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Introduction

Automatic groups were first defined by Cannon, Epstein, Holt, Paterson and Thurston ([CEHPT, 6.1]). As we shall see, an automatic group is a group with an automatic structure, and an automatic structure is a choice of a particular kind of normal form for the elements of the group. This set of normal forms is required to be a regular language, so the definition of an automatic group rests on formal language theory. However, from the very first, the discussion of automatic groups has been strongly geometric in flavor.

In particular, a normal form for a group element can be regarded as an edge path in the Cayley graph of the group. The central issue in proving that a set of normal forms is an automatic structure is to show that it satisfies the "Fellow Traveller Property." This means that the normal forms for two group elements of distance less than or equal to 1 in the Cayley graph remain a bounded distance apart. A result of [CEHPT, 9.9] is that geodesics in the Cayley graph of a negatively curved group (i.e., a "hyperbolic" or "word hyperbolic" group in the sense of [G]) satisfy the Fellow Traveller Property.

The purpose of this paper is to show that Coxeter groups are automatic. This result is not surprising given the result of Moussong [M] that these groups act cocompactly on piecewise Euclidean complexes of nonpositive curvature. It is currently an open question whether all such groups are automatic. There are many examples which are.

Our normal forms for elements of a Coxeter group will be a subset of the geodesics. The key technical device for proving that this set of normal forms is regular and that it enjoys the Fellow Traveller Property is a general fact about Coxeter groups which we call the Parallel Wall Theorem (Theorem 1.7 below). Roughly speaking, this asserts that any geodesic which starts at a given "wall" in the Cayley graph and moves sufficiently far away from it must cross a "parallel wall." (These terms are defined in Section 1.)

The organization of the paper is as follows. Section 1 consists of preliminaries on Cayley graphs and Coxeter groups and culminates in the statement of the Parallel Wall Theorem. In Section 2 we give basic definitions and results concerning regular languages and automatic groups. In Section 3 we use the Parallel Wall Theorem to show that Coxeter groups are automatic. In Section 4 we prove the Parallel Wall Theorem.

1. Preliminaries on Coxeter Groups and Cayley Graphs

We start with a finitely generated group G. We will take a free monoid $\mathcal{A}^*$ generated by $\mathcal{A} = \{g_1, \dots, g_k\}$. Thus $\mathcal{A}^* = \cup_{m=0}^{\infty}\{a_1 \dots a_m \mid a_i \in \mathcal{A}\}$. We refer to the elements of $\mathcal{A}^*$ as *words*. Multiplication in $\mathcal{A}^*$ is given by concatenation of these words, and the identity element of $\mathcal{A}^*$ is the empty word which we denote by ι. The *length* of a word $w = a_1 \dots a_m$ is m. This is denoted by $\ell(w)$.

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We will assume that we have a monoid map of $\mathcal{A}^*$ onto G which we denote by $w \mapsto \overline{w}$. Thus $\{\overline{g_1}, \ldots, \overline{g_k}\}$ generates G as a monoid. We call $\mathcal{A}$ together with the map into G, a *monoid generating set* for G.

Assume now that we have an involution on $\mathcal{A}$, denoted by $g_i \mapsto g_i^{-1}$ so that $\overline{g_i^{-1}} = \overline{g_i}^{-1}$. The *Cayley graph* of G with respect to $\mathcal{A}$ is a labelled directed graph whose vertices are the elements of G and whose edges are $\{(h, h', g) \mid h, h' \in G,\ g \in \mathcal{A},\ h = h'\overline{g}\}$. The edge (h, h', g) is labelled by g. We say that h and h' are *adjacent.* We denote the Cayley graph by $\Gamma_{\mathcal{A}}(G)$.

Our definition of the Cayley graph differs from the usual one by taking the labels in $\mathcal{A}$ rather than in G. For us there will be no practical difference since the map $\mathcal{A} \mapsto \overline{\mathcal{A}}$ will be an injection.

Since $\overline{\mathcal{A}}$ generates G, $\Gamma_{\mathcal{A}}(G)$ is connected. We consider each edge of $\Gamma_{\mathcal{A}}(G)$ to be isometric to the unit interval, and endow $\Gamma_{\mathcal{A}}(G)$ with the path metric. We denote this metric (and its restriction to G) by $d_{\mathcal{A}}(\cdot, \cdot)$. The restriction of $d_{\mathcal{A}}(\cdot, \cdot)$ to G is referred to as the *word metric.* This is because $d_{\mathcal{A}}(h, h') = \min\{\ell(w) \mid w \in \mathcal{A}^*,\ h = h'\overline{w}\}$. We define a length function on G, $\ell_{\mathcal{A}}(\cdot)$, given by $\ell_{\mathcal{A}}(h) = d_{\mathcal{A}}(1, h) = \min\{\ell(w) \mid w \in \mathcal{A}^*, \overline{w} = h\}$. A word w is called *geodesic* if $\ell(w) = \ell_{\mathcal{A}}(\overline{w})$. Notice that each word w labels a unique edge path starting at 1 and ending at $\overline{w}$. Thus we may consider w a map of the interval $[0, \ell(w)]$ into $\Gamma_{\mathcal{A}}(G)$. This map is an isometry if w is geodesic. In any case, we extend this map to the nonnegative reals by setting $w(t) = \overline{w}$ for $t > \ell(w)$. The natural action of G on $\Gamma_{\mathcal{A}}(G)$ induces an action on edge paths. Thus, if $g \in G$ and $w \in \mathcal{A}^*$, gw is a path from g to $g\overline{w}$. Since G acts on $\Gamma_{\mathcal{A}}(G)$ by isometries, gw is geodesic if and only if w is.

In the case where G is a Coxeter group (which we shall define below), it is traditional to denote the group by W and a generating set of elements of the group by S. In this context we will take W to be our group and S to be our monoid generating set. We take $S = \overline{S}$. In particular, $\overline{S} = S = \{\overline{s}_1, \ldots, \overline{s}_k\}$ is a finite set of generators for W. The pair (W, S) is a *Coxeter system* (and W is a *Coxeter group*) if each element of S is of order two and if W has a presentation with generating set S and relations: $(\overline{s_i}^2)$, $(\overline{s_i s_j})^{m_{ij}}$, where m_{ij} denotes the order of $\overline{s_i s_j}$ in W (and where if $m_{ij} = \infty$, then $(\overline{s_i s_j})^\infty$ is regarded as the empty relation).

For the remainder of this section, (W, S) will denote a Coxeter system and Γ will be its Cayley graph.

The set of conjugates of S in W is denoted by R. An element of R is called a *reflection*; an element of S is a *fundamental* reflection.

The edge $(h\overline{s}, h, s)$, $s \in S$, in the Cayley graph is labelled by the element s in S. It could also be labelled by the reflection r, where $r = hsh^{-1} \in R$. Under the natural action of W on Γ, we have $r(h\overline{s}, h, s) = (h, h\overline{s}, s)$. That is to say, r inverts an edge labelled by r.

