POINCARÉ DUALITY GROUPS

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§1. Introduction

A space X is *aspherical* if $\pi_i(X) = 0$ for all $i > 1$. For a space of the homotopy type of a CW-complex this is equivalent to the condition that its universal covering space is contractible.

Given any group Γ, there is an aspherical CW-complex $B\Gamma$ (also denoted by $K(\Gamma, 1)$) with fundamental group Γ; moreover, $B\Gamma$ is unique up to homotopy equivalence (cf. [Hu]). $B\Gamma$ is called the *classifying space* of Γ. ($B\Gamma$ is also called an *Eilenberg-MacLane space* for Γ.) So, the theory of aspherical CW-complexes, up to homotopy, is identical with the theory of groups. This point of view led to the notion of the (co)homology of a group Γ: it is simply the (co)homology of the space $B\Gamma$.

Many interesting examples of aspherical spaces are manifolds. A principal feature of a manifold is that it satisfies Poincaré duality. Thus, one is led to define an *n-dimensional Poincaré duality group* Γ to be a group such that $H^i(\Gamma; A) \cong H_{n-i}(\Gamma; A)$ for an arbitrary $\mathbb{Z}\Gamma$-module A. (There is also a version of this with twisted coefficients in the nonorientable case.) So, the fundamental group of a closed, aspherical n-dimensional manifold M is an n-dimensional Poincaré duality group $\Gamma = \pi_1(M)$. The question of whether or not the converse is true was posed by Wall as Problem G2 in [W3]. As stated it is false: as we shall see in Theorem 7.15, Poincaré duality groups need not be finitely presented, while fundamental groups of closed manifolds must be. However, if we add the requirement that the Poincaré duality group be finitely presented, then the question of whether it must be the fundamental group of an aspherical closed manifold is still the main problem in this area.

Examples of aspherical closed manifolds

1) Low dimensional manifolds.

- *Dimension 1*: The circle is aspherical.
- *Dimension 2*: Any surface other than S^2 or $\mathbb{R}P^2$ is aspherical.
- *Dimension 3*: Any irreducible closed 3-manifold with infinite fundamental group is aspherical. (This follows from Papakyriakopoulos' Sphere Theorem.)

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2) Lie groups. Suppose G is a Lie group and that K is a maximal compact subgroup. Then G/K is diffeomorphic to Euclidean space. If Γ is a discrete, torsion-free subgroup of G, then Γ acts freely on G/K and $G/K \longrightarrow \Gamma\backslash G/K$ is a covering projection. Hence, $\Gamma\backslash G/K$ is an aspherical manifold. For example, if $G = \mathbb{R}^n$ and $\Gamma = \mathbb{Z}^n$, we get the n-torus. If $G = O(n, 1)$, then $K = O(n) \times O(1)$, G/K is hyperbolic n-space and $\Gamma\backslash G/K$ is a hyperbolic manifold. One can also obtain closed infranilmanifolds or closed infrasolvmanifolds in this fashion by taking G to be a virtually nilpotent Lie group or a virtually solvable Lie group. (A group *virtually* has a property if a subgroup of finite index has that property.)

3) Riemannian manifolds of nonpositive curvature. Suppose M^n is a closed Riemannian manifold with sectional curvature ≤ 0. The Cartan-Hadamard Theorem then states that the exponential map, $\exp : T_xM^n \to M^n$, at any point x in M^n, is a covering projection. Hence, the universal covering space of M^n is diffeomorphic to $\mathbb{R}^n$ ($\cong T_xM^n$) and consequently, M^n is aspherical.

During the last fifteen years we have witnessed a great increase in our fund of examples of aspherical manifolds and spaces. In many of these new examples the manifold is tessellated by cubes or some other convex polytope. Some of these examples occur in nature in contexts other than 1), 2) or 3) above, for instance, as the closure of an $(\mathbb{R}^*)^n$ -orbit in a flag manifold or as a blowup of $\mathbb{RP}^n$ along certain arrangements of subspaces. (See [DJS].) Some of these new techniques are discussed below.

4) Reflection groups. Associated to any Coxeter group W there is a contractible simplicial complex Σ on which W acts properly and cocompactly as a group generated by reflections ([D1], [D3], [Mo]). It is easy to arrange that Σ is a manifold (or a homology manifold), so if Γ is a torsion-free subgroup of finite index in W, then Σ/Γ is an aspherical closed manifold. Such examples are discussed in detail in §7.

5) Nonpositively curved polyhedral manifolds. Many new techniques for constructing examples are described in Gromov's paper [G1]. As Aleksandrov showed, the concept of nonpositive curvature often makes sense for a singular metric on a space X. One first requires that any two points in X can be connected by a geodesic segment. Then X is *nonpositively curved* if any small triangle (i.e., a configuration of three geodesic segments) in X is "thinner" than the corresponding comparison triangle in the Euclidean plane. ("Thinner" means that the triangle satisfies the $CAT(0)$-inequality of [G1].) The generalization of the Cartan-Hadamard Theorem holds for a nonpositively curved space X: its universal cover is contractible. Gromov pointed out that there are many polyhedral examples of such spaces equipped with piecewise Euclidean metrics (this means that each cell is locally isometric to a convex cell in Euclidean space). Here are two of Gromov's techniques.

- *Hyperbolization*: In Section 3.4 of [G1] Gromov describes several different techniques for converting a polyhedron into a nonpositively curved space. In all of these hyperbolization techniques the global topology of the polyhedron is changed, but its local topology is preserved. So, if the input is a manifold, then the output is an aspherical manifold. (Expositions and applications of hyperbolization can be found in [CD3] and [DJ].)
- *Branched covers*: Let M be a nonpositively curved Riemannian manifold and Y a union of codimension-two, totally geodesic submanifolds which intersect

orthogonally. Then the induced (singular) metric on a branched cover of M along Y will be nonpositively curved. (See Section 4.4 of [G1], as well as, [CD1].) Sometimes the metrics can be smoothed to get Riemannian examples as in [GT]. As Gromov points out (on pp.125-126 of [G1]) there is a large class of examples where M^n is n-torus and Y is a configuration of codimension-two subtori (see also Section 7 of [CD1]).

Using either the reflection group technique or hyperbolization, one can show that there are examples of aspherical closed (topological) n- manifolds $M^n, n \geq 4$, such that a) the universal covering space of M^n is not homeomorphic to $\mathbb{R}^n$ ([D1], [DJ]) or b) M^n is not homotopy equivalent to a smooth (in [DH]) or piecewise linear manifold (in [DJ]).

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§2. Finiteness conditions

The classifying space $B\Gamma$ of an n-dimensional Poincaré duality group Γ is (homotopy equivalent to) an n-dimensional CW complex. If $B\Gamma$ is a closed manifold, then it is homotopy equivalent to a finite CW complex. We now investigate the cohomological versions for a group Γ (not necessarily Poincaré) of the conditions that $B\Gamma$ is either a) finite dimensional or b) a finite CW-complex. A good reference for this material is Chapter VIII of [Br3].

