf{R}^2$ and a map which is translation by $\tau \in T^2$. If $\alpha: S^1 \times S^1 \to S^1 \times S^1$ is an affine map, then there is a Lie group automorphism $i_\alpha: N \to N$ so that

$$\alpha(n \cdot x) = i_\alpha(n) \cdot \alpha(x)$$

for all $n \in N$ and $x \in S^1 \times S^1$. The formulas for i_α are

$$i_\alpha(t, 1) = (At, 1),$$

$$i_\alpha(t, \omega) = (At + 2\tau, \omega).$$

LEMMA 5.2. *Let $G \subset N$ be a finite subgroup. Any isomorphism $\varphi\colon T^2/G \to T^2/G$ is isotopic (through orbifold diffeomorphisms) to one which lifts to an affine map on T^2.*

Proof. If $G \subset SO(2) \times SO(2)$, then T^2/G is again a torus. Any diffeomorphism of a torus is isotopic to a linear isomorphism, and any linear map on T^2/G lifts to an affine map on T^2.

If G is not contained in $SO(2) \times SO(2)$, then let $G_0 \subset G$ be the subgroup of index two given by $G_0 = G \cap (SO(2) \times SO(2))$. Then T^2/G_0 is a torus and is a double cover of T^2/G. The orbifold T^2/G is isomorphic to S^2, with four points labeled 2. The lemma will be established if we can show that φ can be deformed until it lifts to an affine map on T^2/G_0. First, we may assume that φ is orientation-preserving. Indeed, if $A \in GL(2, \mathbf{Z})$ has determinant -1, then it induces a linear map $L_A\colon T^2/G_0 \to T^2/G_0$ covering $\bar{L}_A\colon T^2/G \to T^2/G$; and either φ or $\varphi \circ \bar{L}_A$ is orientation-preserving. Let $[0] \in T^2/G$ be the image of $0 \in \mathbf{R}^2$ (this is one of the four distinguished points). Secondly, we may assume, by composing φ with the image of a translation of order two, that $\varphi([0]) = [0]$. We can view such a φ as an orientation-preserving homeomorphism of S^2 which fixes $[0]$ and which leaves invariant the other three distinguished points. Birman [1] showed that the group of isotopy classes of such φ is a braid group isomorphic to $PSL(2, \mathbf{Z})$. Furthermore, this isomorphism can be realized as follows. Begin with $A \in SL(2, \mathbf{Z})$, and let $L_A\colon T^2/G_0 \to T^2/G_0$ be its linear map. Then L_A induces an isomorphism $\bar{L}_A\colon T^2/G \to T^2/G$. The correspondence $A \to \bar{L}_A$ factors through $PSL(2, \mathbf{Z})$ to give the isomorphism. It follows that every $\varphi\colon T^2/G \to T^2/G$ is isotopic one which lifts to an affine map $T^2/G_0 \to T^2/G_0$. ■

By using this lemma, we may assume that the identification $\lambda\colon Y \to (T^2/G) \times I$ is such that $\lambda|\partial_1 Y = \lambda_1$ and $\lambda_2 \circ (\lambda|\partial_2 Y)^{-1}$ lifts to an affine mapping α on T^2. This λ defines an action $\mu\colon N \times \tilde{Y} \to \tilde{Y}$, so that $\mu|\partial_1 \tilde{Y} = \psi_1|\partial\tilde{X}_1$ and $\mu|\partial_2\tilde{Y} = (\psi_2 \circ i_\alpha)|\partial\tilde{X}_2$.

Thus we can amalgamate these three actions, ψ_1, μ, and $\psi_2 \circ i_\alpha$, to define an effective action $\psi\colon N \times \tilde{X} \to \tilde{X}$, so that $G \subset N$ and so that $\psi|G$ is the action of the group of covering transformations.

We can write $\tilde{X} \cong T_1 \cup T_2$, where each T_i is an N-invariant solid torus. The standard linear N-action on S^3 has a similar decomposition. By using these decompositions, one can easily construct an N-equivariant diffeomorphism from $\tilde{X}$ to S^3.

Case II. B is a good orbifold. In this case $\tilde{B}$ is isomorphic to S^2 and B is isomorphic to S^2/H for some $H \subset O(3)$. Let $Y \to X$ be the induced covering of X. We know that Y is Seifert-fibered with base $\tilde{B}$. Moreover, since $\tilde{B}$ has no exceptional points, $Y \to \tilde{B}$ is a smooth circle bundle. As a result, the induced Seifert fibration on $\tilde{X}$ is also a smooth circle bundle over S^2.

LEMMA 5.3. *There is a smooth, free S^1-action on $\tilde{X}$ so that the orbits are the fibers of the induced Seifert fibration and so that $G = \pi_1^{\mathrm{orb}}(X)$ normalizes the S^1-action.*

Proof. Each $b \in B$ has a neighborhood U_b isomorphic to D^2/G_b for some $G_b \subset O(2)$. By Lemma 4.2, the preimage of U_b in X is of the form $(S^1 \times D^2)/G_b$, where G_b acts orthogonally on each factor. For each $p \in D^2$ choose a distance function on $S^1 \times \{p\}$, so that this family of distance functions is G_b-invariant and smooth in p and so that the total length of each circle is $2\pi/n$, where n is the order of the kernel of $\pi_1^{\mathrm{orb}}(X) \to \pi_1^{\mathrm{orb}}(B)$. These distance functions define ones on the fibers of $(S^1 \times D^2)/G_b \to D^2/G_b$. Cover B by a finite collection of such open sets U_b and choose product structures and distance functions as above. Choose a smooth partitition of unity subordinate to this cover and use this to average the distance functions on the base.

If we pull back these distance functions to the Seifert fibration $Y \to \tilde{B}$, then we have an S^1-fibration over S^2 with each fiber having total length $2\pi/n$. Pulling back to $\tilde{X}$, the total length of each fiber is 2π. Such a smooth family of distance functions, together with a choice of orientation, gives a free circle action on $\tilde{X}$ whose orbits are the fibers. Since the distance functions are G-invariant, this circle action is normalized by G. This completes the proof of the lemma. ■

Any two smooth, free S^1-actions on S^3 are equivalent (since they give principal S^1-bundles over S^2). In particular, any such action is equivalent to the standard linear action of $S^1 \subset U(2)$, $\zeta \to \left(\begin{smallmatrix}\zeta & 0\\ 0 & \zeta\end{smallmatrix}\right)$. Hence there is an identification of $\tilde{X}$ with S^3 in such a way that the circle action on $\tilde{X}$ normalized by G becomes $S^1 \subset U(2)$. We fix such an identification.

Recall that $\mathscr{D}$ is the group of orientation-preserving diffeomorphisms of S^3, with group multiplication defined by composition. Let $N_{\mathscr{D}}(S^1)$ be the subgroup of $\mathscr{D}$ which normalizes the standard S^1, and let $C_{\mathscr{D}}(S^1)$ be the centralizer (it is a subgroup of index 2 in $N_{\mathscr{D}}(S^1)$). The above identification of $\tilde{X}$ with S^3 gives a representation $G \hookrightarrow N_{\mathscr{D}}(S^1)$, where $G = \pi_1^{\mathrm{orb}}(X)$. We are trying to show that G is actually conjugate to a subgroup of $N_{SO(4)}(S^1)$.

If $f \in N_{\mathscr{D}}(S^1)$, then define a diffeomorphism $\bar{f}\colon S^2 \to S^2$ by

$$\bar{f}(\pi(z)) = \pi(f(z)),$$

where $\pi\colon S^3 \to S^2$ is the Hopf fibration. Let $\varepsilon\colon N_{\mathscr{D}}(S^1) \to \{\pm 1\} = \mathrm{Aut}(S^1)$ be the natural map. If f flips S^1, then, since f is orientation-preserving, $\bar{f}$ must be orientation-reversing. Hence $\varepsilon(f) = \deg(\bar{f})$.

Let $C^\infty(S^2, S^1)$ be the group of smooth mappings from S^2 to S^1. (The group multiplication is pointwise multiplication.) If $\psi \in C^\infty(S^2, S^1)$, then define a diffeomorphism $\hat{\psi} \in C_{\mathscr{D}}(S^1)$ by

$$\hat{\psi}(z) = \psi(\pi(z)) \cdot z,$$

where the dot denotes the standard S^1-action on S^3. There are exact sequences

$$1 \to C^\infty(S^2, S^1) \xrightarrow{i} N_{\mathscr{D}}(S^1) \xrightarrow{p} \mathrm{diff}(S^2) \to 1,$$
$$1 \to C^\infty(S^2, S^1) \xrightarrow{i} C_{\mathscr{D}}(S^1) \xrightarrow{p} \mathrm{diff}_+(S^2) \to 1,$$

where p is the homomorphism $f \to \bar{f}$ and i is $\psi \to \hat{\psi}$.

Lemma 5.4. *If $f \in N_{\mathscr{D}}(S^1)$ and $\psi \in C^\infty(S^2, S^1)$, then $f^{-1} \circ \hat{\psi} \circ f = \hat{\lambda}$,* where $\lambda = (\psi \circ \bar{f})^{\varepsilon(f)}$.

Proof. If $\alpha \in S^1$ and $z \in S^3$, then $f(\alpha \cdot z) = \alpha^{\varepsilon(f)} \cdot f(z)$. Thus

$$\begin{aligned} f \circ \hat{\lambda}(z) &= f(\psi(\bar{f} \circ \pi(z))^{\varepsilon(f)} \cdot z) \\ &= \psi(\bar{f}\pi(z)) \cdot f(z) \\ &= \psi(\pi(f(z)) \cdot f(z) \\ &= \hat{\psi}(f(z)). \quad ■ \end{aligned}$$

Lemma 5.5. *Any homomorphism $\mu: G \to \mathrm{diff}(S^2)$, with G a finite group, is conjugate to a homomorphism with image in* O(3).

Proof. The orbifold $S^2/\mu(G)$ is good and has finite fundamental group. Hence it is smoothly isomorphic to S^2/H for some $H \subset \mathrm{O}(3)$. An isomorphism $S^2/\mu(G) \to S^2/H$ lifts to a diffeomorphism of S^2 which conjugates $\mu(G)$ to H. ■

Let $\mathscr{G} \subset N_{\mathscr{D}}(S^1)$ be the full preimage of $\mathrm{O}(3) \subset \mathrm{diff}(S^2)$ and let $\mathscr{G}^+ \subset \mathscr{G}$ be the full preimage of SO(3). Thus $\mathscr{G}^+$ is the intersection of $\mathscr{G}$ and $C_{\mathscr{D}}(S^1)$.

Corollary 5.6. *Let $G \subset N_{\mathscr{D}}(S^1)$ be a finite group. There is an element $f \in N_{\mathscr{D}}(S^1)$ that conjugates G into $\mathscr{G}$. If $G \subset C_{\mathscr{D}}(S^1)$, then f conjugates G into $\mathscr{G}^+$.*

Proof. By the above lemma, there is $\bar{f} \in \mathrm{diff}(S^2)$ that conjugates $p(G)$ into O(3). If $p(G) \subset \mathrm{diff}_+(S^2)$, then this conjugation sends $p(G)$ into SO(3). Lifting $\bar{f}$ to $f \in N_{\mathscr{D}}(S^1)$ gives the required element. ■

Consider the ladder of groups

$$\begin{array}{ccccccccc} 1 & \longrightarrow & C^\infty(S^2, S^1) & \xrightarrow{i} & \mathscr{G} & \xrightarrow{p} & O(3) & \longrightarrow & 1 \\ & & \uparrow\cup & & \uparrow\cup & & \| & & \\ 1 & \longrightarrow & S^1 & \longrightarrow & N_{SO(4)}(S^1) & \xrightarrow{p} & O(3) & \longrightarrow & 1. \end{array}$$

By applying Corollary 5.6, we conjugate $G = \pi_1^{orb}(X)$ into $\mathscr{G}$. We wish to do a further conjugation by an element of $C^\infty(S^2, S^1)$ to move G into $N_{SO(4)}(S^1)$. We shall first treat the special case in which $G \subset \mathscr{G}^+$.