A *chamber* in Γ is the closed star of a vertex of Γ in the barycentric subdivision of Γ. Thus, the set of chambers is naturally bijective with the vertex set of Γ, that is, with W. Suppose C_h is the chamber containing $h \in W$. The boundary of C_h consists of a collection of edge midpoints. Such a midpoint is called a *mirror* of C_h.

Suppose $r \in R$. The fixed point set of r on Γ is the union of midpoints of all edges which are labelled by r. This fixed point set is denoted by Γ_r and called the *wall of* Γ *corresponding to* r.

An edge path w in Γ is said to *cross the wall* Γ_r (Γ_r is a discrete set) if the intersection of w and Γ_r is nonempty; the *number of times it crosses* Γ_r is the cardinality of $w \cap \Gamma_r$.

Lemma 1.1. ([B , IV, §1.4, Lemma 1]) *Suppose that $h \in W$ and $r \in R$. For any path w from 1 to h in Γ, let $n(w,r)$ be the number of times w crosses Γ_r. Then the parity of $n(w,r)$ is independent of w.*

Thus, the number $(-1)^{n(w,r)}$ depends only on h and r; it is denoted by $\eta(h,r)$.

Suppose next that w is a path form h to h' in Γ. Then $h^{-1}w$ is a path from 1 to $h^{-1}h'$. Moreover, for any $r \in R$, $h^{-1}(w \cap \Gamma_r) = h^{-1}w \cap \Gamma_{h^{-1}rh}$. Thus, the number of times w crosses Γ_r is the same as the number of times $h^{-1}w$ crosses $\Gamma_{h^{-1}rh}$. Define $\mu : W \times W \times R \to \{\pm 1\}$ by $\mu(h,h',r) = \eta(h^{-1}h', h^{-1}rh)$. It follows form the above discussion that w crosses Γ_r an odd number of times if and only if $\mu(h,h',r) = -1$. If this holds we shall say that Γ_r *separates* h from h'.

Lemma 1.2. ([B , IV §1.4 Lemma 2]) *A path w from h to h' is a geodesic if and only if it crosses each wall which separates h from h' exactly once.*

As an illustration of the use of this result, we prove that the "Deletion Condition" holds for Coxeter groups. (Compare [B, p. 14].)

Lemma 1.3 (The Deletion Condition). *Suppose that $u = s_1 \dots s_k \in \mathcal{S}^*$ and that $\ell(\overline{u}) < k$. Then there are indices $i < j$ so that*

$$\overline{u} = \overline{s_1 \dots s_{i-1} s_{i+1} \dots s_{j-1} s_{j+1} \dots s_k}.$$

Proof. Since u is not a geodesic, it must cross some wall, say Γ_r at least twice. Suppose that the first time it crosses Γ_r is in the ith edge and the next time is in the jth edge. Let $x = s_1 \dots s_{i-1}$, $y = s_{i+1} \dots s_{j-1}$, $z = s_{j+1} \dots s_k$. (See Figure 1.)

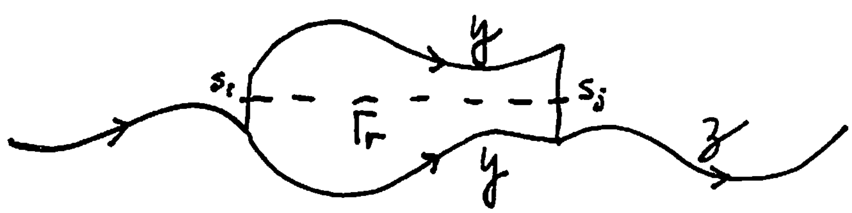

Figure 1.

Consider the path labelled y from $\overline{xs_i}$ to $\overline{xs_iy}$. The reflection r $(= \overline{x^{-1}s_ix})$ takes this path to a path labelled y from $\overline{x}$ to $\overline{xs_iys_j}$. The lemma follows. □

Lemma 1.4. *For each $r \in R$, $\Gamma - \Gamma_r$ has exactly two components.*

Proof. The function $\eta_r : W \to \{\pm 1\}$ defined by $h \to \eta(h,r)$ extends to a continuous function $\Gamma - \Gamma_r \to \{\pm 1\}$. If two vertices h and h' have the same value under η_r, then a geodesic connecting them does not cross Γ_r; hence h and h' belong to the same component of $\Gamma - \Gamma_r$. It follows that η_r induces an identification of $\pi_0(\Gamma - \Gamma_r)$ with $\{\pm 1\}$. □

The closure of a component of $\Gamma - \Gamma_r$ is called a *half-space* bounded by Γ_r.

Next suppose that r_1 and r_2 are distinct elements of R and that D is the subgroup of W generated by $\{r_1, r_2\}$. (So that D is isomorphic to a dihedral group.) Let T denote the set of conjugates of r_1 and r_2 by elements of D. Call two elements h and h' of W, *D-equivalent*, if h and h' are not separated by a wall of Γ which is indexed by an element of T. Call two D-equivalence classes $[h]$ and $[h']$ *adjacent* if h and h' are separated by a single wall indexed by an element of T. Let Λ_D be the graph with vertex set the set of D-equivalence classes of elements in W and with an edge connecting two vertices if and only if they are adjacent. The D-action on W by left translation induces a D-action on Λ_D. Each edge of Λ_D is labelled by an element of T. If v is a vertex of Λ_D, then let S_v denote the set of elements in T which label the edges incident to v.

Lemma 1.5. *Suppose that r_1, r_2 are in R and that $r_1 r_2$ is of infinite order in W. Let D and Λ_D be as above. Then Λ_D is isomorphic to the standard Cayley graph of the infinite dihedral group. That is to say, D acts freely and transitively on the vertex set of Λ_D and Λ_D is isometric to the real line. Moreover, there is a vertex v of Λ_D such that $S_v = \{r_1, r_2\}$.*

Proof. We first make two observations.

(1) Λ_D is connected.

(2) For each $t \in T$, the fixed point set Λ_t separates Λ_D into two components.

Choose a vertex v in Λ_D and let C_v denote the closed star of v in the barycentric subdivision of Λ_D. Recall that S_v is the set of elements of T which label the edges incident to v. Let D' be the subgroup of D generated by S_v. It follows from (1) that $D'C_v = \Lambda_D$. Hence,

(3) S_v generates D (i.e. $D' = D$), and

(4) D acts transitively on the vertex set of Λ_D.

From (2) and very general considerations one can prove that:

(5) (D, S_v) is a Coxeter system

(6) D acts freely on the vertices of Λ_D.