Suppose that R is a nonzero commutative ring and that $R\Gamma$ denotes the group ring of Γ. Regard R as a $R\Gamma$-module with trivial Γ-action.

The *cohomological dimension of* Γ *over* R, denoted $cd_R(\Gamma)$, is the projective dimension of R over $R\Gamma$. In other words, $cd_R(\Gamma)$ is the smallest integer n such there is a resolution of R of length n by projective $R\Gamma$-modules:

$$0 \to P_n \to \cdots \to P_1 \to P_0 \to R \to 0.$$

Our convention, from now on, will be that if we omit reference to R, then $R = \mathbb{Z}$. For example, $cd(\Gamma)$ means the cohomological dimension of Γ over $\mathbb{Z}$.

If Γ acts freely, properly and cellularly on an n-dimensional CW-complex E and if E is acyclic over R, then $cd_R(\Gamma) \leq n$. (Proof: consider the chain complex of cellular chains with coefficients in R:

$$0 \to C_n(E; R) \to \ldots C_0(E; R) \xrightarrow{\varepsilon} R \to 0,$$

where ε is the augmentation.) In particular, if $B\Gamma$ is (homotopy equivalent to) an n-dimensional CW-complex, then $cd(\Gamma) \leq n$.

The *geometric dimension of* Γ, denoted $gd(\Gamma)$, is the smallest dimension of a $K(\Gamma, 1)$ complex (i.e., of any CW-complex homotopy equivalent to $B\Gamma$). We have just seen that $cd(\Gamma) \leq gd(\Gamma)$. Conversely, Eilenberg and Ganea proved in [EG] that $gd(\Gamma) \leq \max\{cd(\Gamma), 3\}$. Also, it follows from Stallings' Theorem [St] (that a group of cohomological dimension one is free) that $cd(\Gamma) = 1$ implies $gd(\Gamma) = 1$. The possibility that there exists a group Γ with $cd(\Gamma) = 2$ and $gd(\Gamma) = 3$ remains open.

A group of finite cohomological dimension is automatically torsion-free. (Proof: a finite cyclic subgroup has nonzero cohomology in every even dimension.) Similarly, if $cd_R(\Gamma) < \infty$, the order of any torsion element of Γ must be invertible in R.

A group Γ is of *type F* if $B\Gamma$ is homotopy equivalent to a finite complex. Γ is of *type* FP_R (respectively, of *type* FL_R) if there is a resolution of R of finite length by finitely generated $R\Gamma$-modules:

$$0 \to P_n \to \cdots \to P_0 \to R \to 0,$$

where each P_i is projective (respectively, free). (The key phrase here is "finitely generated.")

If Γ acts freely, properly, cellularly and cocompactly on an R-acyclic CW-complex E, then Γ is of type FL_R. (Consider the cellular chain complex again.) So, a group of type F is of type FL. Similarly, if $B\Gamma$ is dominated by a finite complex, then Γ is of type FP. Conversely, Wall [W1] proved that if a finitely presented group is of type FL, then it must be of type F.

There is no known example of a group which is of type FP but not of type FL. In fact, it has been conjectured that for any torsion-free group Γ, the reduced projective class group, $\tilde{K}_0(\mathbb{Z}\Gamma)$, is zero, that is, that every finitely generated projective $\mathbb{Z}\Gamma$- module is stably free.

A group Γ is finitely generated if and only if the augmentation ideal is finitely generated as a $\mathbb{Z}\Gamma$-module (Exercise 1, page 12 in [Br3]). Hence, any group of type FP is finitely generated. However, it does not follow that such a group is finitely presented (i.e., that it admits a presentation with a finite number of generators and a finite number of relations). In fact, Bestvina and Brady [BB] have constructed examples of type FL which are not finitely presented. These examples will be discussed further in §7.

§3. Poincaré duality groups

If Γ is the fundamental group of an aspherical, closed n-manifold, M, then it satisfies Poincaré duality:

$$H^i(\Gamma; A) \cong H_{n-i}(\Gamma; D \otimes A)$$

where D is the orientation module and where the coefficients can be any $\mathbb{Z}\Gamma$-module A.

Since the universal covering space $\tilde{M}^n$ of an aspherical M^n is contractible, it follows from Poincaré duality (in the noncompact case) that the cohomology with compact supports of $\tilde{M}^n$ is the same as that of $\mathbb{R}^n$, i.e.,

$$H^i_c(\tilde{M}^n) \cong \begin{cases} 0\,, & \text{for } i \neq n \\ \mathbb{Z}\,, & \text{for } i = n. \end{cases}$$

On the other hand, if Γ acts freely, properly and cocompactly on an acyclic space E, then $H^i(\Gamma; \mathbb{Z}\Gamma) \cong H^i_c(E)$ (by Prop. 7.5, p. 209 in [Br3]). Hence, for $\Gamma = \pi_1(M^n)$,

$$H^i(\Gamma; \mathbb{Z}\Gamma) \cong \begin{cases} 0\,, & \text{for } i \neq n \\ \mathbb{Z}\,, & \text{for } i = n. \end{cases}$$

These considerations led Johnson and Wall [JW] and, independently Bieri [Bi] to the following two equivalent definitions.

Definition 3.1. ([Bi]) A group Γ is a *Poincaré duality group of dimension n over a commutative ring R* (in short, a *PD^n_R-group*) if there is an $R\Gamma$-module D, which is isomorphic to R as an R-module, and a homology class $\mu \in H_n(\Gamma; D)$ (called the *fundamental class*) so that for any $R\Gamma$-module A, cap product with μ defines an isomorphism: $H^i(\Gamma; A) \cong H_{n-i}(\Gamma; D \otimes A)$. D is called the *orientation module* (or *dualizing module*) for Γ. If Γ acts trivially on D, then it is an *orientable PD^n_R-group.*

Definition 3.2. ([JW]). A group Γ is a *Poincaré duality group of dimension n over R* if the following two conditions hold:

(i) Γ is of type FP_R, and

(ii) $H^i(\Gamma; R\Gamma) = \begin{cases} 0 \text{ , for } i \neq n \\ R \text{ , for } i = n \end{cases}$

We note that if $R = \mathbb{Z}$, then (i) implies that Γ is torsion-free and (ii) implies that $cd(\Gamma) = n$.