Case IIA. $G \subset \mathscr{G}^+$. The centralizer of S^1 in SO(4) is $U(2)$. So in this case our ladder becomes

$$\begin{array}{ccccccccc} 1 & \longrightarrow & C^\infty(S^2, S^1) & \xrightarrow{i} & \mathscr{G}^+ & \xrightarrow{p} & SO(3) & \longrightarrow & 1 \\ & & \uparrow\cup & & \uparrow\cup & & \| & & \\ 1 & \longrightarrow & S^1 & \longrightarrow & U(2) & \longrightarrow & SO(3) & \longrightarrow & 1. \end{array}$$

First, we shall define a homomorphism $\rho\colon G \hookrightarrow U(2)$ so that $p \circ \rho = p|G$. Then we shall use a group cohomology argument to construct an element $\psi \in C^\infty(S^2, S^1)$, so that $\hat{\psi}$ conjugates G to $\rho(G)$.

Suppose that $c \in SO(3)$ and that n is an integer. For any $\varphi \in C^\infty(S^2, S^1)$, define a new function $\prod_n(\varphi, c) \in C^\infty(S^2, S^1)$ by

$$\prod_n(\varphi, c) = \prod_{i=0}^{n-1}(\varphi \circ c^i)$$

Similarly, if $\theta \in C^\infty(S^2, \mathbf{R}^1)$, define $\sum_n(\theta, c)$ by

$$\sum_n(\theta, c) = \sum_{i=0}^{n-1}(\theta \circ c^i).$$

Let $\exp\colon\mathbf{R}^1 \to S^1$ be the universal cover. By a *lifting* $\tilde{\varphi}$ of $\varphi \in C^\infty(S^2, S^1)$, we mean that $\tilde{\varphi} \in C^\infty(S^2, \mathbf{R}^1)$ and that $\exp \circ \tilde{\varphi} = \varphi$.

LEMMA 5.7. *Suppose that $f \in \mathscr{G}^+$ has order n. Then there is a $j \in U(2)$ of order n and a $\psi \in C^\infty(S^2, S^1)$ such that*

(i) $f = \hat{\psi} \circ j$ and
(ii) *ψ lifts to $\tilde{\psi}\colon S^2 \to \mathbf{R}^1$ with $\sum_n(\tilde{\psi}, \bar{f}) = 0$.*

Moreover, this decomposition of f is unique.

Proof. We have $\bar{f} \in SO(3)$ and $\bar{f}^n = 1$. Let l be any lift of $\bar{f}$. Then $l^n \in S^1$. Hence there is an element $\zeta \in S^1$ with $l^n = \zeta^{-n}$. Set $k = l\zeta$. Clearly, k is also a lift of $\bar{f}$ and $k^n = 1$. Since $\bar{k} = \bar{f}$, we have that $f = \hat{\varphi} \circ k$ for some $\varphi \in C^\infty(S^3, S^1)$. We have

$$\mathrm{id} = (\hat{\varphi} \circ k)^n = \prod_{i=0}^{n-1} (k^i \circ \hat{\varphi} \circ k^{-i}) \circ k^n = \prod_{i=0}^{n-1} k^i \circ \hat{\varphi} \circ k^{-i}.$$

Lemma 5.4 identifies this last product with the image of $\Pi_n(\varphi, k) = \Pi_n(\varphi, \bar{f})$. Hence $\Pi_n(\varphi, \bar{f}) = 1$. Let $\tilde{\varphi}: S^2 \to \mathbf{R}^1$ be any lifting of φ. Then $\sum_n(\tilde{\varphi}, \bar{f}) = m$ for some integer m. Define $\tilde{\psi} \in C^\infty(S^2, \mathbf{R}^1)$ by $\tilde{\psi}(x) = \tilde{\varphi}(x) - m/n$. Let $\omega \in S^1$ be defined by $\omega = \exp(m/n)$. Finally, let $j = \omega \cdot k$ and let $\psi = \exp \circ \tilde{\psi}$. Clearly, $f = \hat{\psi} \circ j$ is a decomposition as required.

Suppose that $f = \hat{\psi}' \circ j'$ is another such decomposition. Since $\bar{j}' = \bar{j} = \bar{f} \in SO(3)$, we have that $j' = \zeta \cdot j$ for some $\zeta \in S^1$. Thus $\tilde{\psi}'(x) = \tilde{\psi}(x) + d$ for some $d \in \mathbf{R}^1$ that projects onto ζ. Since $\sum_n(\tilde{\psi}', \bar{f}) = \sum_n(\tilde{\psi}, \bar{f}) = 0$ and $nd = \sum_n(\tilde{\psi}', \bar{f}) - \sum_n(\tilde{\psi}, \bar{f})$, it follows that $d = 0$ and, hence, that $\zeta = 1$. ■

For each $f \in \mathscr{G}^+$ of finite order, let $\rho(f) = j$ and $\psi_f = \psi$ be the above unique decomposition. Also, let $\tilde{\psi}_f \in C^\infty(S^2, \mathbf{R}^1)$ be the lifting of ψ_f that sums to zero over the $\bar{f}$-orbits.

Lemma 5.8. *Let $G \subset \mathscr{G}^+$ be a finite group. The map $f \to \rho(f)$ defines a homomorphism $\rho: G \to U(2)$. Furthermore,*

$$\tilde{\psi}_{f \circ g} = \tilde{\psi}_f + \tilde{\psi}_g \circ \bar{f}^{-1}.$$

Proof. We must show that $\rho(e) = e$ and that $\rho(f \circ g) = \rho(f) \circ \rho(g)$. The first is obvious. As for the second, we have $f = \hat{\psi}_f \circ \rho(f)$ and $g = \hat{\psi}_g \circ \rho(g)$. Hence $f \circ g = \hat{\psi}_g \circ \rho(f) \circ \hat{\psi}_g \circ \rho(g)$. Since $\rho(f)$ covers $\bar{f} \in SO(3)$, Lemma 5.4 implies that $\rho(f) \circ \hat{\psi}_g = (\psi_g \circ \bar{f}^{-1})\hat{} \circ \rho(f)$. Thus

$$(*) \qquad \hat{\psi}_{f \circ g} \circ \rho(f \circ g) = \hat{\psi}_f \circ (\psi_g \circ \bar{f}^{-1})\hat{} \circ \rho(f) \circ \rho(g)$$
$$= [\psi_f \cdot (\psi_g \circ \bar{f}^{-1})]\hat{} \circ \rho(f) \circ \rho(g).$$

The elements $\rho(f \circ g)$ and $\rho(f) \circ \rho(g)$ are in $U(2)$ and have the same image in SO(3). Thus there is an element $\zeta \in S^1$, so that

$$\zeta \cdot \psi_{f \circ g} = \psi_f \cdot (\psi_g \circ \bar{f}^{-1}).$$

This means that there is $d \in \mathbf{R}^1$ with $\exp(d) = \zeta$ so that

$$d + \tilde{\psi}_{f \circ g} = \tilde{\psi}_f + (\tilde{\psi}_g \circ \bar{f}^{-1}).$$

Summing over G-orbits gives

$$\mathrm{order}(G) \cdot d + \sum_{\alpha \in G} \tilde{\psi}_{f \circ g} \circ \bar{\alpha} = \sum_{\alpha \in G} \tilde{\psi}_f \circ \bar{\alpha} + \sum_{\alpha \in G} \tilde{\psi}_g \circ \bar{f}^{-1} \circ \bar{\alpha}.$$

Since $\tilde{\psi}_f$ sums over any f-orbit to zero and since a G-orbit is a disjoint union of f-orbits, $\sum_{\alpha\in G}\tilde{\psi}_f\circ\bar{\alpha}=0$. Similarly, $\sum_{\alpha\in G}\tilde{\psi}_{f\circ g}\circ\bar{\alpha}=0$ and

$$\sum_{\alpha\in G}\tilde{\psi}_g\circ\bar{f}^{-1}\circ\bar{\alpha}=0.$$

Thus $d=0$ and $\zeta=1$. This proves that $\zeta\cdot\psi_{f\circ g}=\psi_f\cdot(\psi_g\circ\bar{f}^{-1})$. It follows immediately from $(*)$ that $\rho(f\circ g)=\rho(f)\circ\rho(g)$. ∎

Let $G\subset\mathscr{G}^+$ be a finite group and let $\rho\colon G\to N_{\mathrm{SO}(4)}(S^1)$ be the homomorphism constructed above. Let $\tilde{\Psi}\colon G\to C^\infty(S^2,\mathbf{R}^1)$ be the function that assigns to $g\in G$ the element $\tilde{\psi}_g$. The last part of Lemma 5.8 says that $\tilde{\Psi}$ satisfies the cocycle condition. If we project $\tilde{\Psi}$ to $\Psi\colon G\to C^\infty(S^2,S^1)$, then the resulting cocycle is the "difference cocycle" for the two mappings of G into $\mathscr{G}^+$. Lemma 5.9 shows that if Ψ is a coboundary, then G and $\rho(G)$ are conjugate. We shall complete the proof of Case IIA by showing that $\tilde{\Psi}$ (and hence Ψ) is a coboundary.

LEMMA 5.9. *With notation as above, suppose that $\Psi\colon G\to C^\infty(S^2,S^1)$ is a coboundary, that is, suppose that there exists $\mu\in C^\infty(S^2,S^1)$, so that*

$$(\mu\circ\bar{g}^{-1})\circ\mu^{-1}=\Psi(g)$$

for all $g\in G$. Then $\hat{\mu}$ conjugates G to $\rho(G)$.

Proof.

$$\begin{aligned}(\hat{\mu}\circ g\circ\hat{\mu}^{-1})\circ\rho(g)^{-1}&=\hat{\mu}\circ(g\circ\hat{\mu}^{-1}\circ g^{-1})\circ\hat{\psi}_g\\&=[\mu\cdot(\mu^{-1}\circ\bar{g}^{-1})\cdot\Psi(g)]\hat{}\,.\end{aligned}$$

Thus $\hat{\mu}\circ g\circ\hat{\mu}^{-1}=\rho(g)$ if and only if $\Psi(g)=(\mu\circ\bar{g}^{-1})\cdot\mu^{-1}$. ∎

The next two lemmas establish that $\tilde{\Psi}$ is indeed a coboundary.

LEMMA 5.10. *Let ω be a generator of $\mathbf{Z}/n\mathbf{Z}$ and suppose that ω acts on D^2 by rotation through $2\pi/n$ radians. Let $\theta\in C^\infty(D^2,\mathbf{R}^1)$ be such that $\sum_n(\theta,\omega)=0$. Then there is $\beta\in C^\infty(D^2,\mathbf{R}^1)$ such that $\beta(\omega x)-\beta(x)=\theta(x)$ for all $x\in D^2$.*

Proof. Define

$$\beta(x)=-\sum_{i=0}^{n-2}\frac{n-i-1}{n}\theta(\omega^i x).$$

Since $\sum_n(\theta, \omega) = 0$, we have $-(1/n)\theta(\omega^{n-1}x) = \sum_{i=0}^{n-2}(1/n)\theta(\omega^i x)$. Hence

$$\begin{aligned}\beta(\omega x) - \beta(x) &= -\sum_{i=0}^{n-3}\frac{n-i-1}{n}\theta(\omega^{i+1}x) + \sum_{i=0}^{n-2}\frac{1}{n}\theta(\omega^i x)\\ &\quad + \sum_{i=0}^{n-2}\frac{n-i-1}{n}\theta(\omega^i x)\\ &= -\sum_{i=1}^{n-2}\frac{n-i}{n}\theta(\omega^i x) + \sum_{i=0}^{n-2}\frac{n-i}{n}\theta(\omega^i x)\\ &= \theta(x).\end{aligned}$$

LEMMA 5.11. *Let $\bar{\rho}: G \to \mathrm{SO}(3)$ be a representation of a finite group and suppose that $\tilde{\Psi}: G \to C^\infty(S^2, \mathbf{R}^1)$ satisfies the cocycle condition*

$$\tilde{\Psi}(f \circ g) = \tilde{\Psi}(f) + \tilde{\Psi}(g) \circ \bar{\rho}(f)^{-1}.$$

Then $\tilde{\Psi}$ is a coboundary, i.e., there is a function $\tilde{\mu} \in C^\infty(S^2, \mathbf{R}^1)$, so that $\tilde{\mu} \circ \bar{\rho}(g)^{-1} - \tilde{\mu} = \tilde{\Psi}(g)$ for all $g \in G$.