The argument is essentially the same as the proof of Theorem 1 in [B, V §3.2]. To prove that Λ_D is isometric to the real line it clearly suffices to show that each vertex is incident to exactly 2 edges, i.e., that $\operatorname{Card}(S_v) = 2$. Since S_v generates D, $\operatorname{Card}(S_v) \geq 2$. The product of two distinct elements of T has infinite order in D. Since $S_v \subset T$, the same holds for S_v. It follows that if $\operatorname{Card}(S_v) \geq 3$, then the corresponding Coxeter group has a standard subgroup with Coxeter diagram:

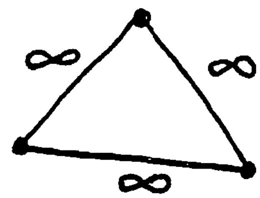

Figure 2.

But such a Coxeter group contains a free group on two generators. Since D contains no such group, we much have $\text{Card}(S_v) = 2$.

It remains to prove the last sentence of the lemma. Choose the vertex v so that $r_1 \in S_v$. Since S_v generates D the other element must be either r_2 or $r_1 r_2 r_1$. If it is $r_1 r_2 r_1$, then $S_{r_1 v} = \{r_1, r_2\}$. Hence, after replacing v by $r_1 v$ if necessary, we will have that $S_v = \{r_1, r_2\}$. □

Complement. *If $r_1 r_2$ has finite order m, then D is the dihedral group of order $2m$. In this case one can prove the analogous result to the above lemma: that Λ_D is isometric to a $2m$-sided polygon.*

The proof (which is omitted) uses the fact that W has a "canonical" representation as a linear reflection group (cf. [B, V §4.4]).

If X is any subset of the Cayley graph Γ, then denote by $W(X)$ the set of vertices which lie in X.

Lemma 1.6. *Suppose that r_1 and r_2 are in R and that $r_1 r_2$ is of infinite order in W. Then there are half-spaces H_1^+ and H_2^+ in Γ bounded by the walls Γ_{r_1} and Γ_{r_2}, respectively, so that $W(H_1^+) \cap W(H_2^+)$ is a single D-equivalence class. (Here D is as in the previous lemma.) Moreover, if for $i = 1, 2$, H_i^- denotes the opposite half-space to H_i^+, then*

$$H_1^- \cap H_2^- = \emptyset,$$
$$H_1^- \subset H_2^+, \quad \text{and}$$
$$H_2^- \subset H_1^+.$$

Proof. By Lemma 1.5 we can find a D-equivalence class v such that the adjacent classes are $r_1 v$ and $r_2 v$. For $i = 1, 2$, let H_i^+ be the half-space bounded by Γ_{r_i} which contains the elements of W in the class v. It follows immediately that

$$W(H_1^+) \cap W(H_2^+) = v.$$

For $i = 1, 2$, let D_i be the set of elements in the infinite dihedral group D which when written as reduced words in $\{r_1, r_2\}$ begin with r_i. Then $D = \{1\} \amalg D_1 \amalg D_2$ and

$$W(H_i^-) = \bigcup_{h \in D_i} hv.$$

It follows that $H_1^- \cap H_2^- = \emptyset$, that $H_1^- \subset H_2^+$ and that $H_2^- \subset H_1^+$. □

Definition. *Suppose that r_1 and r_2 are reflections in a Coxeter group W and that Γ_1 and Γ_2 are the corresponding walls in the Cayley graph. Then Γ_1 and Γ_2 are parallel if $r_1 r_2$ is of infinite order in W.*

Suppose that Γ_1 and Γ_2 are parallel walls in Γ and that $h \in W$. For $i = 1, 2$, let H_i^+ be the half-space bounded by Γ_i which contains h and let H_i^- be the opposite half-space. By Lemma 1.6 there are only three possibilities:

(1) $H_1^+ \cap H_2^+$ is a fundamental domain for the action on Γ of the dihedral group generated by $\{r_1, r_2\}$.
(2) $H_2^+ \subset H_1^+$
(3) $H_1^+ \subset H_2^+$.

If case (2) holds, we say that Γ_2 *separates h from* Γ_1. We note that if case (3) holds for Γ_1 and Γ_2, then case (2) holds for Γ_1 and $r_2\Gamma_1$, i.e., in case (3), $\Gamma_{r_2 r_1 r_2}$ separates h from Γ_1.

We can now state

Theorem 1.7. (The Parallel Wall Theorem) *Let (W, S) be a Coxeter system. Then there is a constant $K(= K(W, S))$ with the property $(*)$ below.*

$(*)$ *For any $r \in R$ and $h \in W$, if the distance from the vertex h to the wall Γ_r is greater than K, then there is another reflection $r' \in R$ such that $\Gamma_{r'}$ is parallel to Γ_r and $\Gamma_{r'}$ separates h from Γ_r.*

2. Preliminaries on Regular Languages and Automatic Structures

Given $\mathcal{A}$, a *language* over $\mathcal{A}$ is a subset of $\mathcal{A}^*$, the free monoid on $\mathcal{A}$. Given two languages $L, M \subset \mathcal{A}^*$, we may perform the standard Boolean operations to find languages $L \cup M$, $L \cap M$ and $\mathcal{A}^* \setminus L$. In addition, the multiplication operator in $\mathcal{A}^*$, namely, concatenation allows us to define $LM = \{uv \mid u \in L,\ v \in M\}$, and for $n \geq 1$, $L^n = \{u_1 \ldots u_n \mid u_i \in L\}$. We take $L^0 = \{\iota\}$, and set $L^* = \cup_{n=0}^{\infty} L^n$. This latter agrees with our usage that $\mathcal{A}^*$ is the free monoid on $\mathcal{A}$.

There are several ways to define the class of regular languages over $\mathcal{A}$. Perhaps the simplest is to say that it is the smallest class which contains all the finite subsets of $\mathcal{A}^*$ and is closed under all the operations of the previous paragraph. Another way is to say that a language is regular if it is the language of a finite state automaton, which we now define.

A *finite state automaton* is a 5–tuple $(\mathcal{A}, T, t_0, \tau, Y)$ where $\mathcal{A}$ is a finite set called an *alphabet*, its elements are called *letters*, T is a finite set of *states*, $t_0 \in T$ is called the *start state*, τ is a function from $\mathcal{A} \times T$ to T, called the *transition function*, and $Y \subset T$ is the set of *accept states*.

Each finite state automaton with alphabet $\mathcal{A}$ defines a language over $\mathcal{A}$ in the following way. Let $w = a_1 \ldots a_n \in \mathcal{A}^*$ with each $a_i \in \mathcal{A}$. We let $p_0 = t_0$, and for $1 \leq j \leq n$ define p_j inductively by $p_j = \tau(a_j, p_{j-1})$. We accept w if and only if $p_n \in Y$.

If we like, we may see a finite state automaton as a finite labelled directed graph with a base point and a set of preferred vertices. The vertices of this graph are the states of the machine, the base point corresponds to t_0. There is an edge from t_i to t_j labelled by $a \in \mathcal{A}$ exactly when $\tau(a, t_i) = t_j$. The preferred vertices are the accept states. In this model, the language accepted by a finite state automaton consists of those words which label edge paths starting at the base point and ending at preferred vertices.