Theorem 3.3. *([BE2], [Br1], [Br3]). Definitions 3.1 and 3.2 are equivalent.*

On pages 220 and 221 of [Br3] one can find three different proofs that the conditions in Definition 3.2 imply those in Definition 3.1. The dualizing module D is $H^n(\Gamma; R\Gamma)$. Conversely, suppose Γ satisfies the conditions of Definition 3.1. Since $D \otimes R\Gamma$ is free (by Cor. 5.7, page 69 of [Br3]), it is acyclic. Hence, $H^i(\Gamma; R\Gamma) \cong H_{n-i}(\Gamma; D \otimes R\Gamma)$ vanishes for $i \neq n$ and is isomorphic to R for $i = n$. The main content of Theorem 3.3 is that Definition 3.1 forces Γ to be of type FP_R. The reason for this is that the statement that Γ is of type FP_R is equivalent to the statement that $cd_R(\Gamma) < \infty$ and that for each i the functor on $R\Gamma$-modules $A \longrightarrow H^i(\Gamma; A)$ commutes with direct limits (see [Br3]). By naturality of cap products and by Poincaré duality, this functor can be identified with $A \to D \otimes A \to H_{n-i}(\Gamma; D \otimes A)$ and this clearly commutes with direct limits.

In line with our convention from §2, for $R = \mathbb{Z}$, denote PD^n_R by PD^n. As Johnson and Wall observed, if Γ is a finitely presented PD^n- group, then $B\Gamma$ is a Poincaré complex in the sense of [W2]. In particular, $B\Gamma$ is finitely dominated.

The principal question in this area (as well as the most obvious one) is if every PD^n-group is the fundamental group of an aspherical closed manifold. As stated the answer is no. For, as we shall see in §7, the Bestvina-Brady examples can be promoted to examples of PD^n-groups, $n \geq 4$, which are not finitely presented. (Kirby and Siebenmann proved that any compact topological manifold is homotopy equivalent to a finite CW-complex; hence, its fundamental group is finitely presented.) So, the correct question is the following.

Question 3.4. *Is every finitely presented PD^n-group the fundamental group of an aspherical closed manifold?*

This is closely related to Borel's Question: are any two aspherical closed manifolds with the same fundamental group homeomorphic? (See [FJ] for a discussion of Borel's Question.) Thus, Question 3.4 asks if any finitely presented PD^n-group corresponds to an aspherical closed manifold and Borel's Question asks if this manifold is unique up to homeomorphism.

A space X is a *homology manifold of dimension* n *over* R if for each point x in X, $H_*(X, X-x; R) \cong H_*(\mathbb{R}^n, \mathbb{R}^n - 0; R)$, i.e., if

$$H_i(X, X-x; R) \cong \begin{cases} 0 \text{ , for } i \neq n \\ R \text{ , for } i = n. \end{cases}$$

The usual proof that manifolds satisfy Poincaré duality also works for homology manifolds. As we shall see in Example 7.4 there are aspherical polyhedra which are homology manifolds over some ring R but not over $\mathbb{Z}$. The fundamental groups of these examples are Poincaré duality groups over R but not over $\mathbb{Z}$. So, when $R \neq \mathbb{Z}$ the appropriate version of Question 3.4 is the following.

Question 3.5. *Is every torsion-free, finitely presented, PD^n_R-group the fundamental group of an aspherical closed R-homology manifold?*

In fact this question is relevant even when $R = \mathbb{Z}$. The reason is this. While it is true that every closed polyhedral homology manifold is homotopy equivalent to a closed manifold, Bryant, Ferry, Mio and Weinberger have shown in [BFMW] that there exist homology manifolds which are compact *ANRs* and which are not homotopy equivalent to closed manifolds. Thus, there is the intriguing possibility that some of these exotic "near manifolds" of [BFMW] could be aspherical.

Ranicki has shown (in Chapter 17 of [R]) that, for Γ of type F, Question 3.4 has an affirmative answer if and only if the "total surgery obstruction" $s(B\Gamma) \in \mathbb{S}_n(B\Gamma)$ is 0 (where $\mathbb{S}_*$ means the relative homotopy groups of the assembly map in algebraic L-theory). A similar remark applies to Question 3.5 using the "4-periodic total surgery obstruction" in Chapter 25 of [R]. In fact, as explained on page 275 of [R], the strongest form of the Novikov Conjecture is equivalent (in dimensions ≥ 5) to the conjecture that both Question 3.4 and Borel's Question have affirmative answers. As explained on page 298 of [R], a slightly weaker version of the Novikov Conjecture (that the assembly map is an isomorphism) is equivalent to allowing the possibility that there exist aspherical ANR homology manifolds which are not homotopy equivalent to manifolds, as in [BFMW].

Duality groups. There are many interesting groups which satisfy Definition 3.1 except for the requirement that D be isomorphic to R. The proof of Theorem 3.3 also gives the following result.

Theorem 3.6. *(Bieri-Eckmann [BE2], Brown [Br1]). The following two conditions are equivalent.*

(i) *There exists an $R\Gamma$-module D and a positive integer n such that for any $R\Gamma$-module A there is a natural isomorphism (i.e., induced by cap product with a fundamental class): $H^i(\Gamma; A) \cong H_{n-i}(\Gamma; D \otimes A)$.*

(ii) *Γ is of type FP_R and*

$$H^i(\Gamma; R\Gamma) \cong \begin{cases} 0 \text{ , for } i \neq n \\ D \text{ , for } i = n. \end{cases}$$

If either of these conditions hold, then Γ is a *duality group of dimension n over* R (in short, a *D^n_R-group).*

Theorem 3.7. *(Farrell [F]). Suppose R is a field and that Γ is a D^n_R-group. If $\dim_R(D) < \infty$, then $\dim_R(D) = 1$ (and consequently Γ is a PD^n_R-group).*

It follows that if Γ is a duality group over $\mathbb{Z}$, then either $D \cong \mathbb{Z}$ or D is of infinite rank. Conjecturally, D must be free abelian.

Examples of duality groups

1) Finitely generated free groups are duality groups of dimension 1.

2) Suppose that M is a compact aspherical n-manifold with nonempty boundary. Let $\partial_1 M, \dots, \partial_m M$ denote the components of of ∂M and suppose that each $\partial_j M$ is aspherical and that $\pi_1(\partial_j M) \to \pi_1(M)$ is injective. Then $\pi_1(M)$ is a duality group of dimension $n-1$. For example, any knot group (that is, the fundamental group of the complement of a nontrivial knot in S^3) is a duality group of dimension two.

3) Let Γ be a torsion-free arithmetic group and G/K the associated symmetric space. Then Γ is a duality group of dimension $n - \ell$, where $n = \dim(G/K)$ and ℓ is the $\mathbb{Q}$-rank of Γ. (See [BS].)

4) Let S_g denote the closed surface of genus g. The group of outer automorphisms $\mathrm{Out}(\pi_1(S_g))$ is the *mapping class group* of S_g. It is a virtual duality group of dimension $4g-5$ (i.e., any torsion-free subgroup of finite index in $\mathrm{Out}(\pi_1(S_g))$ is a duality group). (See [Ha].)

§4. Subgroups, extensions and amalgamations

We have the following constructions for manifolds:

1) Any covering space of a manifold is a manifold.

2) If $F \to E \to B$ is a fiber bundle and if F and B are manifolds, then so is E.