Proof. Suppose that $\bar{\rho}(f) = 1$. Then

$$(*) \qquad \tilde{\Psi}(f \circ g) = \tilde{\Psi}(f) + \tilde{\Psi}(g).$$

This means that, restricted to the kernel of $\bar{\rho}$, $\tilde{\Psi}$ is a homomorphism. Since $\ker \bar{\rho}$ is a finite group and $C^\infty(S^2, \mathbf{R}^1)$ has no elements of finite order, $\tilde{\Psi}|\ker \bar{\rho} \equiv 0$. By invoking (*) once again, we see that $\tilde{\Psi}$ factors through $\bar{\rho}(G)$ to define a cocycle on that group. In view of this it suffices to solve the problem for the group $\bar{\rho}(G)$. We may therefore assume that $\bar{\rho}$ in injective.

We can cover S^2 by G-invariant open sets $U_1, \ldots, U_k$, where each U_i has the form

$$U_i \cong G \times_{H_i} D^2.$$

where $H_i \subset G$ is a cyclic isotropy group. By the previous lemma we can find $\tilde{\mu}_i \in C^\infty(D^2, \mathbf{R}^1)$, so that $\tilde{\Psi}(g) = \tilde{\mu}_i \circ \bar{\rho}(g)^{-1} - \tilde{\mu}_i$ for all $g \in H_i$. Extend $\tilde{\mu}_i$ to U_i by using the same formula for all $g \in G$.

Let $\{\lambda_i\}$ be a G-invariant smooth partitition of unity subordinate to $\{U_i\}$ and define

$$\tilde{\mu} = \sum_{i=1}^k \lambda_i \tilde{\mu}_i.$$

Clearly, $\tilde{\Psi}(g) = \tilde{\mu} \circ \rho(g)^{-1} - \tilde{\mu}$. ■

Setting $\mu = \exp \circ \tilde{\mu}$, we have that $\hat{\mu}$ conjugates $G \subset \mathscr{G}^+$ to $\rho(G) \subset U(2)$. This completes the proof of Case IIA.

Case IIB. $G \subset \mathscr{G}$, *but* G *is not contained in* $\mathscr{G}^+$. Let $G' = G \cap \mathscr{G}^+$. Then G' is a subgroup of index 2 in G. By case IIA, we can conjugate G by an element of $C^\infty(S^2, S^1)$ so that G' is contained in $U(2)$. Choose arbitrarily an element $h \in G - G'$ and an element $k \in N_{SO(4)}(S^1)$ with $\bar{k} = \bar{h} \in O(3)$. There is an element $\varphi \in C^\infty(S^2, S^1)$, so that $k = \hat{\varphi} \circ h$.

LEMMA 5.12. $\varphi(x) = \varphi(\bar{h}^{-1}x)$ *for all* $x \in S^2$.

Proof. Both k^2 and h^2 are elements in $U(2)$ with the same image in SO(3). Thus there is $\zeta \in S^1$ with $\hat{\zeta} \circ h^2 = k^2$. Since $k = \hat{\varphi} \circ h$, by using Lemma 5.4, we see that $\hat{\zeta} = \hat{\varphi} \circ \hat{\lambda}$, where $\lambda = (\varphi \circ \bar{h}^{-1})^{-1}$. Hence $\zeta = \varphi(x) \cdot \varphi(\bar{h}^{-1}x)^{-1}$ for all x. Therefore, it suffices to show that $\zeta = 1$. Let $\tilde{\varphi} \in C^\infty(S^2, \mathbf{R}^1)$ be a lifting of φ. Then $\tilde{\zeta} = \tilde{\varphi}(x) - \tilde{\varphi}(\bar{h}^{-1}x)$ is a constant that projects to ζ. By summing over an $\bar{h}$-orbit, we see that $\tilde{\zeta} = 0$ and hence that $\zeta = 1$. ■

LEMMA 5.13. *If* $g \in G' = G \cap U(2)$, *then* $hgh^{-1} = kgk^{-1}$, *where* h *and* k *are as above.*

Proof. Again, hgh^{-1} and kgk^{-1} are both elements of $U(2)$ with the same projection in SO(3). Hence, their difference is an element $\zeta \in S^1$. Since $k = \hat{\varphi} \circ h$, $k \circ g \circ k^{-1} = \hat{\varphi} \circ (h \circ g \circ h^{-1}) \circ \hat{\varphi}^{-1}$. Set $f = h \circ g \circ h^{-1}$. Then

$$\zeta = f \circ (\hat{\varphi} \circ f \circ \hat{\varphi}^{-1})^{-1} = (f \circ \hat{\varphi} \circ f^{-1}) \circ \hat{\varphi}^{-1} = \hat{\lambda} \circ \hat{\varphi}^{-1},$$

where $\hat{\lambda} = \varphi \circ \bar{f}^{-1}$. Hence $\zeta = (\varphi \circ \bar{f}^{-1}) \cdot \varphi^{-1}$. By applying this equality at $x = \bar{h}y$, we find that $\zeta = \varphi(\bar{f}^{-1}\bar{h}y) \cdot \varphi(\bar{h}y)^{-1} = \varphi(\bar{h}\bar{g}^{-1}y) \cdot \varphi(\bar{h}y)^{-1}$. Since $\varphi \circ \bar{h} = \varphi$, we have $\zeta = (\varphi \circ \bar{g}^{-1})^{-1} \cdot \varphi$. Choose a lifting $\tilde{\varphi} \in C^\infty(S^2, \mathbf{R}^1)$ for φ. Then $\tilde{\zeta} = \tilde{\varphi} - \tilde{\varphi} \circ \bar{g}^{-1}$ is a lifting for ζ. Summing over a $\bar{g}$-orbit shows that $\tilde{\zeta} = 0$ and hence that $\zeta = 1$. ■

COROLLARY 5.14. *With notation as above,* φ *and its lifting* $\tilde{\varphi}$ *are* G-*invariant.*

Proof. The last step in the proof of Lemma 5.13 shows that $\tilde{\varphi}$ is invariant under $G' \subset G$. Lemma 5.12 shows that $\tilde{\varphi}$ is invariant under h. Since G' and h together generate G, it follows that $\tilde{\varphi}$, and hence φ, is G-invariant. ■

Now we define $\rho: G \to N_{SO(4)}(S^1)$. Restricted to G' it is the identity. It sends an element of the form $g \circ h$, $g \in G'$, to $g \circ k$. By using Lemmas 5.12 and 5.13, the following becomes a straightforward calculation.

LEMMA 5.15. *The map* $\rho: G \to N_{SO(4)}(S^1)$ *is a homomorphism.*

LEMMA 5.16. *There is an element* $\mu \in C^\infty(S^2, S^1)$ *that conjugates* G *to* $\rho(G) \subset N_{SO(4)}(S^1)$.

Proof. According to Corollary 5.14, the element $\varphi \in C^\infty(S^2, S^1)$ has a G-invariant lifting $\tilde{\varphi} \in C^\infty(S^2, \mathbf{R}^1)$. Define μ by $\mu(x) = \exp(1/2\, \tilde{\varphi}(x))$. We claim that $\hat{\mu} \circ g \circ \hat{\mu}^{-1} = \rho(g)$ for all $g \in G$. Notice that $\hat{\mu}$ is also G-invariant. Thus

$$\hat{\mu} \circ g \circ \hat{\mu}^{-1} = \hat{\mu} \circ \hat{\mu}^{-\varepsilon(\alpha)} \circ g = (\hat{\mu}^{1-\varepsilon(\alpha)})\hat{} \circ g.$$

If $\varepsilon(g) = 1$, then $\hat{\mu} \circ g \circ \hat{\mu}^{-1} = g = \rho(g)$. If $\varepsilon(g) = -1$, then

$$\hat{\mu} \circ g \circ \hat{\mu}^{-1} = (\mu^2)\hat{} \circ g = \hat{\varphi} \circ g = \hat{\varphi} \circ (g \circ h^{-1}) \circ h = (g \circ h^{-1}) \circ \hat{\varphi} \circ h$$
$$= (g \circ h^{-1}) \circ k = \rho(g). \quad \blacksquare$$

This completes the proof of Case IIB.

6. Statement of the Main Result

We shall establish the following result.

THEOREM 6.1. *Let* X *be a closed, orientable, three-dimensional orbifold with finite fundamental group. Suppose that* X *is of cyclic type and that at least one point in* X *has a local group of order* > 5. *Then there is a Seifert-fibered orbifold* X_1 *and a homotopy* 3*-sphere* Σ, *so that* X *is diffeomorphic to* $X_1 \# \Sigma$.

This theorem, combined with Theorem 5.1, yields the following corollary.

COROLLARY 6.2. *Let* $G \times \tilde{\Sigma} \to \tilde{\Sigma}$ *be a finite, smooth, orientation-preserving group action on a homotopy* 3*-sphere. Suppose that all isotropy groups for this action are cyclic and at least one has order* > 5. *Then the action is essentially linear.*

Proof of the Corollary Assuming the Theorem. Let X be the quotient orbifold $\tilde{\Sigma}/G$. Since G is an orientation-preserving action X is orientable. The local group of X at x is isomorphic to the isotropy group $G_{\tilde{x}}$ at any $\tilde{x} \in \tilde{\Sigma}$ that projects onto x. Thus X is of cyclic type and at least one local group of X has order > 5. The fundamental group of X is isomorphic to G. Thus Theorem 6.1 implies that $X \cong X_1 \# \Sigma$, where X_1 is a Seifert-fibered orbifold and Σ is a homotopy 3-sphere. According to Theorem 5.1, X_1 is diffeomorphic to S^3/H for some $H \subset SO(4)$. Thus, $\tilde{\Sigma}/G$ is diffeomorphic to $(S^3/H) \# \Sigma$. This means that the action of G on $\tilde{\Sigma}$ is essentially linear. (See Chapter I for a discussion of essentially linear actions.) ■

According to Proposition 3.3, if X is a closed, orientable, three-dimensional orbifold of cyclic type with finite fundamental group, then it is isomorphic to $X_1 \# \Sigma$, where X_1 is prime and Σ is a homotopy 3-sphere. Thus we can reformulate Theorem 6.1 as follows.

THEOREM 6.3. *Let X be a closed, orientable, prime, three-dimensional orbifold of cyclic type with finite fundamental group. Suppose that there is a point $x \in X$ whose local group has order > 5. Then X is a Seifert-fibered orbifold.*

The above remarks show that Theorem 6.3 implies Theorem 6.1. The rest of this chapter is devoted to proving Theorem 6.3.

7. A Special Case

Let X be an orbifold as in the hypothesis of Theorem 6.3. Let (Q, K, ρ) be the underlying triple of X, In this section we shall prove Theorem 6.3 under the additional hypotheses that

(i) Q is simply connected, and
(ii) $K \subset Q$ is connected.