Before defining an automatic structure, we will need the notion of a product language. We will wish to study pairs of words (w, w') where $w, w' \in \mathcal{A}^\star$. To do this, we suppose that $\$ \notin \mathcal{A}$ and consider $\mathcal{A}_\$ = (\mathcal{A} \cup \{\$\}) \times (\mathcal{A} \cup \{\$\}) \setminus \{(\$,\$)\}$. We include $\mathcal{A}^\star \times \mathcal{A}^\star$ into $\mathcal{A}_\$^\star$ as follows. Given $w = a_1 \dots a_m$, $w' = b_1 \dots b_n$, we let $p = \max\{m, n\}$. If $m < p$, for $m < j \leq p$ we let $a_j = \$$. If $n < p$, for $n < j \leq p$ we let $b_j = \$$. We now take $(w, w') \in \mathcal{A}^\star \times \mathcal{A}^\star$ to $(a_1, b_1) \dots (a_p, b_p) \in \mathcal{A}_\$^\star$. Note that this inclusion is not a monoid map as its image need not be closed under concatenation. We will suppress reference to $\mathcal{A}_\$^\star$. However, whenever we state that a collection of pairs of words is regular, we will mean that its image is regular in $\mathcal{A}_\$^\star$.

It is now easy to state the definition of an automatic structure.

Definition. ([CEHPT, 6.1]) *Given a finite set $\mathcal{A}$ together with a monoid map $\mathcal{A}^\star \mapsto G$ and a language L over $\mathcal{A}$, we will say that $(\mathcal{A}, L)$ is an automatic structure for G if*

(1) *L is regular.*
(2) *$\overline{L} = G$.*
(3) *$\{(w, w') \mid w, w' \in L,\ \overline{w} = \overline{w'}\}$ is regular.*
(4) *For each $g \in \mathcal{A}$, $\{(w, w') \mid w, w' \in L,\ \overline{w} = \overline{w'g}\}$ is regular.*

If G has an automatic structure, we say G is automatic.
If in addition L satisfies
(4′) For each $g \in \mathcal{A}$, $\{(w, w') \mid w, w' \in L,\ \overline{w} = \overline{gw'}\}$ is regular,
we say that $(\mathcal{A}, L)$ is a biautomatic structure. If G has a biautomatic structure, we say that G is biautomatic.

(This definition of a biautomatic structure is a paraphrase of [CEHPT, 24.1].)

If $\mathcal{A}$ is a monoid generating set for G and N is a constant, we say that u, v are *N-fellow travellers* if for all t, $d_{\mathcal{A}}(u(t), v(t)) \leq N$.

We will use the following as our criterion for automaticity.

Theorem 2.1. ([CEHPT, 6.7, 24.2]) *Let $\mathcal{A}$ be a monoid generating set for G, and let $L \subset \mathcal{A}^\star$ be a regular language which surjects onto G. Then*

(1) L is the language of an automatic structure for G if and only if there is a constant N such that whenever $u, v \in L$, $g \in \mathcal{A} \cup \{\iota\}$ with $\overline{u} = \overline{vg}$, it follows that u and v are N-fellow travellers.

(2) L is the language of a biautomatic structure for G if and only if L is the language of an automatic structure for G and there is a constant N such that whenever $u, v \in L$, $g \in \mathcal{A}$ with $\overline{u} = \overline{gv}$, it follows that u and gv are N-fellow travellers.

Definition. *A partial order $\prec$ on $\mathcal{A}^\star$ is regular if $\prec = \{(u, v) \mid u \prec v\}$ is a regular product language.*

(As usual, this definition abuses notation by identifying $\prec \subset \mathcal{A}^\star \times \mathcal{A}^\star$ with its image in $\mathcal{A}_\$^\star$.)

Given a monoid map of $\mathcal{A}^\star$ onto G and a partial order relation $\prec$, we define

$$L_\prec = \{u \mid u \text{ is } \prec \text{ minimal for } \overline{u}\}$$

The following lemma encodes a standard line of argument which might be dubbed "falsification by a fellow traveller."

Lemma 2.2. *Suppose that $\prec$ is a regular partial order. Suppose that there is a N so that if u is not $\prec$ minimal for $\overline{u}$, then there is u' so that $\overline{u}' = \overline{u}$, $u' \prec u$, and u' and u are N fellow travellers. Then $L_\prec$ is regular.*

Proof. We first use the "standard comparator automata" of [CEHPT, 6.5] to show that $L_N = \{(u,v) \mid \overline{u} = \overline{v}$ and u and v are N fellow travellers$\}$ is regular. We build a machine M_N whose states are the vertices of $B(N) = \{g \in G \mid \ell_{\mathcal{A}}(g) \leq N\}$ together with a fail state. The start state is 1. For each state $g \in B(N)$ and each pair of letters (a_1, a_2) such that $\overline{a_1^{-1} g \overline{a_2}} \in B(N)$, there is an edge from g to $\overline{a_1^{-1} g \overline{a_2}}$ labelled by (a_1, a_2). (We take $\overline{\$} = 1$.) For each state g and each pair of letters (a_1, a_2) such that $\overline{a_1^{-1} g \overline{a_2}} \notin B(N)$, there is an edge from g to the fail state. There is no edge out of the fail state. The sole accept state is 1. The reader may check that M_N accepts a pair (u,v) if and only if $(u,v) \in L_N$.

We let p_2 denote projection on the second factor. It is a theorem that the projection of a regular language is regular (see, for example [CEHPT, 2.2].) We have

$$L_\prec = \mathcal{A}^* \setminus p_2(L_N \cap \prec).$$

Consequently $L_\prec$ is regular. □

We set $u \prec_1 v$ if $\ell(u) < \ell(v)$. Thus $L_{\prec_1}$ is the language of geodesics. We fix an ordering of $\mathcal{A}$. This induces a lexicographic ordering on $\mathcal{A}^i$ for each $i \geq 0$. We then have a total ordering $\prec_2$ of $\mathcal{A}^*$ by taking $\prec_2$ to be the lexicographic ordering on each $\mathcal{A}^i$ and taking $\mathcal{A}^0 \prec_2 \mathcal{A}^1 \prec_2 \mathcal{A}^2 \ldots$. Thus $L_{\prec_2}$ contains the lexicographically least geodesic for each element of G. We will refer to $L_{\prec_1}$ and $L_{\prec_2}$ as L_{geo} and L_{lex}, respectively.

The following lemma is a standard result. The proof is left to the reader.

Lemma 2.3. *The partial orderings $\prec_1$ and $\prec_2$ are regular.*

3. Coxeter Groups are Automatic

Theorem 3.1. *Finitely generated Coxeter groups are automatic.*

We fix a monoid generating set S so that (W, S) is a Coxeter System, where $\overline{S} = S$, and we fix an ordering on S.