3) Suppose M is a manifold with boundary and that the boundary consists of two components $\partial_1 M$ and $\partial_2 M$ which are homeomorphic. Then the result of gluing M together along $\partial_1 M$ and $\partial_2 M$ via a homeomorphism is a manifold. (In this construction one usually has one of two situations in mind: either a) M is connected or b) M consists of two components with $\partial_1 M$ and $\partial_2 M$ their respective boundaries.)

In this section we consider the analogous constructions for Poincaré duality groups.

Subgroups. The following analog of construction 1), is proved as Theorem 2 in [JW]. It follows fairly directly from Definition 3.2.

Theorem 4.1. *([Bi], [JW]). Suppose that Γ is a torsion -free group and that Γ' is a subgroup of finite index in Γ. Then Γ is a PD^n_R-group (R a commutative ring) if and only if Γ' is.*

By way of contrast, there is the following result of Strebel [Str].

Theorem 4.2. *([Str]). If Γ is a PD^n-group and Γ' is a subgroup of infinite index, then $cd(\Gamma') < n$.*

Extensions. The next result, the analog of construction 2), is Theorem 3 of [JW].

Theorem 4.3. *([Bi], [JW]). Suppose that Γ is an extension of Γ'' by Γ':*

$$1 \to \Gamma' \to \Gamma \to \Gamma'' \to 1$$

If both Γ' and Γ'' are Poincaré duality groups over R, then so is Γ. Conversely, if Γ is a Poincaré duality group over R and if both Γ' and Γ'' are of type FP_R, then Γ' and Γ'' are both Poincaré duality groups over R.

The corresponding result for Poincaré spaces was stated in [Q] and proved in [Go].

Theorem 4.4. *([Go], [Q]). Suppose that $F \to E \to B$ is a fibration and that F and B are dominated by finite complexes. Then E is dominated by a finite complex and E satisfies Poincaré duality if and only if both F and B do.*

Corollary 4.5. *Suppose that Γ is a finitely presented, torsion-free group of type FP and that Γ acts freely, properly and cocompactly on a manifold M. If M is dominated by a finite complex, then Γ is a Poincaré duality group.*

Proof. Consider the fibration $M \to M \times_\Gamma E\Gamma \to B\Gamma$, where $E\Gamma$ denotes the universal covering space of $B\Gamma$. Since $M \times_\Gamma E\Gamma$ is homotopy equivalent to the closed manifold M/Γ, it satisfies Poincaré duality. $\square$

For example the corollary applies to the case where $M \cong S^k \times \mathbb{R}^n$. (See [CoP].)

Amalgamations. Suppose that M is a compact manifold with boundary, with boundary components $(\partial_j M)_{j\in I}$, that M, as well as each boundary component is aspherical, and that for each $j \in I$ the inclusion $\partial_j M \subset M$ induces a monomorphism $\pi_1(\partial_j M) \to \pi_1(M)$. Set $\Gamma = \pi_1(M)$, let S_j denote the image of $\pi_1(\partial_j M)$ in Γ, and let $\boldsymbol{S}$ denote the family of subgroups $(S_j)_{j\in I}$. Then, following [BE3] and [E], the fact that $(M, \partial M)$ satisfies Poincaré-Lefschetz duality can be reformulated in terms of group cohomology as follows.

Let Γ be a group and $\boldsymbol{S} = (S_j)_{j\in I}$ a finite family of subgroups. For any subgroup H of Γ let $\mathbb{Z}(\Gamma/H)$ denote the free abelian group on Γ/H with $\mathbb{Z}\Gamma$-module structure induced from left multiplication. Let $\Delta = \ker(\bigoplus \mathbb{Z}(\Gamma/S_j) \xrightarrow{\epsilon} \mathbb{Z})$, where ϵ is defined by $\epsilon(\gamma S_j) = 1$ for all $j \in I$ and $\gamma \in \Gamma$. Set

$$H_i(\Gamma, \boldsymbol{S}; A) = H_{i-1}(\Gamma; \Delta \otimes A)$$

$$H^i(\Gamma, \boldsymbol{S}; A) = H^{i-1}(\Gamma; \mathrm{Hom}(\Delta, A)).$$

Definition 4.6. ([BE3], [E]). The pair $(\Gamma, \boldsymbol{S})$ is a *Poincaré duality pair of dimension n* (in short a *PD^n-pair*) with *orientation module D* (where D is isomorphic to $\mathbb{Z}$ as an abelian group) if there are natural isomorphisms:

$$H^i(\Gamma; A) \cong H_{n-i}(\Gamma, \boldsymbol{S}; D \otimes A)$$

$$H^i(\Gamma, \boldsymbol{S}; A) \cong H^{n-i}(\Gamma; D \otimes A).$$

(It follows that each S_j is a PD^{n-1}-group.)

As observed in [JW] one can then use Mayer-Vietoris sequences to prove the analogs of construction 3). For example, suppose that (Γ_1, H) and (Γ_2, H) are PD^n-pairs. (Here $\boldsymbol{S}$ consists of a single subgroup H.) Then the amalgamated product $\Gamma_1 *_H \Gamma_2$ is a PD^n-group. Similarly, suppose $(\Gamma, \boldsymbol{S})$ is a PD^n-pair where $\boldsymbol{S}$ consists of two subgroups S_1 and S_2 and that $\theta : S_1 \to S_2$ is an isomorphism. Then the HNN-extension $\Gamma *_\theta$ is a PD^n-group. If a group can be written as an amalgamated product or HNN-extension over a subgroup H, then it is said to *split over H.)* From these two constructions and induction one can prove the following more general statement.

Theorem 4.7. *Suppose that Γ is the fundamental group of a finite graph of groups. Let Γ_v denote the group associated to a vertex v and S_e the group associated to an edge e. For each vertex v, let $E(v)$ denote the set of edges incident to v and let $\boldsymbol{S}_v = (S_e)_{e \in E(v)}$ be the corresponding family of subgroups of Γ_v. If $(\Gamma_v, \boldsymbol{S}_v)$ is a PD^n-pair for each vertex v, then Γ is a PD^n-group.*

For the definition of a "graph of groups" and its "fundamental group", see [Se2] or [SW].

Kropholler and Roller in [KR1,2,3] have made an extensive study of when a PD^n-group can split over a subgroup H which is a PD^{n-1}-group.

If a PD^n-group Γ splits over a subgroup H, there is no reason that H must be a PD^{n-1}-group. (We shall give examples where it is not in Example 7.3.) However, it follows from the Mayer-Vietoris sequence that $cd(H) = n - 1$. In particular, for $n \geq 3$, a PD^n-group cannot split over a trivial subgroup or an infinite cyclic subgroup. We restate this as the following lemma, which we will need in §6.

Lemma 4.8. *For $n \geq 3$, a PD^n-group is not the fundamental group of a graph of groups with all edge groups trivial or infinite cyclic.*

§5. Dimensions one and two

Question 3.4 has been answered affirmatively in dimensions ≤ 2.