The hypothesis that some local group has order > 5 means that $\rho(K) = n > 5$. The hypothesis that $\pi_1^{orb}(X)$ is finite means that the cyclic n-sheeted branch cover of Q branched over K has finite fundamental group. We shall show that under all these assumptions Q is Seifert-fibered (in the classical sense), with K being a union of fibers. According to 4.4, this means that the smooth orbifold X is Seifert-fibered of restricted type.

Our argument begins in exactly the same way as that of the solution to the Smith conjecture. Let $\nu(K)$ be the interior of a disk bundle neighborhood of K in Q.

LEMMA 7.1. *With notation and assumptions as above, one of the following is true:*

(a) $Q - \nu(K)$ *is Seifert-fibered.*
(b) $Q - \nu(K)$ *has an incompressible torus which is not peripheral (i.e., not parallel to the boundary).*
(c) $\text{int}(Q - \nu(K))$ *has a complete metric of finite total volume all of whose sectional curvatures are* -1. *(Such a structure is called hyperbolic.)*

This lemma follows from the Jaco–Shalen, Johannson theorem and Thurston's uniformization theorem exactly as in the case of the Smith conjecture (see Chapter IV).

LEMMA 7.2. *$Q - \nu(K)$ does not have a closed, incompressible surface that is not peripheral.*

This follows from the Meeks–Yau result exactly as in the case of the Smith conjecture (see Chapter VII).

COROLLARY 7.3. *Either case (a) or case (c) of Lemma 7.1 obtains. If case (c) obtains, then the representation $\pi_1(Q - \nu(K)) = \Gamma \hookrightarrow PSL_2(\mathbf{C})$ coming from a hyperbolic structure on $Q - \nu(K)$ is conjugate to a representation $\Gamma \to PSL_2(A) \subset PSL_2(\mathbf{C})$, where A is a ring of algebraic integers in a number field.*

Proof. The first statement follows immediately from Lemmas 7.1 and 7.2. The second follows from an application of Bass's theorem on subgroups of $PSL_2(\mathbf{C})$ [Chapter VI]. The reasoning is the same as that which occurs in the case of the Smith conjecture [Chapter IV]. ■

To prove Theorem 6.3 in the special case under consideration here, we first show that $Q - \nu(K)$ is Seifert-fibered. To do this it suffices to assume that $Q - K$ has a complete hyperbolic structure of finite volume and deduce a contradiction.

We make this assumption. By Corollary 7.3 we can choose the holonomy representation for the hyperbolic structure, so that $\pi_1(Q - K) = \Gamma$ is represented in $PSL_2(A)$.

Let $\mu \in \Gamma$ be the class of the meridian about K. Since $\pi_1(Q) = \{e\}$, $\Gamma/\{\mu\} = \{e\}$, i.e., μ is a normal generator for Γ. The group $\pi_1^{orb}(X) = G$ is $\Gamma/\{\mu^n\}$.

The situation which we have can be summarized as follows:

(i) There is a group $\Gamma \subset PSL_2(\mathbf{C})$ that is discrete and torsion-free and acts on hyperbolic 3-space so that the quotient has finite volume.

(ii) Γ is actually contained in $PSL_2(A)$, where A is a ring of algebraic integers.

(iii) Γ is normally generated by an element μ of trace ± 2.

(iv) $\Gamma/\{\mu^n\}$ is a finite group for some $n > 5$.

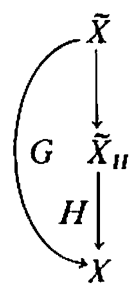
We shall show that the only torsion-free, discrete group in $PSL_2(\mathbf{C})$, satisfies conditions (ii)–(iv), is a cyclic group. Since no cyclic group acts with quotient having finite volume, this will yield a contradiction and will establish that $Q - K$ cannot be hyperbolic.

Choose a prime ideal $\mathfrak{p} \subset A$ so that $n \in \mathfrak{p}$. Let p be the rational prime below $\mathfrak{p}$, i.e., $\mathfrak{p} \cap \mathbf{Z} = (p)$. Consider $\Gamma \subset \mathrm{PSL}_2(A) \to \mathrm{PSL}_2(A/\mathfrak{p})$. Let $H \subset \mathrm{PSL}_2(A/\mathfrak{p})$ be the image of Γ.

Lemma 7.4. (a) *H is normally generated (over itself) by $[\mu]$ and*
(b) *μ becomes an element of order p (or 1) in $\mathrm{PSL}_2(A/\mathfrak{p})$.*

The first fact follows immediately, since μ normally generates Γ. The second follows from the fact that any matrix in $\mathrm{PSL}_2(A/\mathfrak{p})$ of trace ± 2 is conjugate to a matrix of the form $\pm\begin{pmatrix}1&\lambda\\0&1\end{pmatrix}$ for some $\lambda \in A/\mathfrak{p}$. Clearly, any such upper triangular matrix is of order p (or 1) since $A/\mathfrak{p}$ is a finite field of characteristic p. As a consequence of Lemma 7.4(b), the element μ^n is sent to 1 in $\mathrm{PSL}_2(A/\mathfrak{p})$ and hence $\Gamma \to H \subset \mathrm{PSL}_2(A/\mathfrak{p})$ factors through $\Gamma/\{\mu^n\} = G$.

Let $\tilde{X}$ be the universal cover of X, and let $\tilde{X}_H$ be the covering corresponding to the quotient group H:



Let P be the underlying space of $\tilde{X}_H$. Also, let $\tilde{\Gamma}$ be the kernel of $\Gamma \to H$,

$$1 \to \tilde{\Gamma} \to \Gamma \to H \to 1.$$

By Dickson's theorem [see the appendix] on subgroups of PSL_2 of finite fields, the only possibilities for H are

(7.5) (α) *H is cyclic (of order p of 1).*
(β) *H is conjugate to $\mathrm{PSL}_2(F)$, where F is a subfield of $A/\mathfrak{p}$.*
(γ) *$p = 2$ and H is a dihedral group of order $(2k + 1) \cdot 2$.*
(δ) *$p = 3$ and H is isomorphic to the icosahedral group A_5.*

Case α. H is cyclic. If H is cyclic, then it is generated by $[\mu]$, which has order p or 1. Thus $\tilde{X}_H \to X$ is either the trivial covering or the p-sheeted cover of Q branched along K. The group $\tilde{\Gamma}$ is isomorphic to $\pi_1(P - K)$. Clearly, under the representation $\Gamma \subset \mathrm{PSL}_2(A)$, the group $\tilde{\Gamma}$ is the subgroup of all matrices in Γ congruent to $\pm\begin{pmatrix}1&0\\0&1\end{pmatrix}$ modulo $\mathfrak{p}$.

Lemma 7.6. $\tilde{\Gamma}/[\tilde{\Gamma}, \tilde{\Gamma}] \otimes \mathbf{Z}/p\mathbf{Z} \cong \mathbf{Z}/p\mathbf{Z}$.

Proof. The group $\tilde{\Gamma}/[\tilde{\Gamma}, \tilde{\Gamma}]$ is isomorphic to $H_1(P - K)$. Therefore, the lemma states that $H_1(P - K; \mathbf{Z}/p\mathbf{Z}) \cong \mathbf{Z}/p\mathbf{Z}$. If H is trivial, then $P = Q$ and the result is immediate. If $H = \mathbf{Z}/p\mathbf{Z}$, then $P - K$ is a cyclic, p-fold covering of $Q - K$. We know that $H_1(Q - K) \cong \mathbf{Z}$. Let Z be the infinite cyclic cover of $Q - K$ and let $T: Z \to Z$ be the covering transformation corresponding to a generator of $\mathbf{Z}$. Then $P - K \cong Z/T^p$. We have an exact sequence

$$H_2(Q - K) \longrightarrow H_1(Z) \xrightarrow{1 - T_*} H_1(Z) \longrightarrow H_1(Q - K) \longrightarrow \mathbf{Z} \longrightarrow 0.$$

Hence $1 - T_*: H_1(Z) \to H_1(Z)$ is an isomorphism. But $1 - T_*^p \equiv (1 - T_*)^p$ mod p. Thus $1 - T_*^p = (1 - T_*)^p: H_1(Z; \mathbf{Z}/p\mathbf{Z}) \to H_1(Z; \mathbf{Z}/p\mathbf{Z})$. Hence $1 - T_*^p$ is an isomorphism modulo p. It follows that $H_1(P - K; \mathbf{Z}/p\mathbf{Z}) \cong \mathbf{Z}/p\mathbf{Z}$. This completes the proof of Lemma 7.6. ■

Let $H_n \subset \mathrm{PSL}_2(A/\mathfrak{p}^n)$ be the image of $\tilde{\Gamma}$ under reduction modulo $\mathfrak{p}^n$. By the previous lemma, $(H_n/[H_n, H_n]) \otimes \mathbf{Z}/p\mathbf{Z}$ is either 0 or $\mathbf{Z}/p\mathbf{Z}$. On the other hand, $H_n \subset \mathrm{PSL}_2(A/\mathfrak{p}^n)$ consists of a group of matrices congruent to $\pm\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$ mod $\mathfrak{p}$.

Lemma 7.7. *The group of matrices in* $\mathrm{PSL}_2(A/\mathfrak{p}^n)$ *congruent to* $\pm\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$ mod $\mathfrak{p}$ *is a nilpotent p-group.*

Proof. Let $C_1 \subset \mathrm{PSL}_2(A/\mathfrak{p}^n)$ be the group of these matrices. Let $C_{n-1} \subset C_1$ be the subgroup of matrices congruent to $+\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$ mod $\mathfrak{p}^{n-1}$. We claim that C_{n-1} is contained in the center of C_1. The proof is a simple computation:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \equiv \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \quad \text{mod } \mathfrak{p}^n,$$

if $a \equiv d = \pm 1(\mathfrak{p})$, $b \equiv c \equiv 0(\mathfrak{p})$, $\alpha \equiv \delta \equiv 1(\mathfrak{p}^{n-1})$, and $\beta \equiv \gamma \equiv 0(\mathfrak{p}^{n-1})$. By induction on n, one easily establishes that C_1 is a nilpotent group and, in fact, has order a power of p. ■

It follows immediately from Lemma 7.7 that $H_n \subset \mathrm{PSL}_2(A/\mathfrak{p}^n)$ is a nilpotent p-group.

Since the abelianization of H_n tensored with $\mathbf{Z}/p\mathbf{Z}$ is cyclic, it follows that H_n itself is a cyclic group. Thus if we reduce $\tilde{\Gamma}$ modulo any power of $\mathfrak{p}$, then the result is a cyclic group. But every nontrivial element in $\tilde{\Gamma}$ is nontrivial modulo some power of $\mathfrak{p}$. Hence every nontrivial element of $\tilde{\Gamma}$ is detected in a cyclic image of $\tilde{\Gamma}$. This means that the commutator subgroup $[\tilde{\Gamma}, \tilde{\Gamma}]$ is trivial, i.e., that $\tilde{\Gamma}$ is abelian. This is a contradiction since by hypothesis $\tilde{\Gamma}$ is the fundamental group of a complete hyperbolic manifold of finite volume. This shows that Case (α) never occurs.

Case β (for $p > 5$). H is conjugate to $\mathrm{PSL}_2(F)$, *F a field of characteristic $p > 5$.* Let μ_H and μ_G denote the images of $\mu \in \Gamma$ in H and G, respectively. Consider the action of G on $\tilde{X}$. The element μ_G has fixed point set a circle lying above $K \subset Q$. Consequently, the action of $\mu_H \in H$ on $\tilde{X}_H$ has as fixed point set a union of circles $S_1, \ldots, S_t$, each projecting onto K. (Here $t \geq 1$.) Case β for $p > 5$ is ruled out by the following lemma applied to P, the underlying space of $\tilde{X}_H$.