We will prove Theorem 3.1 by showing that L_{geo} and L_{lex} are regular, and that L_{lex} satisfies the Fellow Traveller Property (1) of Theorem 2.1.

Lemma 3.2. *Let $N = 2K + 2$, where K is the constant of Theorem 1.7. Suppose that there are elements $s_1, s_2 \in S$ and a geodesic u such that $\overline{us_2} = \overline{s_1 u}$. Put $v = s_1 u s_2$. Then u and v are N-fellow travellers.*

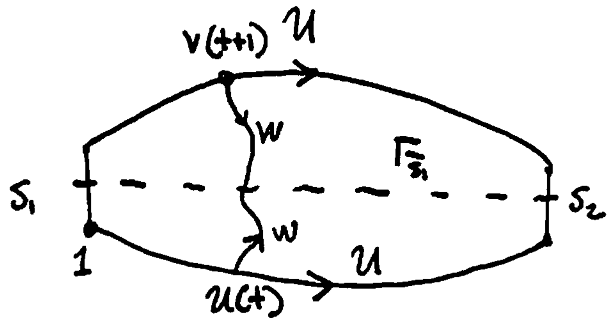

Figure 3.

Proof. Consider the wall $\Gamma_{\overline{s_1}}$ ($= \overline{u}\Gamma_{\overline{s_2}}$). We claim that for each t, $0 \le t \le \ell(u)$, the distance from $u(t)$ to $\Gamma_{\overline{s_1}}$ is $\le K + \frac{1}{2}$. For otherwise, by Theorem 1.7, there would be a parallel wall separating $u(t)$ from $\Gamma_{\overline{s_1}}$; but u would cross such a wall at least twice, contradicting the supposition that it is geodesic. Hence there is a path w of length $\le K + \frac{1}{2}$ from $u(t)$ to $\Gamma_{\overline{s_1}}$. The reflection of w across $\Gamma_{\overline{s_1}}$ is a path from $\Gamma_{\overline{s_1}}$ to $v(t+1)$. Thus w together with its reflected image gives a path of length $\le 2K + 1$ from $u(t)$ to $v(t+1)$. Consequently, $d(u(t), v(t)) \le d(u(t), v(t+1)) + 1 \le 2K + 2 = N$.

Lemma 3.3. *Let $N = 2K + 2$, where K is the constant of Theorem 1.7.*

(1) *If u is not $\prec_1$ minimal for $\overline{u}$,then there is u' such that $\overline{u'} = \overline{u}$, $u' \prec_1 u$, and u' and u are N–fellow travellers.*

(2) *If u is not $\prec_2$ minimal for $\overline{u}$, then there is u' such that $\overline{u'} = \overline{u}$, $u' \prec_2 u$, and u' and u are N–fellow travellers.*

Proof. (1) Suppose u is not $\prec_1$ minimal for $\overline{u}$, that is to say, u is not geodesic. Then by the Deletion Condition (Lemma 1.3), we can write $u = xs_1ys_2z$, where $s_1, s_2 \in \mathcal{S}$, y is geodesic, and $\overline{u} = \overline{xyz}$. Put $u' = xyz$. Then $u' \prec_1 u$. For $t \le \ell(x)$, $u(t) = u'(t)$, while for $t \ge \ell(x) + \ell(y) + 2$, $u(t) = u'(t-2)$. Finally, for $\ell(x) < t < \ell(x) + \ell(y) + 2$, Lemma 3.2 implies that $d(u(t), u'(t)) \le N$. Hence u and u' N–fellow travel.

(2) Suppose that u is geodesic, but not lexicographically least for $\overline{u}$. We then have $\overline{u} = \overline{u''}$, where $u = xy$, $u'' = xy''$ where $y'' \prec_2 y$ and $\overline{y''} = \overline{y}$. (See Fig. 4.)

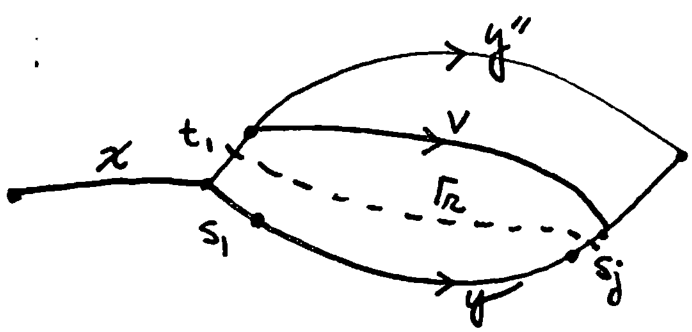

Figure 4.

Let $y = s_1 \dots s_k$, $y'' = t_1 \dots t_k$, where each s_i and t_i are elements of $\mathcal{S}$. In particular, $t_1 \prec_2 s_1$. Let Γ_r be the wall containing the t_1 mirror of $\overline{x}$. Now y and y'' must cross the same set of walls, so there is j, $1 < j \le k$ so that the s_j mirror of $\overline{xs_1 \dots s_{j-1}}$ is also Γ_r.

Let v label the path given by the reflection of $s_1 \dots s_{j-1}$ across Γ_r. Then $\overline{t_1 v} = \overline{s_1 \dots s_j}$. Both of these are geodesic, so by Lemma 3.2, $t_1 v$ and $s_1 \dots s_k$ are N–fellow travellers. Let $u' = xt_1 v s_{j+1} s_k$. Then $\overline{u'} = \overline{u}$, $u' \prec_2 u$ and u and u' are N–fellow travellers as required. □

From Lemmas 2.2, 2.3, and 3.3 we immediately get the following.

Corollary 3.4. *Let (W, S) be a Coxeter system. Then L_{geo} and L_{lex} are regular.*

Finally, we must show the following.

Lemma 3.5. *Suppose $w, w' \in L_{\text{lex}}$, and $s \in S$. If $\overline{w} = \overline{w's}$ then w and w' are N–fellow travellers.*

Proof. We first suppose that $\overline{w} = \overline{w's}$, and without loss of generality assume that $\ell(w) < \ell(w')$. We then have $w = xs_1 \dots s_k$ and $w' = xt_1 \dots t_{k+1}$, where each s_i and s_j are in S, and $s_1 \neq t_1$. Since w' is $\prec_2$ minimal for $\overline{w'}$ and ws is also a geodesic for $\overline{w'}$, it follows that $t_1 \prec_2 s_1$.

Let Γ_r be the wall containing the s mirror of $\overline{w}$. Then for some j, Γ_r also contains the t_j mirror of $\overline{xt_1 \dots t_j}$. We distinguish two cases.

Case 1. $j = 1$. (See Fig. 5.) In this case $\overline{s_1 \dots s_k} = \overline{t_2 \dots t_{k+1}}$. Since both must be $\prec_2$ minimal, it follows that $s_1 \dots s_k = t_2 \dots t_{k+1}$. The path labelled $s_1 \dots s_k$ starting at $\overline{x}$ and the path labelled $t_2 \dots t_{k+1}$ starting at $\overline{xt_1}$ are reflections of each other in Γ_r. Since they are both geodesics, by Lemma 3.2, they are N–fellow travellers.