A PD^1-group is infinite cyclic. (Since $H^1(\Gamma; \mathbb{Z}\Gamma) = \mathbb{Z}$, Γ has two ends and since Γ is torsion-free, a result of Hopf [H] implies that $\Gamma \cong \mathbb{Z}$.)

The affirmative answer to Question 3.4 in dimension two is the culmination of several papers by Eckmann and his collaborators, Bieri, Linnell and Müller, see [BE1], [BE2], [EL], [EM], [M] and especially [E].

Theorem 5.1. *A PD^2-group is isomorphic to the fundamental group of a closed surface.*

A summary of the proof can be found in [E]. In outline, it goes as follows: 1) using a theorem of [M] one shows that if a PD^2-group splits as an amalgamated product or HNN extension over a finitely generated subgroup, then the theorem holds and then 2) using the Hattori-Stallings rank, it is proved in [EL] that any PD^2-group has positive first Betti number and hence, that it splits.

Combining Theorem 5.1 with Theorem 4.1 we get the following.

Corollary 5.2. *Suppose that a torsion-free group Γ contains a surface group Γ' as a subgroup of finite index. Then Γ is a surface group.*

For a discussion of the situation in dimension three see Thomas' article [T].

§6. Hyperbolic groups

Any finitely generated group Γ can be given a "word metric," $d : \Gamma \times \Gamma \to \mathbb{N}$, as follows. Fix a finite set of generators T. Then $d(\gamma, \gamma')$ is the smallest integer k such that $\gamma^{-1}\gamma'$ can be written as a word of length k in $T \cup T^{-1}$. If Γ is a discrete group of isometries of a metric space Y and Y/Γ is compact, then the word metric on Γ is quasi-isometric to the induced metric on any Γ-orbit in Y.

Rips defined the notion of a "hyperbolic group" in terms of the word metric (For the definition, see [G1].) This idea was then developed into a vast and beautiful theory in Gromov's seminal paper [G1]. It is proved in [G1] that the property of being hyperbolic depends only on the quasi-isometry type of the word metric, in particular, it is independent of the choice of generating set. The idea behind the definition is this: in the large, Γ should behave like a discrete, cocompact group of isometries of a metric space Y which is simply connected and "negatively curved" in some sense (for example, a space Y which satisfies the CAT(ε)-inequality for some $\varepsilon < 0$). In particular, the fundamental group of a closed Riemannian manifold of (strictly) negative sectional curvature is hyperbolic. Many more examples can be found in [G1].

Rips proved that given a hyperbolic group Γ there is a contractible simplicial complex E on which Γ acts properly and cocompactly, see [G1]. In particular, if Γ is torsion-free, then E/Γ is a $K(\Gamma, 1)$ complex. So, torsion-free hyperbolic groups are automatically of type F.

Associated to any hyperbolic group Γ, there is a space $\partial\Gamma$, called the "ideal boundary" of Γ. The points in $\partial\Gamma$ are certain equivalence classes of sequences $(\gamma_i)_{i \epsilon \mathbb{N}}$ in Γ which go to infinity in an appropriate sense. (The definition of $\partial\Gamma$ can be found on page 98 of [G1].) If Γ is the fundamental group of a negatively curved, closed Riemannian n-manifold, then $\partial\Gamma$ is homeomorphic to S^{n-1} (in this case, $\partial\Gamma$ can be identified with the space of all geodesic rays in the universal covering space emanating from some base point).

In [BM] Bestvina and Mess proved that the Rips complex E can be compactified to a space $\overline{E}$ by adding $\partial\Gamma$ as the space at infinity; moreover, $\partial\Gamma$ is homotopically inessential in $\overline{E}$ in a strong sense. (In technical terms, $\overline{E}$ is a Euclidean retract and $\partial\Gamma$ is a Z-set in $\overline{E}$.) It follows that $H^*_c(E) \cong \check{H}^{*-1}(\partial\Gamma)$, where $\check{H}^{*-1}(\partial\Gamma)$ denotes the reduced Cěch cohomology of $\partial\Gamma$. Since we also have $H^*(\Gamma; \mathbb{Z}\Gamma) \cong H^*_c(E)$, this gives the following theorem.

Theorem 6.1. *(Bestvina-Mess [BM]). Let Γ be a torsion-free hyperbolic group and*

R a commutative ring. Then $H^*(\Gamma; R\Gamma) \cong \check{H}^{*-1}(\partial\Gamma; R)$.

Remark. Suppose Γ is a (not necessarily hyperbolic) group of type F (so that $B\Gamma$ is a finite complex). Then Γ *has a Z-set compactification* if $E\Gamma$ can be compactified to a Euclidean retract $\overline{E\Gamma}$ so that $\partial\overline{E\Gamma}(= \overline{E\Gamma} - E\Gamma)$ is a Z-set in $\overline{E\Gamma}$. It is quite possible that every group of type F admits a Z-set compactification. (The Novikov Conjecture is known to hold for such groups, see [CaP].) We note that Theorem 6.1 holds for any such group. Thus, such a group Γ is a D^n-group if and only if the Cěch cohomology of $\partial\overline{E\Gamma}$ is concentrated in dimension $n-1$; it is a PD^n-group if and only if $\partial\overline{E\Gamma}$ has the same Cěch cohomology as S^{n-1}.

Theorem 6.2. *(Bestvina [Be]). Suppose that a hyperbolic group Γ is a PD^n_R-group. Then $\partial\Gamma$ is a homology $(n-1)$-manifold over R (with the same R-homology as S^{n-1}).*

However, as shown in [DJ], even when $R = \mathbb{Z}$, for $n \geq 4$, there are examples where $\partial\Gamma$ is not homeomorphic to S^{n-1}; $\partial\Gamma$ need not be simply connected or locally simply connected (so, in these examples $\partial\Gamma$ is not even an ANR). In dimension three Theorem 6.2 has the following corollary.

Corollary 6.3. *(Bestvina-Mess [BM]). If Γ is a hyperbolic PD^3- group, then $\partial\Gamma$ is homeomorphic to S^2.*

In this context, Cannon has proposed the following version of Thurston's Geometrization Conjecture.

Conjecture 6.4. *If a PD^3-group is hyperbolic (in the sense of Rips and Gromov), then it is isomorphic to the fundamental group of a closed hyperbolic 3-manifold (i.e., a 3-manifold of constant curvature -1).*

A proof of this would constitute a proof of a major portion of the Geometrization Conjecture. The issue is to show that the action of the group Γ on $\partial\Gamma(= S^2)$ is conjugate to an action by conformal transformations. Cannon and his collaborators seem to have made progress on an elaborate program for proving this (see [C]).

The group of outer automorphisms of a hyperbolic PD^n-group. A proof of the following theorem is outlined on page 146 of [G1]. A different argument using work of Paulin [P] and Rips [Ri] can be found in [BF].