LEMMA 7.8. *Let M be a closed, orientable 3-manifold with finite fundamental group. Let F be a finite field of characteristic $p > 5$. There is no action of* $\mathrm{PSL}_2(F)$ *on M, so that*

(i) *all isotropy groups are cyclic and*
(ii) *an element of order p, $g \in \mathrm{PSL}_2(F)$, has fixed points.*

Proof. Suppose there were an action of $\mathrm{PSL}_2(F)$ on M satisfying (i) and (ii). Since M has a finite fundamental group and $p \neq 2$, the dimension of $H_1(M; \mathbf{Z}/p)$ is ≤ 1. By Smith theory [2, p. 126], if $\mathbf{Z}/p$ acts on M with fixed point set W in M, then the dimension of $H_1(W; \mathbf{Z}/p)$ is ≤ 2. Thus the fixed point set of g, denoted by $W(g)$, is either one circle or two. The normalizer $N\langle g \rangle \subset \mathrm{PSL}_2(F)$ of $\langle g \rangle$ acts on $W(g)$. If $\alpha \in N\langle g \rangle$ fixes a component S of $W(g)$, then the group $\langle \alpha, g \rangle$ generated by α and g acts on S and hence on some tubular neighborhood of S. Since all isotropy groups are cyclic, it is possible to choose this neighborhood ν so that $\langle \alpha, g \rangle$ acts freely on $\nu - S$. In particular, $\langle \alpha, g \rangle$ acts freely and in an orientation-preserving manner on $\partial \nu$. This implies that $\langle \alpha, g \rangle$ is an abelian group. Thus α is in the centralizer $Z\langle g \rangle$ of $\langle g \rangle$. This proves that the order of $N\langle g \rangle / Z\langle g \rangle$ is bounded above by the number of components of $W(g)$. Thus the order of $N\langle g \rangle / Z\langle g \rangle$ is ≤ 2.

Consider the element $h_\lambda = \pm\left(\begin{smallmatrix} 1 & \lambda \\ 0 & 1 \end{smallmatrix}\right)$, $\lambda \neq 0$ in F. Its normalizer contains all elements of the form $\pm\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{smallmatrix}\right)$, where α is a nonzero element of the prime field of characteristic p. None of these elements, save $\pm$identity, commute with h_λ. Thus $N\langle h_\lambda \rangle / Z\langle h_\lambda \rangle$ has order divisible by $(p-1)/2$. On the other hand, every element of order p in $\mathrm{PSL}_2(F)$ is conjugate to an element h_λ for some λ. Thus for any element of order p in $\mathrm{PSL}_2(F)$, g, $N\langle g \rangle / Z\langle g \rangle$ has has order divisible by $(p-1)/2$. If $p > 5$, this contradicts the fact, established above, that $N\langle g \rangle / Z\langle g \rangle$ has order at most 2. ■

Case β (for $p = 2, 3,$ or 5), Case γ, and Case δ. We shall need the following lemma.

LEMMA 7.9. *Let X be a three-dimensional orbifold of cyclic type with finite fundamental group. It is impossible for the exceptional set to have four or more components labeled 2 or three or more components labeled by n, with $n > 2$.*

Proof. Let us consider the second case. Suppose there are three components of the exceptional set of X all labeled $n > 2$. Take the orbifold cover Y of X corresponding to the universal topological cover of the underlying space of X. In Y there are at least three components of the exceptional set labeled n—say $\tilde{K}_1$, $\tilde{K}_2$, and $\tilde{K}_3$. Let $\tilde{Q}$ be the topological space underlying Y. There is a regular branched covering of $\tilde{Q}$ branched over $\tilde{K}_1 \amalg \tilde{K}_2$ with group of covering transformations $\mathbf{Z}/n\mathbf{Z} \times \mathbf{Z}/n\mathbf{Z}$. This corresponds to an orbifold covering of Y. The preimage of $\tilde{K}_3$ in this covering is at least n-circles. We can repeat this process ad infinitum. This contradicts the fact that $\pi_1^{\mathrm{orb}}(X)$ is finite.

A similar argument works in the case of 4-circles labeled 2. Details are left to the reader. ■

We have $p < 5$, F is a finite field of characteristic p, and $H \subset \mathrm{PSL}_2(F)$ is noncyclic and normally generated by an element μ_H of order p. Consider the regular covering $\tilde{X}_H \to X$. Let (P, J, τ) be the underlying triple for $\tilde{X}_H$, and let $\pi\colon P \to Q$ be the ramified covering of underlying spaces. The map π is ramified over K, with index of ramification p. Since $p \leq 5$ and $\rho(K) = n > 5$, we see that J is $\pi^{-1}(K)$ and that $\tau(J) = n/p$.

LEMMA 7.10. *The number of components of J is divisible by the number of cosets of $Z\langle\mu_H\rangle$ in H.*

Proof. H acts on P leaving J invariant. Since $J/H = K$, H acts transitively on the components of J. Let J_1 be one of these components, and let $\mathrm{stab}(J_1)$ be its stabilizer. Thus the number of components is the order of $H/\mathrm{stab}(J_1)$. As we have seen in the proof of Lemma 7.8, $\mathrm{stab}(J_1) \subset Z\langle\mu_H\rangle$. Therefore, the number of components of J is divisible by the order of $H/Z\langle\mu_H\rangle$. ■

If $p = 3$ or 5 and H is isomorphic to $\mathrm{PSL}_2(F)$, order$(F) = p^s$, then the centralizer of an element μ_H of order p has order p^s and thus J has at least $(p^{2s} - 1)/2 \geq 4$ components; $\tau(J) = n/p$.

If $p = 3$ and H is isomorphic to A_5, then the centralizer of μ_H has order 3, and J has at least 20 components; $\tau(J) = n/3$.

If $p = 2$ and H is isomorphic to $\mathrm{PSL}_2(F)$, order$(F) = 2^s$, then J has at least $2^{2s} - 1 \geq 3$ components; $\tau(J) = n/2 \geq 3$.

If $p = 2$ and H is isomorphic to a dihedral group of order$(2k + 1)\cdot 2$, then J has at least $(2k + 1) \geq 3$ components; $\tau(J) = n/2 \geq 3$.

All these possibilities are ruled out by Lemma 7.9.

At this point we have ruled out the possibility that $Q - K$ has a complete hyperbolic structure of finite volume. Thus by Lemma 7.1, $Q - \nu(K)$ must be Seifert-fibered. To complete the proof of Theorem 6.3 in the special case under consideration we need only show that this Seifert fibration structure extends over $\overline{\nu(K)}$ with K being a fiber. The next proposition guarantees this.

PROPOSITION 7.11. *Let X be a three-dimensional orbifold with finite fundamental group. Let K_1 be a component of the exceptional set of X. Let $\nu(K_1) \subset X$ be a neighborhood of K_1. Suppose that the orbifold $X - \nu(K_1)$ is Seifert-fibered. Then X is Seifert-fibered.*

Proof. Let B be the base orbifold for the Seifert fibration on $X - \nu(K_1)$. There is a boundary circle C for B corresponding to $\partial\nu(K_1)$. Let μ_1 be the meridian in $\partial\nu(K_1)$. Then μ_1 projects to some multiple of C in B. Unless this multiple is 0, Lemma 4.4 shows that the Seifert fibration extends over X.

If the multiple is zero, then $\pi_1^{\text{orb}}(X) = \pi_1^{\text{orb}}(X - \nu(K_1))/\{\mu^{\rho(K_1)}\}$ has $\pi_1^{\text{orb}}(B)$ as a quotient. Hence $\pi_1^{\text{orb}}(B)$ is finite. This means that B is diffeomorphic to D^2/G for some $G \subset O(2)$. Since ∂B has a component that is a circle, G is actually a subgroup of SO(2). As a result, the underlying topological space of $X - \nu(K_1)$ is a solid torus, and the exceptional set of $X - \nu(K_1)$ is either empty or the core of this solid torus. This means that the topological space underlying X is the union of two solid tori, and the exceptional set of X is either the union of the two cores or one of the cores. In these cases it is easy to construct a Seifert fibration of the underlying topological space, so that the exceptional set is a union of fibers. By Lemma 4.4 this implies that X is a Seifert-fibered orbifold. ■

8. Completion of the Proof

In this section we deduce Theorem 6.3 from the special case proved in Section 7. Basically, the argument is by induction on the order of the group. It is based on a result for orbifolds (Theorem 8.1) that generalizes a theorem for 3-manifolds proved by Waldhausen [10] and Gordon and Heil [5].

Recall that if X^3 is Seifert-fibered, then the class of a generic fiber generates a normal, cyclic subgroup N of $\pi_1^{\text{orb}}(X)$. This group is nontrivial unless X is diffeomorphic to S^3/G. The following theorem shows that often the existence of such a normal subgroup is also sufficient for X to be Seifert-fibered when $\partial X \neq \varnothing$.

THEOREM 8.1. *Let X^3 be a prime, good, orientable three-dimensional orbifold with nonempty boundary. X is Seifert-fibered if and only if $\pi_1^{\text{orb}}(X)$ contains a nontrivial, normal, cyclic subgroup.*

The proof of this theorem is contained in [6]. Basically, one follows the original Waldhausen argument in [10].

A subgroup $N \subset \Gamma$ is said to be *characteristic* if any automorphism of Γ leaves N invariant.

Another result of [6] that is needed here follows.

PROPOSITION 8.2. *Let $N \subset \pi_1^{orb}(X)$ be the normal, cyclic subgroup generated by the fiber of a Seifert fibration of X. Unless $X \cong T^2 \times I$, N is characteristic.*

This proposition is not difficult. The main idea is that if N is not characteristic, then there is another normal, cyclic group in $\pi_1^{orb}(X)$. This group projects to a nontrivial, normal cyclic subgroup of the base of the Seifert fibration. This limits the base severely. An examination of the possible base spaces completes the argument.

By using these two results, we complete the proof of Theorem 6.3. The argument is by induction on the order of the fundamental group.

Let X be an orbifold that satisfies the hypothesis of Theorem 6.3. This means that X is a closed, orientable, prime orbifold of cyclic type with finite fundamental group. Let Q be the underlying space of X, and let $K \subset Q$ be the exceptional set. We know that $Q - K$ is irreducible. Let K_1 be a component of K for which $\rho(K_1) > 5$.

Case 1. Q is not simply connected. Let $\tilde{Q} \to Q$ be the universal topological cover. There is an orbifold covering $Y \to X$, so that on the level of spaces the projection is $\tilde{Q} \to Q$. The exceptional set of Y is the preimage of K in $\tilde{Q}$. We call it $\tilde{K}$. Let $\tilde{K}_1$ be the preimage of K_1. Since $\tilde{Q} - \tilde{K}$ is a covering of $Q - K$, it follows that $\tilde{Q} - \tilde{K}$ is irreducible. This means that Y satisfies the hypothesis of Theorem 6.3. By induction, Y is Seifert-fibered. By Lemma 4.6, $\tilde{K}_1$ must be a union of fibers. Let $\nu(K_1)$ be a disk bundle neighborhood of K_1 in Q and let $\nu(\tilde{K}_1)$ be its preimage in $\tilde{Q}$. We can easily arrange for $Y - \nu(\tilde{K}_1)$ to be Seifert-fibered.

Case 1a. $Y - \nu(\tilde{K}_1)$ is not isomorphic (as an orbifold) to $T^2 \times I$. In this case the normal, cyclic group $N \subset \pi_1^{orb}(Y - \nu(\tilde{K}_1))$ generated by the class of a generic fiber is nontrivial (by Lemma 4.3) and characteristic (by Proposition 8.2). Hence it forms a nontrivial, normal, cyclic subgroup of $\pi_1^{orb}(X - \nu(K_1))$. Since X is prime, $X - \nu(K_1)$ is also prime. Thus, according to Theorem 8.1, $X - \nu(K_1)$ is Seifert-fibered. By Proposition 7.11, we can choose this Seifert fibration so that it extends to one on all of X.