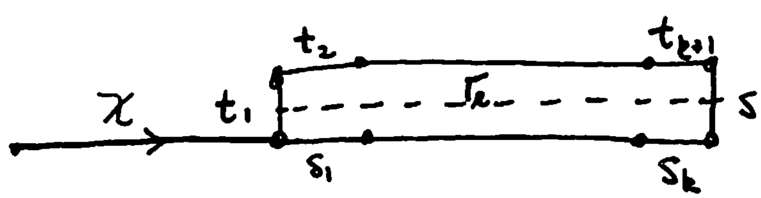

Figure 5.

Case 2 $j \neq 1$. (See Fig. 6.) Applying the Deletion Condition again, we see that $\overline{t_1 \dots t_{j-1} t_{j+1} \dots t_{k+1}} = \overline{s_1 \dots s_k}$. But, $t_1 \dots t_{j-1} t_{j+1} \dots t_{k+1} \prec_2 s_1 \dots s_k$, and hence $xt_1 \dots t_{j-1} t_{j+1} \dots t_{k+1} \prec_2 xs_1 \dots s_k = w$, contradicting the $\prec_2$ minimality of w. □

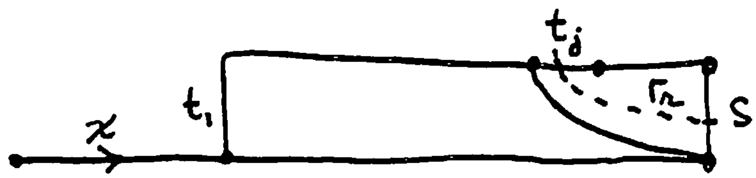

Figure 6.

This concludes the proof of Theorem 3.1 modulo the proof of the Parallel Wall Theorem. We prove the Parallel Wall Theorem in Section 4.

Remark 1 We have seen that the Parallel Wall Theorem implies that L_{geo} is regular. This is closely connected with the question of *cones* and *cone types*.

Suppose that $w = s_1 \dots s_n \in L_{\text{geo}}$ and that $\overline{w} = g$. For $0 \leq i \leq n$ we let $r_i = \overline{s_1 \dots s_i s_{i-1} \dots s_1}$, and let H_i be the half-space corresponding to r_i which does not contain the identity. It then follows from Lemma 1.2 that the set $\mathcal{H} = \{H_i\}$ depends only on g and not on w. We define the *cone* at g to be

$$\mathrm{c}(g) = \bigcap_{H \in \mathcal{H}} H.$$

(When $g = 1$, we take the empty intersection to be all of Γ.)

Notice that $W(\mathrm{c}(g))$ (the set of vertices of Γ in $\mathrm{c}(g)$) consists of those $h \in W$ which are *outbound* from g. That is to say $W(\mathrm{c}(g))$ consists of those h for which there is a geodesic from 1 to h which passes through g. Notice also that if C denotes the fundamental chamber then $\mathrm{c}(g) = W(\mathrm{c}(g))C$.

While $\mathrm{c}(g)$ is defined to be the intersection of $\ell(g)$ half-spaces, in fact $\mathrm{c}(g)$ is the intersection of a bounded number of half-spaces where the bound is independent of $\ell(g)$. For suppose that $h \notin \mathrm{c}(g)$. We let u be a geodesic from 1 to g and v be a geodesic from g to h. Then uv is not geodesic, and must cross some wall twice. Let Γ_r be the first wall which is crossed twice. As is shown in the proof of Lemma 3.2, $d(g, \Gamma_r) \leq K + \frac{1}{2}$ where K is the constant of the Parallel Wall Theorem. Letting $B_g(K + \frac{1}{2})$ denote the ball of radius $K + \frac{1}{2}$ around g and $\mathcal{H}' = \{H_i \in \mathcal{H} \mid \Gamma_{r_i} \cap B_g(K + \frac{1}{2}) \neq \emptyset\}$, we have

$$\mathrm{c}(g) = \bigcap_{H \in \mathcal{H}'} H.$$

Left multiplication induces an equivalence relation on cones. That is, $\mathrm{c}(g)$ and $\mathrm{c}(g')$ are equivalent if $g^{-1}\mathrm{c}(g) = g'^{-1}\mathrm{c}(g')$. We call the equivalence class of $\mathrm{c}(g)$ the *cone type* of g. (This does not agree with standard usage. However, it suffices to carry the standard argument in the present case.) Since a fixed number of walls meet any ball of radius $K + \frac{1}{2}$, there are only finitely many cone types.

One may take these cone types (together with a fail state) as the states of a finite state automaton whose language is L_{geo}. For further details (and a correct definition of cone types!) see [C].

Remark 2 One might ask whether the automatic structure $(\mathcal{S}, L_{\text{lex}})$ is always a biautomatic structure. That is to say, we would like to know whether for each $s \in \mathcal{S}$, the language $\{(w, w') \mid w, w' \in L_{\text{lex}} \text{ and } \overline{w} = \overline{sw'}\}$ is regular. The answer is no. Consider the Coxeter system $(W, \mathcal{S})$, where $\mathcal{S} = \{a, b, c, x, y\}$, and W is the direct product of

$$W_1 = \langle \overline{a}, \overline{b}, \overline{c} \mid \overline{a}^2 = \overline{b}^2 = \overline{c}^2 = 1 \rangle$$

and

$$W_2 = \langle \overline{x}, \overline{y} \mid \overline{x}^2 = \overline{y}^2 = 1 \rangle.$$

Notice that in W_1 and W_2, there is exactly one geodesic word for each element. Thus, in W geodesics for any given element differ only by commuting generators of W_1 past generators of W_2.

If we take the lexicographical ordering determined by

$$a \prec_2 b \prec_2 c \prec_2 x \prec_2 y,$$

Then $L_{\text{lex}}(W) = L_{\text{lex}}(W_1)L_{\text{lex}}(W_2)$. (The notation here has the obvious meaning.) The reader may check that this is indeed the language of a biautomatic structure. However, if we take the ordering

$$a \prec_2 b \prec_2 x \prec_2 y \prec_2 c,$$

we no longer have a biautomatic structure. To see this, choose n and consider the word $w = (ab)^n(xy)^n$. This word is the lexicographically least representative for $\overline{w}$. On the other hand, the lexicographically least representative for $\overline{cw} = \overline{c(ab)^n(xy)^n}$ is $w' = (xy)^n c(ab)^n$. But now $d(\overline{c}w(n), w'(n)) = 4n$. Since we may choose n to be arbitrarily large, for any given N we can find w and w' so that $\overline{cw} = \overline{w'}$ but cw and w' are not N fellow travellers. Thus $L_{\text{lex}}(W)$ is not the language of a biautomatic structure.