Theorem 6.6. *(Gromov). Let Γ be a hyperbolic PD^n- group with $n \geq 3$. Then $Out(\Gamma)$, its group of outer automorphisms, is finite.*

Sketch of Proof. (See [BF].) Paulin [P] proved that if Γ is hyperbolic and $\mathrm{Out}(\Gamma)$ is infinite, then Γ acts on an $\mathbb{R}$-tree with all edge stabilizers either virtually trivial or virtually infinite cyclic. A theorem of Rips [R] then implies that there is a Γ-action on a simplicial tree with the same type of edge stabilizers. Since Γ is torsion-free this implies that Γ splits as an amalgamated free product or HNN extension over a trivial group or an infinite cyclic group. By Lemma 4.8 such a group cannot satisfy Poincaré duality if $n \geq 3$. $\square$

Remarks. i) The theorem is false for $n = 2$, i.e., for surface groups.

ii) The theorem is also false in the presence of 0 curvature. For example, $\mathrm{Out}(\mathbb{Z}^n) = GL(n, \mathbb{Z})$, which is infinite if $n > 1$.

iii) Suppose M_1 and M_2 are two aspherical manifolds with boundary with $\partial M_1 = \partial M_2 = T^{n-1}$. Let M be the result of gluing M_1 to M_2 along T^{n-1} and let $\Gamma = \pi_1(M)$. (If $n \geq 3$, then Γ is not hyperbolic.) The homotopy class of any closed loop in T^{n-1} then defines a Dehn twist about T^{n-1}. In this way we get a monomorphism $\mathbb{Z}^{n-1} \to \mathrm{Out}(\Gamma)$, so the outer automorphism group is infinite.

iv) When Γ is the fundamental group of a closed hyperbolic manifold (either a real, complex or quaternionic hyperbolic manifold) of dimension > 2, then Theorem 6.6 follows from the Mostow Rigidity Theorem.

§7. Examples

Right-angled Coxeter groups. Given a simplicial complex L, we shall describe a simple construction of a cubical cell complex P_L so that the link of each vertex in P_L is isomorphic to L. If L satisfies a simple combinatorial condition (that it is a "flag complex"), then P_L is aspherical.

Let S denote the vertex set of L and for each simplex σ in L let $S(\sigma)$ denote its vertex set. Define P_L to be the subcomplex of the cube $[-1,1]^S$ consisting of all faces parallel to $\mathbb{R}^{S(\sigma)}$ for some simplex σ in L (such a face is defined by equations of the form: $x_s = \varepsilon_s$, where $s \in S - S(\sigma)$ and $\varepsilon_s \in \{\pm 1\}$). There are 2^S vertices in P_L and the link of each of them is naturally identified with L. Hence, if L is homeomorphic to S^{n-1}, then P_L is a closed n-manifold. Similarly, if L is an $(n-1)$-dimensional homology manifold over R with the same R-homology as S^{n-1}, then P_L is a R-homology n-manifold.

For each s in S let r_s be the linear reflection on $[-1,1]^S$ which sends the standard basis vector e_s to $-e_s$ and which fixes $e_{s'}$ for $s' \neq s$. The group generated by these reflections is isomorphic to $(\mathbb{Z}/2)^S$. The subcomplex P_L is $(\mathbb{Z}/2)^S$- stable. A fundamental domain for the action on $[-1,1]^S$ is $[0,1]^S$; moreover, the orbit space $[-1,1]^S/(\mathbb{Z}/2)^S$ is naturally identified with this subspace. Set $K = P_L \cap [0,1]^S$ and for each s in S let K_s be the subset of K defined by $x_s = 0$. (K_s is called the *mirror* associated to r_s.) The cell complex K is homeomorphic to the cone on L; the subcomplex K_s is the closed star of the vertex s in the barycentric subdivision of L. In order to describe the universal covering space Σ_L of P_L, we first need to discuss Coxeter groups.

A *Coxeter matrix* M on a set S is a symmetric $S \times S$ matrix (m_{st}) with entries in $\mathbb{N} \cup \{\infty\}$ such that $m_{st} = 1$ if $s = t$ and $m_{st} \geq 2$ if $s \neq t$. Associated to M there is a *Coxeter group* W defined by the presentation

$$W = \langle S \mid (st)^{m_{st}} = 1,\ (s,t) \in S \times S \rangle$$

A Coxeter matrix is *right-angled* if all of its off-diagonal entries are 2 or ∞. Similarly, a Coxeter group is *right-angled* if its Coxeter matrix is.

One can associate to L a right-angled Coxeter matrix M_L and a right-angled

Coxeter group W_L as follows. M_L is defined by

$$m_{st} = \begin{cases} 1, & \text{if } s = t \\ 2, & \text{if } \{s,t\} \text{ spans an edge in } L \\ \infty, & \text{otherwise.} \end{cases}$$

W_L is the associated Coxeter group. Let $\Theta : W_L \to (\mathbb{Z}/2)^S$ be the epimorphism which sends s to r_s and let Γ_L be the kernel of Θ. Then Γ_L is torsion-free. (Γ_L is the commutator subgroup of W_L.)

The complex Σ_L can be defined as $(W_L \times K)/\backsim$. The equivalence relation $\backsim$ on $W_L \times K$ is defined by: $(w,x) \backsim (w',x')$ if and only if $x = x'$ and $w^{-1}w' \in W_x$, where W_x is the subgroup generated by $\{s \in S \mid x \in K_s\}$. It is easy to see that $P_L \cong ((\mathbb{Z}/2)^S \times K)/\backsim$, where the equivalence relation is defined similarly. The natural Θ-equivariant map $\Sigma_L \to P_L$ is a covering projection. Moreover, Σ_L is simply connected (by Corollary 10.2 in [D1]). Consequently, Σ_L is the universal covering space of P_L and $\pi_1(P_L) = \Gamma_L$.

We now turn to the question of when Σ_L is contractible. A simplicial complex L with vertex set S is a *flag complex* if given any finite set of vertices S', which are pairwise joined by edges, there is a simplex σ in L spanned by S' (i.e., $S(\sigma) = S'$). For example, the derived complex (also called the "order complex") of any poset is a flag complex. In particular, the barycentric subdivision of any simplicial complex is a flag complex. Hence, the condition that L is a flag complex does not restrict its topological type; it can be any polyhedron.

Theorem 7.1. *([D1], [D3]). The complex Σ_L is contractible if and only if L is a flag complex.*

Gromov gave a different proof from that of [D1] for the above theorem (in Section 4 of [G1]); he showed that the natural piecewise Euclidean metric on a cubical complex is nonpositively curved if and only if the link of each vertex is a flag complex. Since Σ_L is a cubical complex with the link of each vertex isomorphic to L, the theorem follows. (An exposition of this method is given in [D3].)