Case 1b. $Y - \nu(\tilde{K}_1)$ is isomorphic to $T^2 \times I$. In this case $\tilde{K} = \tilde{K}_1$, and hence $K = K_1$ in Q. Thus $Q - \nu(K_1)$ is finitely covered by $T^2 \times I$. The only orientable manifold with one boundary component that is covered by $T^2 \times I$ is the twisted I-bundle over the Klein bottle. This manifold has two S^1-fibrations. At least one of them extends to a Seifert fibration of Q with K being a fiber. According to Proposition 4.4, this implies that X is a Seifert-fibered orbifold.

Case 2. Q is simply connected and K has at least two components. Let K_2 be a component of K distinct from K_1. Let p be a prime dividing $\rho(K_2)$. Since Q is simply connected, there is a p-sheeted, branched, cyclic covering of Q branched over K_2. Let $P \to Q$ be this branched, cyclic covering. Since $p|\rho(K_2)$, there is a corresponding covering of orbifolds $Y \to X$. The exceptional set is the preimage of $K - K_2$ in P if $p = \rho(K_2)$. Otherwise, it is the preimage of all of K in P. Let $\tilde{K}$ be the exceptional set, and let $\tilde{K}_1$ be the preimage of K_1 in P.

The orbifold Y satisfies all the hypotheses of Theorem 6.3, except possibly the condition that Y is prime. By Proposition 3.3 and induction, Y is isomorphic to $Y_1 \# \Sigma$ where Y_1 is Seifert-fibered and Σ is a homotopy 3-sphere. By Lemma 4.6, $\tilde{K}_1$ must be a union of fibers. Let $\nu(K_1)$ be a disk bundle neighborhood of $K_1 \subset Q$, and let $\nu(\tilde{K}_1)$ be its preimage in P. The orbifold $Y_1 - \nu(\tilde{K}_1)$ is Seifert fibered.

Case 2a. $Y_1 - \nu(\tilde{K}_1)$ is not isomorphic to $T^2 \times I$. In this case, as in Case la, $\pi_1^{orb}(Y - \nu(\tilde{K}_1)) = \pi_1^{orb}(Y_1 - \nu(\tilde{K}_1))$ has a nontrivial, characteristic, cyclic subgroup N. This subgroup N is a normal subgroup in $\pi_1^{orb}(X - \nu(K_1))$. Since X is prime, so is $X - \nu(K_1)$. Hence Theorem 8.1 says that $X - \nu(K_1)$ is Seifert-fibered. By Proposition 7.11, this means that X itself is Seifert-fibered.

Case 2b. $Y_1 - \nu(\tilde{K}_1)$ is isomorphic to $T^2 \times I$. Since $Y - \nu(\tilde{K}_1)$ has two boundary components and $X - \nu(K_1)$ has one, the group of covering transformations of Y over X has even order. By construction this group is a cyclic group of prime order. Hence the group is isomorphic to $\mathbf{Z}/2\mathbf{Z}$. It acts by interchanging the boundary components. Thus we have an extension

$$1 \to \mathbf{Z} \times \mathbf{Z} \to \pi_1^{orb}(X - \nu(K_1)) \to \mathbf{Z}/2\mathbf{Z} \to 1.$$

It follows that $\pi_1^{orb}(X - \nu(K_1))$ has a nontrivial, cyclic, normal subgroup. Since X is prime, so is $X - \nu(K_1)$. Thus Theorem 8.1 implies that $X - \nu(K_1)$ is Seifert-fibered. As before, Proposition 7.11 allows us to find a Seifert fibration on all of X.

Case 3. Q is simply connected and K has only one component. This is exactly the special case dealt with in Section 7.

This completes the proof of Theorem 6.3 and hence of Theorem A.

Appendix

Let F be a field with p^n elements. Let $\mu \in PSL_2(F)$ be the element $\pm\begin{pmatrix}1&1\\0&1\end{pmatrix}$. We shall give Dickson's argument [3] classifying groups $G \subset PSL_2(F)$ that contain μ. Two such groups are considered equivalent if there are a finite extension $\tilde{F}$ of F and an element $\alpha \in PSL_2(\tilde{F})$ that normalizes $PSL_2(F)$, commutes with μ, and conjugates one to the other. This leads to a classification of groups $G \subset PSL_2(F)$ that contain a nontrivial element of trace ± 2 (or, equivalently, that contain an element of order p). Once again the classification is up to conjugation by elements in $PSL_2(\tilde{F})$, $\tilde{F}$ a finite extension of F, that normalize $PSL_2(F)$. The reason is that any nontrivial element of trace ± 2 in $PSL_2(F)$ is conjugate to $\pm\begin{pmatrix}1&\lambda\\0&1\end{pmatrix}$, $\lambda \neq 0$. We let $\tilde{F}$ be the extension of F obtained by adjoining $x = \sqrt{\lambda}$ to F. In $PSL_2(\tilde{F})$ the matrix $\pm\begin{pmatrix}x&0\\0&x^{-1}\end{pmatrix}$ normalizes $PSL_2(F)$ and conjugates $\pm\begin{pmatrix}1&\lambda\\0&1\end{pmatrix}$ to $\pm\begin{pmatrix}1&1\\0&1\end{pmatrix}$.

Let us begin then with $G \subset PSL_2(F)$ containing μ. There are three classes of such groups, as we shall see

Class I. Subgroups of upper triangular matrices in $PSL_2(F)$;

Class II. groups conjugate to $PSL_2(F')$ or a $\mathbf{Z}/2\mathbf{Z}$-extension of $PSL_2(F')$ for some subfield $F' \subset F$; and

Class III. exceptional groups for F a field of order a power of 2 or 3.

We shall discuss in more detail later the various groups in these classes.

Let us set up some notation. *A maximal unipotent subgroup*, or MU subgroup for short, is any subgroup conjugate in $PSL_2(F)$ to the group of strictly upper triangular matrices

$$B_\infty = \left\{\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} \quad \text{for } x \in F\right\},$$

Let $G \subset PSL_2(F)$, and let $B \subset PSL_2(F)$ be a MU subgroup. If $G \cap B$ is nontrivial, then we say that it is a MU *subgroup of* G. Any MU subgroup of G has order p^l for some $1 \leq l \leq n$ (where l might depend on the MU subgroup).

LEMMA A1. *Any* MU *subgroup of* G *is a* p*-Sylow subgroup of* G.

Proof. A subgroup is a p-Sylow subgroup if it is a maximal p-group (i.e., if it is not contained in any larger p-group). Any MU subgroup of $PSL_2(F)$ is a p-Sylow subgroup, and any two distinct MUs in $PSL_2(F)$ have trivial intersection. From this it follows easily that any maximal p-group in G must be of the form $G \cap B$, and, conversely, that if $G \cap B$ is nontrivial, then it is a maximal p-group of G. Thus the p-Sylow subgroups of G are the groups $G \cap B$ that are nontrivial. ■

COROLLARY A2. *Any two* MU *subgroups of G are conjugate.*

Case I. *G has only one* MU *subgroup. Then G is a group of upper triangular matrices.* If G has only one MU subgroup, then that subgroup must be B_∞ (since $\mu \in G$) and B_∞ must be normal in G. This implies that every element of G normalizes B_∞, i.e., that G is a group of upper triangular matrices. This completes Case I.

For the rest of this appendix we assume that G has more than one MU *subgroup.*

LEMMA A3. *The number of* MU *subgroups of G is* $fp^r + 1$, *where* p^r *is the order of* $G \cap B_\infty$ *and* $f \geq 1$.

Proof. B_∞, acting by conjugation on the MU subgroups of $\mathrm{PSL}_2(F)$, fixes B_∞ and acts freely and transitively on the others. Thus $G \cap B_\infty$, acting by conjugation on the MU subgroups of G, has one fixed point—$G \cap B_\infty$—and acts freely on the remaining ones. Thus the number of MU subgroups of G is $fp^r + 1$, where p^r is the order of $G \cap B_\infty$. ■

We adhere to the following notation:

order$(G \cap B_\infty) = p^r$.

number of MU subgroups of $G = fp^r + 1, f \geq 1$.

N_∞ = normalizer in G of $G \cap B_\infty$.

order$(N_\infty) = dp^r$.

order $G = dp^r(fp^r + 1)$.

Of course, N_∞ consists of all upper triangular matrices in G. The MU subgroup $G \cap B_\infty$ is naturally identified with a subgroup of the additive group of F. This identification sends $\pm\left(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\right)$ in $G \cap B_\infty$ to $x \in F$. Call the image of $G \cap B_\infty$ under this identification V. It is automatically a vector space over the prime field $F_p \subset F$. Let $F' \subset F$ be the subset of all $a \in F$ such that $a \cdot V \subset V$. One sees easily that $F' \subset F$ is a subfield. Clearly, V is a vector space over F'. Let the order of F' be p^l, $l|n$. Since the order of V is p^r, we see that $l \leq r$. Since $1 \in V$, $F' \subset V$.

Consider the sequence

$$1 \to G \cap B_\infty \to N_\infty \xrightarrow{\psi} F^*/\{\pm 1\}, \qquad \text{where} \quad \psi\left(\pm\begin{pmatrix}\alpha & x\\ 0 & \alpha^{-1}\end{pmatrix}\right) = \pm\alpha.$$

The image of ψ is a subgroup of $F^*/\{\pm 1\}$ of order d. Such a group is cyclic and generated by $\pm\eta_0$, where η_0 is a primitive dth root of -1. The action of $\pm\alpha \in F^*/\{\pm 1\}$ on $G \cap B_\infty = V$ is by multiplication by α^2

$$\begin{pmatrix}\alpha & Y \\ 0 & \alpha^{-1}\end{pmatrix}\begin{pmatrix}1 & x \\ 0 & 1\end{pmatrix}\begin{pmatrix}\alpha^{-1} & -Y \\ 0 & \alpha\end{pmatrix} = \begin{pmatrix}1 & \alpha^2 x \\ 0 & 1\end{pmatrix}.$$

Thus $\eta_0^2 \in (F')^*$. This shows that $d \mid (p^l - 1)$.

We wish to arrange that N_∞ meets the diagonal matrices, $\{\pm(\begin{smallmatrix}\alpha & 0 \\ 0 & \alpha^{-1}\end{smallmatrix})\}$, in exactly the cyclic group generated by

$$\pm\begin{pmatrix}\eta_0 & 0 \\ 0 & \eta_0^{-1}\end{pmatrix}.$$

Lemma A4. *There is an element of B_∞ that conjugates G to $G' \subset \mathrm{PSL}_2(F)$, so that $G' \cap \{\pm(\begin{smallmatrix}\alpha & 0 \\ 0 & \alpha^{-1}\end{smallmatrix})\}$ is the cyclic group generated by*

$$\pm\begin{pmatrix}\eta_0 & 0 \\ 0 & \eta_0^{-1}\end{pmatrix}.$$

(Notice that such conjugation leaves fixed $G \cap B_\infty$.)