4. Proof of the Parallel Wall Theorem

Theorem 1.7. *(The Parallel Wall Theorem) Let (W, S) be a Coxeter system. Then there is a constant $K(= K(W, S))$ with the property $(*)$ below.*

$(*)$ *For any $r \in R$ and $h \in W$, if the distance from the vertex h to the wall Γ_r is greater than K, then there is another reflection $r' \in R$ such that $\Gamma_{r'}$ is parallel to Γ_r and $\Gamma_{r'}$ separates h from Γ_r.*

The proof is based on Lemmas 4.1 and 4.2 below. In order to state these lemmas we first need to develop some more terminology.

Any Coxeter system (W, S) can be decomposed as a direct product of irreducible Coxeter systems, cf. [B, IV §1.9]. This means that S can be partitioned as $S = S_1 \amalg \cdots \amalg S_n$, where for $i \neq j$ each element of S_i commutes with each element of S_j and where if W_i denotes the subgroup generated by S_i, then (W_i, S_i) is an irreducible system. Let S_{fin} be the union of those S_i such that W_i is finite and let S_{inf} be the union of the remaining S_i's. Let W_{fin} and W_{inf} be the subgroups generated by S_{fin} and S_{inf}, respectively, so that $W = W_{fin} \times W_{inf}$.

Let I denote the set of conjugacy classes of elements in S. Since no element of S_{fin} is conjugate to one of S_{inf}, the partition $S = S_{fin} \amalg S_{inf}$ induces a partition $I = I_{fin} \amalg I_{inf}$. By definition, any element r in R is conjugate to an element of S; the image of this element in I is well-defined and denoted by Type(r). We say that r *belongs to a finite factor* if Type$(r) \in I_{fin}$.

Lemma 4.1. *Suppose that r does not belong to a finite factor. Then there is a reflection r' such that Γ_r and $\Gamma_{r'}$ are parallel.*

In order to state the next lemma we need some more definitions.

Suppose that $S = \{s_0, \ldots, s_n\}$. Let m_{ij} denote the order of $\overline{s_i s_j}$ in W. The $(n+1)$ by $(n+1)$ matrix (m_{ij}) is called the *Coxeter matrix* of (W, S). Associated to the Coxeter matrix we have the *cosine matrix* $C = (c_{ij})_{0 \le i,j \le n}$ defined by

$$c_{ij} = -\cos(\pi/m_{ij})$$

(where π/∞ is interpreted to be 0).

Definition. *Suppose that C is the cosine matrix associated to Coxeter system (W, S) of rank $n+1$ (where the rank is the number of elements in S). Then W is of hyperbolic type if the cosine matrix has n positive eigenvalues and 1 negative one. It is of Euclidean type if C is positive semidefinite and precisely one eigenvalue is 0.*

Remarks. The group W is finite if and only if its cosine matrix is positive definite (cf Théorème 2 in [B, V §4.8]). In particular, W is infinite if it is of Euclidean or hyperbolic type. Since $c_{ij} = 0$ if and only if $m_{ij} = 2$, it is obvious that a decomposition of (W, S) into direct factors is equivalent to a decomposition of its cosine matrix into blocks (possibly after renumbering the elements of S). If (W, S) is of Euclidean type (respectively, hyperbolic type), then clearly the cosine matrix of any direct factor must be either positive definite or of Euclidean type (respectively, hyperbolic type); moreover, (W, S) cannot have more than one direct factor of Euclidean type (respectively, hyperbolic type). It follows that (W_{inf}, S_{inf}) is irreducible and of Euclidean type (respectively, hyperbolic type).

The second lemma is the following special case of Lemma 4.1.

Lemma 4.2. *Suppose that (W, S) is of Euclidean or hyperbolic type and that r is a reflection which does not belong to a finite factor. Then there is a reflection r' so that Γ_r and $\Gamma_{r'}$ are parallel walls.*

We shall now show that Lemma 4.2 $\Rightarrow$ Lemma 4.1 $\Rightarrow$ The Parallel Wall Theorem. Then we will prove Lemma 4.2.

Proof that Lemma 4.2 $\Rightarrow$ Lemma 4.1. Let (W, S) be an arbitrary Coxeter system and r a reflection in W. After replacing r by a conjugate, we may assume that $r \in S$. Furthermore, without loss of generality we may assume that $S = S_{inf} \neq \emptyset$ and that $W = W_{inf}$. (Lemma 4.1 only concerns r which do not belong to a finite factor.)

Let $\mathcal{P}$ be the poset of all subsets T of S satisfying the following three conditions:

(1) $r \in T$
(2) Not every element of T commutes with r.
(3) W_T is infinite.

(For any subset X of S, W_X denotes the subgroup generated by X.) The poset $\mathcal{P}$ is nonempty, since $S \in \mathcal{P}$. Choose a minimal element T in $\mathcal{P}$. We can write $T = \{s_0, s_1, \ldots, s_n\}$ where $s_0 = r$. For $i = 1, 2, \ldots n$, put $S_i = T - \{s_i\}$. If $n = 1$ then since W_T is infinite, $m_{01} = \text{order}(s_0 s_1) = \infty$ and therefore Γ_{s_1} and $\Gamma_{s_0} = \Gamma_r$ are parallel and we are done. Therefore, we assume $n \geq 2$. Since T is minimal, either property (2) or (3) fails for each S_i. But property (2) cannot fail for more than one S_i. Hence, there exists an S_j such that (3) fails, i.e., such that W_{S_j} is finite. The cosine matrix of (W_{S_j}, S_j) is therefore positive definite. Since the matrix for (W_{S_j}, S_j) is a principal minor of the cosine matrix for (W_T, T) it follows that (W_T, T) is either of Euclidean or hyperbolic type. By Lemma 4.2, there is a reflection r' in W_T (and then *a fortiori* in W) such that rr' is of infinite order, i.e., Γ_r and $\Gamma_{r'}$ are parallel. □

Proof that Lemma 4.1 ⇒ The Parallel Wall Theorem. For each reflection r in R, let H_r^+ denote the half-space in the Cayley graph Γ which is bounded by Γ_r and which contains the vertex 1. For each $s \in S_{inf}$ let $R(s)$ denote the set of r in R such that Γ_r and Γ_s are parallel and such that $H_s^+ \subset H_r^+$. We note that if Γ_r and Γ_s are parallel, then by Lemma 1.6 either $H_s^+ \subset H_r^+$ or $H_s^+ \subset H_{srs}^+$. Hence, Lemma 4.1 implies that $R(s)$ is nonempty for all $s \in S_{inf}$. The distance from a vertex h in Γ to a wall Γ_r is denoted by $d(h, \Gamma_r)$ (a half-integer). Put

$$m(s) = \inf_{r \in R(s)} d(1, \Gamma_r), \quad \text{and}$$

$$M = \sup_{s \in S_{inf}} m(s).$$

Since S_{inf} is a finite set, $M < \infty$. Let M' be the length of the element of longest length in W_{fin}. We note that if $s \in S_{fin}$, then the distance from any vertex h to Γ_s is bounded by M'. (In fact, it is $\leq [M'/2] + \frac{1}{2}$). We claim that

$$K = \max(M, M')$$

is the desired constant in the Parallel Wall Theorem.