Example 7.2. (Topological reflection groups on $\mathbb{R}^n$). Suppose that L is a PL triangulation of the sphere S^{n-1} as a flag complex. (To insure that L is a flag complex we could take it to be the barycentric subdivision of an arbitrary PL triangulation of S^{n-1}.) Then K is homeomorphic to the n-disk (since it is homeomorphic to the cone on L). The K_s are the dual cells to the vertices of L. If L is the boundary complex of a convex polytope, then K is the dual polytope. In fact this is the correct picture to keep in mind: K closely resembles a convex polytope. In various special cases W_L can be represented as a group generated by reflections on Euclidean n-space or hyperbolic n-space so that K is a fundamental domain. The right-angledness hypothesis on the Coxeter group should be thought of as the requirement that the hyperplanes of reflection intersect orthogonally, i.e., that if two of the K_s intersect, then their intersection is orthogonal. For example, suppose L is a subdivision of the circle into m edges. L is a flag complex if and only if $m \geq 4$. K is an m-gon; we should view it as a right-angled m-gon in the Euclidean plane (if $m = 4$) or the hyperbolic plane (if $m > 4$). P_L is the orientable surface of Euler characteristic $2^{m-2}(4-m)$; its universal cover Σ_L is the Euclidean or hyperbolic plane as $m = 4$ or $m > 4$. For another example, suppose that L is the boundary complex of an n-dimensional octahedron (the n-fold join of S^0 with itself). Then K is an n-cube, the K_s are its codimension-one faces, P_L is an n-torus, Σ_L is $\mathbb{R}^n$ and its tiling by copies of K is the standard cubical tessellation of $\mathbb{R}^n$. If L is a more random PL triangulation of S^{n-1}, then we may no longer have such a nice geometric interpretation; however, the basic topological picture is the same ([D1]): K is an n-cell, the K_s are $(n-1)$-cells and they give a cellulation of the boundary of K, P_L is a closed PL n-manifold and its universal cover Σ_L is homeomorphic to $\mathbb{R}^n$. (The proof that Σ_L is homeomorphic to $\mathbb{R}^n$ uses the fact that the triangulation L is PL; otherwise, it need not be, cf., Remark (5b.2) in [DJ].) We can think of K as an orbifold; each point has a neighborhood which can be identified with the quotient space of $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$ by the action of $(\mathbb{Z}/2)^k \times 1$, for some $k \leq n$. Its orbifoldal fundamental group is W_L and its orbifoldal universal cover is Σ_L.

Example 7.3. (Splittings). Suppose that L is triangulation of S^{n-1} as a flag complex and that L_0, L_1 and L_2 are full subcomplexes such that $L_1 \cup L_2 = L$ and $L_1 \cap L_2 = L_0$. Then P_L is an aspherical n-manifold, $P_{L_1} \cup P_{L_2} = P_L$, and $P_{L_1} \cap P_{L_2} = P_{L_0}$. Thus, Γ_L splits as an amalgamated product of Γ_{L_1} and Γ_{L_2} along Γ_{L_0}. It follows from Theorem B in [D4] that Γ_{L_0} is a PD^{n-1}-group if and only if L_0 is a homology $(n-2)$-manifold with the same homology as S^{n-2}, and if this is the case, then P_{L_0} is an aspherical homology $(n-1)$- manifold. On the other hand, the complexes L_0, L_1 and L_2 can be fairly arbitrary. For example, we could choose L_0 to be a triangulation of any piecewise linear submanifold of codimension one in S^{n-1}. (L_0 then separates the sphere into two pieces L_1 and L_2.) By Theorem A in [D4], for $1 < i < n$, $H^i(\Gamma_{L_0}; \mathbb{Z}\Gamma_{L_0})$ is an infinite sum of copies of $H^{i-1}(L_0)$. So, if L_0 is not a homology sphere, then Γ_{L_0} will not be a PD^{n-1}-group and $(\Gamma_{L_1}, \Gamma_{L_0})$ and $(\Gamma_{L_2}, \Gamma_{L_0})$ will not be PD^n-pairs. Hence, there are many examples of splittings of PD^n-groups over subgroups which do not satisfy Poincaré duality.

By allowing L to be a homology sphere we can use Theorem 7.1 to get many examples of aspherical homology manifolds P_L over various rings R.

Example 7.4. (The fundamental group at infinity, [D1]). Suppose L is a triangulation of a homology $(n-1)$-sphere as a flag complex. Then P_L is an aspherical homology n-manifold and hence, Γ_L is a PD^n-group. For $n \geq 4$, there are homology $(n-1)$-spheres which are not simply connected. If we choose L to be such a homology sphere, then it is proved in [D1] that Σ_L is not simply connected at infinity; its fundamental group at infinity (an invariant of Γ_L) is the inverse limit of k-fold free products of $\pi_1(L)$. (This answered Question F16 of [W3] in the negative.)

Example 7.5. (Nonintegral Poincaré duality groups, [D3], [D4], [DL]). Suppose that $R = \mathbb{Z}[\frac{1}{m}]$ and that L is a triangulation of a lens space, $S^{2k-1}/(\mathbb{Z}/m)$ as a flag complex. With coefficients in R, L has the same homology as does S^{2k-1}. Hence, P_L is an aspherical R-homology manifold of dimension $2k$ and consequently Γ_L is a PD^n_R-group, for $n = 2k$. Furthermore, one can show (as in Example 11.9 of [D3] or Example 5.4 of [D4]) that $H^n(\Gamma_L; \mathbb{Z}\Gamma_L) \cong H^n_c(\Sigma_L; \mathbb{Z}) \cong \mathbb{Z}$, while for $1 < i < n, H^i(\Gamma_L; \mathbb{Z}\Gamma_L)$ is a countably infinite sum of $H^{i-1}(L; \mathbb{Z})$ which is m-torsion whenever i is odd and $i \geq 3$. So, Γ_L is not a Poincaré duality group over

$\mathbb{Z}$. By taking L to be the suspension of a lens space we obtain a similar example for n odd. Therefore, we have proved the following.

Theorem 7.6. *For $R = \mathbb{Z}[\frac{1}{m}]$ and for any $n \geq 4$ there are PD^n_R-groups which are not PD^n-groups.*

The construction in Example 7.5 suggests the following.

Question 7.7. *If Γ is a PD^n_R-group for a nonzero ring R, then is $H^n(\Gamma; \mathbb{Z}\Gamma) = \mathbb{Z}$?*

There is also the following weaker version of this question.

Question 7.8. *For any PD^n_R group Γ, is it true that the image of the orientation character $w_1 : \Gamma \to Aut(R)$ (defined by the action of Γ on $H^n(\Gamma; R\Gamma)$) is contained in $\{\pm 1\}$?*

The Bestvina-Brady example. There is a similar construction to the one given above involving "right-angled Artin groups" (also called "graph groups"). Given a flag complex L with vertex set S, let Q_L be the subcomplex of the torus T^S consisting of all subtori $T^{S(\sigma)}$, where σ is a simplex in L. Then, as shown in [CD4], [D3], Q_L is aspherical and its fundamental group is the right-angled Artin group defined by the presentation:

$$A_L = \langle S \mid [s,t] = 1, \text{ if } \{s,t\} \textit{ spans an edge of } L \rangle$$

Let $\rho : A_L \to \mathbb{Z}$ be the homomorphism defined by $\rho(s) = 1$, for all $s \in S$ (or ρ could be any other "generic" homomorphism). Denote the kernel of ρ by H_L.