Proof. If $\eta_0 = \pm 1$, then no conjugation is necessary since $N_\infty = G \cap B_\infty$. Suppose that $\eta_0 \neq \pm 1$. Let $\pm(\begin{smallmatrix}\eta_0 & x \\ 0 & \eta_0\end{smallmatrix})$ be an element of N_∞. Since $\eta_0 \neq \pm 1$, we see that $\eta_0 \neq \eta_0^{-1}$. Conjugate G by

$$\pm\begin{pmatrix}1 & x/(\eta_0 - \eta_0^{-1}) \\ 0 & 1\end{pmatrix}.$$

The conjugate group contains

$$\pm\begin{pmatrix}1 & \dfrac{x}{\eta_0 - \eta_0^{-1}} \\ 0 & 1\end{pmatrix}\begin{pmatrix}\eta_0 & x \\ 0 & \eta_0^{-1}\end{pmatrix}\begin{pmatrix}1 & \dfrac{x}{\eta_0 - \eta_0^{-1}} \\ 0 & 1\end{pmatrix} = \pm\begin{pmatrix}\eta_0 & 0 \\ 0 & \eta_0^{-1}\end{pmatrix}. \quad \blacksquare$$

We assume that we have made the required conjugation and have renamed the new group G. At this point we have

(A5) (a) *a subfield $F' \subset F$ with p^l elements,*
(b) *an F'-vector subspace $V \subset F$ of order p^r containing F', and*
(c) *an integer d such that $d \mid (p - 1)$, and such that*
(d) *$N_\infty \subset G$ consists of all products*

$$\left\{\pm\begin{pmatrix}1 & v \\ 0 & 1\end{pmatrix}\begin{pmatrix}\eta_0^i & 0 \\ 0 & \eta_0^{-i}\end{pmatrix} \quad \text{for } v \in V \text{ and } \eta_0 \text{ a primitive } d\text{th-root of } -1\right\}.$$

LEMMA A6. *There are $(fp^r + 1)$ right cosets of N_∞ in G. Each nonidentity coset contains at most 2d elements of order p (at most d if $p = 2$).*

Proof. Since the order of N_∞ is $p^r d$ and the order of G is $(1 + fp^r)dp^r$, the first statement follows immediately.

Let $V_j = \left(\begin{smallmatrix}\alpha_j & \beta_j\\ \gamma_j & \delta_j\end{smallmatrix}\right)$; $j = 1, \ldots, fp^r$, be coset representatives for the nonidentity cosets (thus $\gamma_j \neq 0$). The condition that a product

$$\pm\begin{pmatrix}\alpha_j & \beta_j\\ \gamma_j & \delta_j\end{pmatrix}\begin{pmatrix}\eta & \eta v\\ 0 & \eta^{-1}\end{pmatrix},$$

with $v \in V$ and $\eta^2 \in (F')^*$, be an element of order p is that its trace $\eta\alpha_j + \gamma_j\eta v + \eta^{-1}\delta$ be ± 2. By fixing j and η, there are at most two solutions (one if $p = 2$) to

$$\eta\alpha_j + \gamma_j\eta v + \eta^{-1}\delta = \pm 2. \quad \blacksquare$$

On the other hand, G has $(1 + fp^r)$ MU subgroups. These groups have trivial intersection and each contains $(p^r - 1)$ nontrivial elements of trace ± 2. Thus $G - N_\infty$ has $fp^r(p^r - 1)$ elements of trace ± 2. In light of Lemma A6, this implies that

$$fp^r(p^r - 1) \leq 2dfp^r \qquad \text{if} \quad p \text{ is odd}$$

and

$$fp^r(p^r - 1) \leq dfp^r \qquad \text{if} \quad p = 2.$$

Since $d \mid (p^l - 1)$ and $l \leq r$, this yields

(A7) $l = r$ and $d = (p^r - 1)/2$ or $d = p^r - 1$, if p is odd.
$l = r$ and $d = p^r - 1$, if $p = 2$.
$F' = V$.

(A8) *If $d = (p^r - 1)/2$, then $N_\infty = B_\infty \cap \mathrm{PSL}_2(F')$. This is also true for $p = 2$. If $d = p^r - 1$ and $p > 2$, then N_∞ equals $\{\pm\left(\begin{smallmatrix}\eta & \eta\lambda\\ 0 & \eta^{-1}\end{smallmatrix}\right) \mid \eta^2 \in (F')^*$ and $\lambda \in F'\}$. Furthermore, if $d = p^r - 1$ and $p > 2$, then the degree $[F : F']$ is even.*

Case IIA. If p is odd and $d = (p^r - 1)/2$, then $G = \mathrm{PSL}_2(F')$, unless $p^r = 3$, in which case G is conjugate to $\mathrm{PSL}_2(F')$. As we have seen (A6) each nonidentity coset of N_∞ contains at most $2d = (p^r - 1)$ elements of order p. This gives a maximum of $fp^r(p^r - 1)$. On the other hand, this is exactly the number of elements of order p outside N_∞. We conclude that for each j and each $\eta \in (F')^*$ there are two solutions to

$$\alpha_j\eta + \gamma_j\eta v + \delta_j\eta^{-1} = \pm 2.$$

Since each coset of N_∞ contains an element of order p, and hence of trace 2, we can assume that we have chosen $v_j = \left(\begin{smallmatrix}\alpha_j & \beta_j\\ \gamma_j & \delta_j\end{smallmatrix}\right)$ so that $\alpha_j + \delta_j = 2$. By substituting, we see that

$$\alpha_j(\eta - \eta^{-1}) + \gamma_j\eta v = \pm 2 - 2\eta^{-1}$$

has two solutions for each $\eta \in (F')^*$. (Here, α_j and γ_j are fixed in F, and "solution" means $v \in F'$ satisfying the equation.) If we take $\eta = \pm 1$, then above equation reduces to $\gamma_j v = \pm 4$. For there to be two $v \in F'$ solving this equation, it is necessary that $\gamma_j \in (F')^*$. If the order of F' is not 3, then choose $\eta \neq \pm 1, \eta \in (F')^*$. There are solutions $v \in F'$ for $\alpha_j(\eta - \eta^{-1}) + \gamma_j\eta v = \pm 2 - 2\eta^{-1}$. Since $\eta - \eta^{-1} \in (F')^*$ and $\gamma_j \in (F')^*$, it follows that $\alpha_j \in F'$. Consequently, $\delta_j = 2 - \alpha_j$ is in F'. Finally, $\beta_j = (\alpha_j\delta_j - 1)/\gamma_j$ is also in F'. This proves that except in the case for which F' has order 3, $G \subset \mathrm{PSL}_2(F')$.

If F' has order 3, then the argument above shows that each nonidentity coset of N_∞ contains an element of the form

$$\pm\begin{pmatrix}\alpha_j & \beta_j\\ \pm 1 & 2 - \alpha_j\end{pmatrix}.$$

Multiplying by $\pm\left(\begin{smallmatrix}1 & \pm 1\\ 0 & 1\end{smallmatrix}\right) \in N_\infty$, we change this coset representative to

$$\pm\begin{pmatrix}\langle\alpha_j\rangle & \mp(1 + \alpha_j^2)\\ \pm 1 & -\alpha_j\end{pmatrix}.$$

Conjugate G by $\pm\left(\begin{smallmatrix}1 & \pm\alpha\\ 0 & 1\end{smallmatrix}\right)$. As the reader can easily check, the resulting group (which we continue to call G) has the following properties.

(a) The lower left-hand entry of every element of G is contained in F' and
(b) $\pm\left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right) \in G$.

Any group with these two properties is easily seen to be contained in $\mathrm{PSL}_2(F')$.

We have shown that in all cases under Case IIA, G is contained in $\mathrm{PSL}_2(F')$ (after conjugation). The order of G is $(1 + fp^r)p^r(p^r - 1)/2$, and the order of $\mathrm{PSL}_2(F')$ is $(p^{2r} - 1)p^r/2$. Thus $f = 1$ and $G = \mathrm{PSL}_2(F')$. This completes Case IIA.

Case IIB. If $p > 3$ and $d = (p^r - 1)$, then n/r is even and G consists of all products $\pm\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)\left(\begin{smallmatrix}x & 0\\ 0 & x^{-1}\end{smallmatrix}\right)$ where $\pm\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)$ is in $\mathrm{PSL}_2(F')$ and $x \in F^*$ with $x^2 \in F'$. Thus G is a $\mathbf{Z}/2\mathbf{Z}$-extension of $\mathrm{PSL}_2(F')$. We saw in (A8) that if $d = (p^r - 1)$ with p odd, then n/r is even. We also saw that N_∞ is the group of all products

$$\left\{\pm\begin{pmatrix}\eta & 0\\ 0 & \eta^{-1}\end{pmatrix}\begin{pmatrix}1 & \lambda\\ 0 & 1\end{pmatrix} \mid \lambda \in F' \text{ and } \eta^2 \in (F')^*\right\}.$$

Since n/r is even, every element of $(F')^*$ has a square root in F. In particular, $\eta = \sqrt{-1}$ is in F. Thus

$$t = \pm\begin{pmatrix}\sqrt{-1} & 0 \\ 0 & -\sqrt{-1}\end{pmatrix}$$

is an element of $N_\infty \subset G$. Its normalizer in G is all diagonal matrices $\{\pm\left(\begin{smallmatrix}\eta & 0 \\ 0 & \eta^{-1}\end{smallmatrix}\right) \mid \eta^2 \in (F')^*\}$ unless G contains an element of the form $+\left(\begin{smallmatrix}0 & -\tau^{-1} \\ \tau & 0\end{smallmatrix}\right)$. In the latter case, the normalizer of t is the $\mathbf{Z}/2\mathbf{Z}$-extension of the group of diagonal matrices generated by any such element. Depending on which of the two cases obtains, there are either $(1 + fp)p^r$ or $(1 + fp^r)p^r/2$ elements in G conjugate to t. (In the second case, note that f must be odd.) Of these conjugates, fp^{2r} or $(fp^r - 1)p^r/2$ are outside N_∞. All conjugates of t have trace 0. If an element in the coset V_jN_∞ has trace 0, then it is a product

$$\begin{pmatrix}\alpha_j & \beta_j \\ \gamma_j & \delta_j\end{pmatrix}\begin{pmatrix}\eta & \eta\lambda \\ 0 & \eta^{-1}\end{pmatrix},$$

where $\alpha_j\eta + \gamma_j\eta\lambda + \delta_j\eta^{-1} = 0$. For j fixed and η fixed, there is at most one solution $\lambda \in F'$. Thus $G - N_\infty$ contains at most $(fp^r)(p^r - 1)$ elements of trace 0. This means t has $(fp^r - 1)p^r/2$ conjugates in G, and hence in G there is an element of the form $\pm\left(\begin{smallmatrix}0 & -\tau^{-1} \\ \tau & 0\end{smallmatrix}\right)$.

We now count the elements of trace 0 (i.e., the elements of order 2) in $G - N_\infty$. For each coset V_jN_∞ that contains such an element we choose the coset representative V_j to be of trace 0:

$$V_j = \begin{pmatrix}\alpha_j & \beta_j \\ \gamma_j & -\alpha_j\end{pmatrix}, \qquad \gamma_j \neq 0.$$

Some of these cosets can contain more than one element of trace 0. For $V_j \cdot N_\infty$ to contain two such elements, there must be $\eta \in F^*$ with $\eta^2 \in F'$ and $\lambda \in F'$ with either $\lambda \neq 0$ or $\eta \neq \pm 1$ so that

(A9) $$\alpha_j(\eta - \eta^{-1}) + \gamma_j\eta\lambda = 0.$$

If there are such η and λ, then $\alpha_j/\gamma_j = \eta^2\lambda(1 - \eta^2)$. Thus $\alpha_j/\gamma_j \in F'$. Conversely, if $\alpha_j/\gamma_j \in F'$ and $\eta \in F^*$ is any element with $\eta^2 \in F'$, then there is a unique solution to A9 with $\lambda \in F'$. This proves

(A10) *Each nonidentity right coset of N_∞ that contains two distinct elements of trace 0 contains exactly $(p^r - 1)$ such elements.*

Let A be the number of nonidentity cosets of N_∞ that contain $(p^r - 1)$ elements of tr 0, and let B be the number that contain exactly 1. Then

(A11) $$(fp^r - 1)p^r/2 \leq A(p^r - 1) + B \leq A(p^r - 1) + fp^r - A.$$

LEMMA A12. *Let $V = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}$ and $V' = \begin{pmatrix} \alpha' & \beta' \\ \gamma' & -\alpha' \end{pmatrix}$ be elements of trace 0 in G. Suppose that α/γ and α'/γ' both belong to F'. The elements V and V' are in the same coset of N_∞ if and only if $\alpha/\gamma = \alpha'/\gamma'$.*

Proof. The only if part of the lemma is clear. Conversely, if V and V' belong to different cosets of N_∞, then $VV' = V(V')^{-1}$ is not an element of N_∞. Thus

$$\begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}\begin{pmatrix} \alpha' & \beta' \\ \gamma' & -\alpha' \end{pmatrix} = \begin{pmatrix} \alpha\alpha' + \beta\gamma' & \beta'\alpha - \beta\alpha' \\ \alpha'\gamma - \alpha\gamma' & \gamma\beta' + \alpha\alpha' \end{pmatrix}$$

is not upper triangular. Hence $\alpha'\gamma \neq \alpha\gamma'$. This implies that $\alpha'/\gamma' \neq \alpha/\gamma$. ■

Applying Lemma A12 we see that $A \leq p^r$, and hence

$$A(p^r - 1) + fp^r - A \leq p^{2r} - p^r + fp^r - p^r = p^r(p^r + f - 2).$$

In light of inequality A11 we have

$$(fp^r - 1)p^r/2 \leq p^r(p^r + f - 2)$$

or

$$(fp^r - 1)/2 \leq (p^r + f - 2)$$

Since f is odd, there are only two possibilities: either $f = 1$, or $f = 3$ and $p^r = 3$. In the hypothesis of Case IIB we assume $p > 3$. Thus we conclude that in this case $f = 1$.