To check this we suppose that h is a vertex of Γ, that $r \in R$ and that $d(h, \Gamma_r) > K$. Choose a geodesic from h to Γ_r which realizes this distance. Denote the last vertex it passes through before it gets to Γ_r by g. Then g and rg are adjacent vertices and the midpoint of the edge between them is in Γ_r. It follows that $g^{-1}\Gamma_r = (rg)^{-1}\Gamma_r$ is a fundamental wall, i.e., $g^{-1}rg \in S$. Put $s = g^{-1}rg$. Translating the geodesic from h to Γ_r by $(rg)^{-1}$ we see that

(1) Γ_s separates 1 from $(rg)^{-1}h$
(2) $d((rg)^{-1}h, \Gamma_s) = d(h, \Gamma_r) > K$.

Since $K \geq M'$, s cannot belong to S_{fin}. Since $K \geq m(s)$ there is a $r'' \in R(s)$ such that Γ_s and $\Gamma_{r''}$ are parallel and such that $d(1, \Gamma_{r''}) = m(s) \leq K$. Since $r'' \in R(s)$

and $d((rg)^{-1}h, \Gamma_s) > K$ it follows that $\Gamma_{r''}$ separates $(rg)^{-1}h$ from Γ_s. Putting $r' = (rg)r''(rg)^{-1}$ and translating back by rg we find that $\Gamma_{r'}$ $(= rg\Gamma_{r''})$ separates h from Γ_r $(= rg\Gamma_s)$, as was to be proved. □

Before embarking on the proof of Lemma 4.2, let us sketch the general idea behind the proof. The Coxeter system (W, S) is of Euclidean or hyperbolic type. As before, we may assume that $(W, S) = (W_{inf}, S_{inf})$. Then (W, S) is an irreducible Coxeter system and it acts as a discrete group generated by isometric reflections on M^n, where M^n denotes either Euclidean space $\mathbb{R}^n$ or hyperbolic space $\mathbb{H}^n$ and where $n = \text{Card}(S) - 1$. The fixed point set of a reflection in R is a hyperplane in M^n. A fundamental domain (= "chamber") for W on M^n is an intersection of half-spaces indexed by the reflections in S.

Suppose that r is the given reflection and that P_r is the corresponding hyperplane in M^n. We seek another reflection r' such that $r'r$ is of infinite order. This is clearly equivalent to finding an r' so that the hyperplanes $P_{r'}$ and P_r are disjoint. The idea for doing this is to find an element $h \in W$ so that hP_r and P_r are disjoint. Noting that $hP_r = P_{hrh^{-1}}$, we will then be done by setting $r' = hrh^{-1}$.

Proof of Lemma 4.2 in the Euclidean case. Since W is a discrete group of isometries of $\mathbb{R}^n$, the subgroup consisting of translations is of finite index in W. Since (W, S) is irreducible, it follows from Proposition 10, p. 101, in [**B**, V §4.9] that a fundamental chamber for W on $\mathbb{R}^n$ is a simplex; in particular, it is compact. Thus, the set of directions of translations in W span $\mathbb{R}^n$. Hence, we can find a translation h in W in a direction not parallel to the hyperplane $P_r \subset \mathbb{R}^n$. Then P_r and hP_r will be disjoint as was to be proved. □

Before turning to the hyperbolic case, we first recall a few basic facts of hyperbolic geometry. Two geodesic rays in $\mathbb{H}^n$ are *asymptotic* if they remain a bounded distance apart. This is an equivalence relation on the set of rays and the set of equivalence classes is naturally an $(n-1)$-sphere, denoted by S_∞, and called the "sphere at infinity". The sphere at infinity can be used to compactify $\mathbb{H}^n$: there is a topology on $\mathbb{H}^n \cup S_\infty$ so that it is homeomorphic to the n-disk. Every isometry of $\mathbb{H}^n$ extends to a homeomorphism of the compactification; it turns out that this defines an action of $\text{Isom}(\mathbb{H}^n)$ on S_∞ as a group of conformal transformations. If h is an orientation-preserving isometry of $\mathbb{H}^n$ and if it fixes no points in $\mathbb{H}^n$, then it fixes either one or two points in S_∞. If it fixes one point it is called *parabolic*. In this case, if K is any compact subset of S_∞, then as $m \to \pm\infty$, $h^m K$ approaches the fixed point. If h fixes two points in S_∞, then it is *hyperbolic*. The two fixed points determine a geodesic in $\mathbb{H}^n$ called the *axis* of h. The isometry h stabilizes its axis and acts as a translation on it. In this case, for any compact subset K of S_∞, $h^m K$ goes to one fixed point as $m \to +\infty$ and to the other as $m \to -\infty$. If W is a discrete subgroup of $\text{Isom}(\mathbb{H}^n)$, then let $L(W)$ be the closure of the subset of S_∞ consisting of all fixed points of hyperbolic or parabolic elements of W.

If X^m is an m-dimensional totally geodesic subspace of $\mathbb{H}^n$, then its sphere at infinity, denoted by $S_\infty(X^m)$ is an $(m-1)$-dimensional subsphere of S_∞. This sets up a bijective correspondence between totally geodesic subspaces of $\mathbb{H}^n$ an subspheres of S_∞.

Proof of Lemma 4.2 in the hyperbolic case. Let S^{m-1} be the smallest subsphere of S_∞ which contains $L(W)$. Then S^{m-1} is clearly W-stable. It follows that the subspace X^m corresponding to this subsphere is also W-stable. If $X^m \neq \mathbb{H}^n$ then we get a decomposition of the linear representation of W on $\mathbb{R}^{n+1}$ into linear subspaces (one subspace being the $(m+1)$-dimensional linear subspace corresponding to X^m). Since (W,S) is irreducible there is no such nontrivial decomposition. Thus, $S^{m-1} = S_\infty$. That is to say, $L(W)$ is not contained in any proper subsphere of S_∞.

Now let r be the reflection in question and P_r the corresponding hyperplane in $\mathbb{H}^n$. By the previous paragraph there is a parabolic or hyperbolic element $h \in W$ whose fixed set is not contained in $S_\infty(P_r)$. For a sufficiently large m, $h^m(S_\infty(P_r))$ will then be contained in a single component of $S_\infty - S_\infty(P_r)$, and therefore, $h^m(P_r) \cap P_r = \emptyset$. As before, this implies that the walls corresponding to r and $r' = h^m r\, h^{-m}$ are parallel. □

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