Recall that there is an element of the form $+\begin{pmatrix} 0 & -\tau^{-1} \\ \tau & 0 \end{pmatrix}$ in G. Let us consider its p^r conjugates by all elements of $G \cap B_\infty$:

$$\left\{ \pm \begin{pmatrix} x\tau & -x^2\tau - \tau^{-1} \\ \tau & -x\tau \end{pmatrix} \middle| x \in F' \right\}.$$

These are all elements of trace 0. The ratios α_j/γ_j run through the p^r elements of F'. By Lemma A12, this implies that these p^r elements are in p^r distinct right cosets of N_∞. Since $f = 1$, there are exactly p^r such cosets. Thus we can use these elements as the coset representatives for all the nonidentity cosets of N_∞. It follows that G is generated by N_∞ and $\pm\begin{pmatrix} 0 & -\tau^{-1} \\ \tau & 0 \end{pmatrix}$. Also notice that each nonidentity coset of N_∞ in G has a representative of the form $\pm\begin{pmatrix} \alpha_j & \beta_j \\ \gamma_j & \delta_j \end{pmatrix}$, where α_j/γ_j is in F'. Consequently, all elements of $G - N_\infty$ are of the form $\pm\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, where α/γ is in F'. In particular,

$$\pm\begin{pmatrix} 0 & -\tau^{-1} \\ \tau & 0 \end{pmatrix}\begin{pmatrix} \tau & -\tau - \tau^{-1} \\ \tau & -\tau \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ \tau^2 & -1 - \tau^2 \end{pmatrix}$$

has this property. This proves that $\tau^2 \in F'$. Clearly, then, both N_∞ and $\pm\begin{pmatrix} 0 & -\tau^{-1} \\ \tau & 0 \end{pmatrix}$ are contained in the $\mathbf{Z}/2\mathbf{Z}$-extension of $PSL_2(F')$ described in Case IIB. This implies that G is contained in this extension. The order of G

is twice that of $PSL_2(F')$. This implies that G is equal to this extension. This completes Case IIB.

Case IIIA. If $p = 2$ and $p^r > 2$, then $G = PSL_2(F')$. We know by A8 that N_∞ consists of all upper triangular matrices in $PSL_2(F')$, order $F' = 2^r$ for $r > 1$. We also know, by A6, that each coset of N_∞ contains at most $(p^r - 1)$ elements of order 2. Since there are a total of $fp^r(p^r - 1)$ such elements in $G - N_\infty$, each coset contains exactly $(p^r - 1)$ elements of order 2. For each nonidentity coset of N_∞ choose a representative of order 2:

$$V_j = \begin{pmatrix} \alpha_j & \beta_j \\ \gamma_j & \alpha_j \end{pmatrix}, \qquad \gamma_j \neq 0, \quad j = 1, \ldots, fp^r.$$

If $V_j \cdot N_\infty$ contains $p^r - 1$ elements of order 2 (i.e., of trace 0), then for each $\eta \in (F')^*$ there is an element $x \in F'$, so that

$$\alpha_j\eta + \gamma_j\eta x + \alpha_j\eta^{-1} = 0.$$

Since F' has more than two elements, there is $\eta \in (F')^*$ with $\eta \neq \eta^{-1}$. For any such η, we have

$$\alpha_j/\gamma_j = \eta x/(\eta + \eta^{-1}).$$

This proves that $\alpha_j/\gamma_j \in F'$ for $j = 1, \ldots, fp^r$. Thus for every $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in G - N_\infty$ the quotient α/γ is in F'. If $i \neq j$, then $V_i V_j = V_i V_j^{-1}$ is in $G - N_\infty$. Hence

$$\begin{aligned}(\alpha_i\alpha_j &+ \beta_j\gamma_i)/(\gamma_j\alpha_u + \alpha_j\gamma_i) \\ &= (\alpha_i/\gamma_i)(\alpha_j/\gamma_j)\cdot[(\alpha_i/\gamma_i) + (\alpha_j/\gamma_j)]^{-1} + [(\alpha_i/\gamma_i) + (\alpha_j/\gamma_j)]^{-1}\beta_j/\gamma_j\end{aligned}$$

is in F'. It follows easily that β_j/γ_j is in F'. Since $(\alpha_j)^2 - \beta_j\gamma_j = 1$ and α_j/γ_j and β_j/γ_j belong to F', it results that $(\gamma_j)^2 \in F'$. Since F' is of characteristic 2, γ_j also belongs to F'. Thus α_j, β_j, and γ_j belong to F'. This proves that $G \subset PSL_2(F')$. On the other hand, the order of G is $(1 + fp^r)(p^r)(p^r - 1)$ and the order of $PSL_2(F)$ is $(1 + p^r)p^r(p^r - 1)$. Hence $G = PSL_2(F')$.

Case IIIB. If $p^r = 2$, then G is a dihedral group of order $2(1 + 2f)$. (If $f = 1$, then $G = PSL_2(F_2)$.) We know that $d = 1$, and hence that the order of G is $2(1 + 2f)$. The subgroup $G \cap B_\infty$ is cyclic of order 2 and is its own normalizer N_∞. Thus G contains $(1 + 2f)$ conjugates of $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad V_1, \ldots, V_{2f}.$$

Let

$$V_j = \begin{pmatrix} \alpha_j & \beta_j \\ \gamma_j & \delta_j \end{pmatrix}, \text{ where } \gamma_j \neq 0.$$

Since V_j is of order 2, $\delta_j = \alpha_j$. We claim that each nonidentity coset of N_∞ contains exactly one element of order 2. If this is true, then the V_j are representatives for the nonidentity cosets of N_∞. The reason that each nonidentity coset of N_∞ contains at most one element of trace 0 is that

$$\begin{pmatrix}\alpha_j & \beta_j\\ \gamma_j & \alpha_j\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} = \begin{pmatrix}\alpha_j & \alpha_j+\beta_j\\ \gamma_j & \gamma_j+\alpha_j\end{pmatrix},$$

which does not have trace 0 if $\gamma_j \neq 0$. This proves that $U_j = V_j\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$ is not of order two.

Consider now a product $V_i \cdot V_j$, $i \neq j$. We claim that this product is not of order 2. If it is, say $V_i V_j = V_l$, then since all the V_i are conjugate to $V_0 = \left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$, we have

$$(g_i V_0 g_i^{-1}) \cdot (g_j V_0 g_j^{-1}) = g_l V_0 g_l^{-1}$$

for some g_i, g_j, and g_l in G. Conjugating by g_l^{-1} gives

$$(g_l^{-1} g_i) V_0 (g_i^{-1} g_l)(g_l^{-1} g_j) V_0 (g_j^{-1} g_l) = V_0$$

or $V_i' \cdot V_j' = V_0$. As we have already seen, this is impossible. Thus we have a homomorphism from G onto $\mathbf{Z}/2\mathbf{Z}$ that sends each V_j nontrivially and each U_j to the identity. In particular, the $\{U_0, \ldots, U_{2f}\}$ form a normal subgroup of G. The action of V_0 on this subgroup sends U_i to U_i^{-1}. For the function $U_i \to U_i^{-1}$ to be a homomorphism of the group of the U_i, that group must be abelian. Thus the group of the U_i is an abelian subgroup of odd order in $\mathrm{PSL}_2(F)$. Since F is of characteristic 2, the only such groups are cyclic. Thus G itself is dihedral.

Case IIIC. If $p = 3$ and $d = p^r - 1$, then G is either the $\mathbf{Z}/2\mathbf{Z}$-extension of $\mathrm{PSL}_2(F')$ *or an icosahedral group.* As we saw in Case IIB, either $f = 1$, or $f = 3$ and $p^r = 3$. If $f = 1$, then the argument given in Case IIB is valid to show that G is the given $\mathbf{Z}/2\mathbf{Z}$-extension of $\mathrm{PSL}_2(F)$. The remaining case is $p^r = 3$ and $f = 3$. The order of G is $(1 + 3 \cdot 3)(3)(2) = 60$. A straightforward argument shows that G is isomorphic to A_5. All such groups turn out to be conjugate by an upper triangular matrix of the form $\pm\left(\begin{smallmatrix}1 & \alpha\\ 0 & 1\end{smallmatrix}\right)$ for $\alpha^2 \in F$. This completes all possible cases and finishes the classification.

Notice that if $G \subset \mathrm{PSL}_2(F)$ is normally generated by $\mu = \pm\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$, then G is of one of the following types:

(I′) G is cyclic.
(II′) G is conjugate to $\mathrm{PSL}_2(F')$ for F' a subfield of F.
(III′) $p = 3$ and G is isomorphic to A_5 or $p = 2$ and G is isomorphic to a dihedral group of order $(1 + 2f) \cdot 2$.

References

[1] Birman, J., Braids, links, and mapping class groups, *Ann. of Math. Studies*, Vol. 82, Princeton Univ. Press, Princeton, New Jersey, 1974.

[2] Bredon, G., "Introduction to Compact Transformation Groups." Academic Press, New York and London, 1972.

[3] Dickson, L., "Linear Groups." Dover, New York, 1958.

[4] Du Val, P., "Homographies, Quaternions and Rotations." Oxford Univ. Press (Clarendon), London and New York, 1964.

[5] Gordon, C. Mc A., and Heil, W., Cyclic normal subgroups of fundamental groups of 3-manifolds, *Topology* **14** (1975), 305–309.

[6] Morgan, J. W., 3-dimensional Seifert-fibered orbifolds, to appear.

[7] Orlik, P., Seifert manifolds, *Lecture Notes in Mathematics*, Vol. **291**, Springer-Verlag, Berlin and New York, 1972.

[8] Satake, I., On a generalization of the notion of manifold, *Proc. Nat. Acad. Sci. U.S.A.* **42** (1956), 359–363.

[9] Seifert, H., and W. Threlfall, Topologische Untersuchung der Diskontinuitätsbereiche endlicher Bewegungsgruppen des dreidimensionalen sphärischen Raumes I, *Math. Ann.* **104** (1930), 1–70; II, *Math. Ann.* **107** (1932), 543–586.

[10] Waldhausen Q. F., Gruppen mit zentrum und 3-dimensionale Mannigfaltigkeiten, *Topology* **6** (1967), 505–517.