The universal covering space $\tilde{Q}_L$ of Q_L is naturally a cubical complex. One can find a ρ-equivariant map $f : \tilde{Q}_L \to \mathbb{R}$, such that its restriction to each cube is an affine map. Choose a real number λ and let $\tilde{Y}$ denote the level set $f^{-1}(\lambda)$, and $Y = \tilde{Y}/H_L$. Then Y is a finite CW-complex of the same dimension as L.

Bestvina and Brady prove in [BB] that (i) if L is acyclic, then so is $\tilde{Y}$ and (ii) if L is not simply connected, then H_L is not finitely presented. So, if L is any complex which is acyclic and not simply connected, then H_L is a group of type FL which is not finitely presented.

Theorem 7.9. *(Bestvina-Brady [BB]). There are groups of type FL which are not finitely presented.*

Remark. In fact, the right angled Artin group A_L is a subgroup of finite index in a right-angled Coxeter group W. To see this, for each $s \in S$, introduce new generators r_s and t_s and new relations: $(r_s)^2 = 1 = (t_s)^2$ and whenever $s \neq s'$, $(r_s r_{s'})^2 = 1$, $(r_s t_{s'})^2 = 1$, and $(t_s t_{s'})^2 = 1$ if $\{s, s'\}$ spans an edge in L. Let W be the right-angled Coxeter group generated by the r_s and t_s and let $\theta : W \to (\mathbb{Z}/2)^S$ be the epimorphism which, for each $s \in S$, sends both r_s and t_s to the corresponding generator of $(\mathbb{Z}/2)^S$. Then A_L is the kernel of θ. The generators of A_L can be identified with $\{r_s t_s\}_{s \in S}$.

The reflection group trick. Example 7.2 can be generalized as follows. Suppose that X is an n-dimensional manifold with boundary and that L is a PL triangulation of its boundary. The cellulation of ∂X which is dual to L gives X the structure of an orbifold. (For example, think of X as a solid torus with ∂X being cellulated by polygons, three meeting at each vertex.) As in Example 7.2 each point in X has a neighborhood of the form $\mathbb{R}^k/(\mathbb{Z}/2)^k \times \mathbb{R}^{n-k}$ for some $k \leq n$. The orbifoldal fundamental group G of X is an extension of the right-angled Coxeter group W_L defined previously. The epimorphism $G \to W_L \to (\mathbb{Z}/2)^S$ defines an orbifoldal covering space P of X and an action of $(\mathbb{Z}/2)^S$ on P as a group generated by reflections. Since X is an orbifold, P is a closed n-manifold. Its fundamental group Γ is the kernel of $G \to (\mathbb{Z}/2)^S$. It is not hard to see (cf., Remark 15.9 in [D1]) that if X is aspherical and if L is a flag complex, then P is aspherical. Hence, its fundamental group Γ is a PD^n-group.

Example 7.10. (The reflection group trick, first version). Suppose that π is a group of type F. "Thicken" the finite complex $B\pi$ into a compact manifold with boundary X (e.g., embed $B\pi$ in Euclidean space and take X to be a regular neighborhood). Let L be a PL triangulation of ∂X as a flag complex. Then P is an aspherical closed manifold with fundamental group Γ. Since X is a fundamental domain for the $(\mathbb{Z}/2)^S$-action on P, the orbit map $P \to X$ is a retraction. Hence, on the level of fundamental groups, there is a retraction from Γ onto π. If π has some property which holds for any group that retracts onto it, then Γ will be a PD^n-group with the same property. In [Me], Mess used this construction to show that PD^n-groups need not be residually finite (answering Wall's Question F6 of [W3] in the negative).

Theorem 7.11. *(Mess [Me]). There are aspherical closed n-manifolds, $n \geq 4$, the fundamental groups of which are not residually finite.*

Next we want to give some more detail about the reflection group trick and at the same time weaken the hypotheses in two ways. First, we will not require X to be a manifold with boundary. Second, we will not require X to be aspherical, but, rather, only that it has a covering space $\tilde{X}$ which is acyclic. So, suppose we are given the following data:

(i) a finite CW-complex X,

(ii) a group π and an epimorphism $\varphi : \pi_1(X) \to \pi$ so that the induced covering space $\tilde{X} \to X$ is acyclic,

(iii) a subcomplex of X and a triangulation of it by a flag complex L.

From this data we will construct a virtually torsion-free group G and an action of it on an acyclic complex Ω with quotient space X.

Let $\tilde{L}$ denote the inverse image of L in $\tilde{X}$ and let $\tilde{S}$ be its vertex set. Let $W_{\tilde{L}}$ be the right-angled Coxeter group defined by $\tilde{L}$. The group π acts on $\tilde{S}$ (by deck transformations) and hence, on $W_{\tilde{L}}$ (by automorphisms). Define G to be the semidirect product, $W_{\tilde{L}} \rtimes \pi$. There is a natural epimorphism $\theta : G \to W_L$. Set $\Gamma = \theta^{-1}(\Gamma_L)$. Then Γ is torsion-free (since Γ_L and π are) and of finite index in G. Since G is a semidirect product, so is Γ. In particular, the natural map $\Gamma \to \pi$ is a retraction.

For each s in $\tilde{S}$ let $\tilde{X}_s$ denote the closed star of s in the barycentric subdivision of $\tilde{L}$ and define $\Omega = (W_{\tilde{L}} \times \tilde{X})/\backsim$ as in the first part of this section. The groups $W_{\tilde{L}}$ and π act on Ω. ($W_{\tilde{L}}$ acts by left multiplication on the first factor; π acts by automorphisms on the first factor and deck transformations on the second.) These actions fit together to define a G-action on Ω. This G-action is proper and cellular and $\Omega/G = X$. Let $P = \Omega/\Gamma$. As in [D1] or [DL], it can be shown that Ω is acyclic. (Ω is constructed by gluing together copies of $\tilde{X}$, one for each element of $W_{\tilde{L}}$. If we order the elements of $W_{\tilde{L}}$ compatibly with word length, then each copy of $\tilde{X}$ will be glued to the union of the previous ones along a contractible subspace.) The above discussion gives the following result.

Proposition 7.12. *([D4]).*

(i) Γ *is of type* FL.

(ii) If $\pi_1(X) = \pi$ *and* φ *is the identity (so that* $X = B\pi$*), then* Ω *is contractible and hence,* Γ *is of type* F.

The next proposition, which is stated in [DH], follows easily from the results of [D1].

Proposition 7.13.

(i) If X *is a compact* n*-manifold with* $L = \partial X$*, then* Ω/Γ *is a closed* n*-manifold.*

(ii) If X *is a compact homology* n*-manifold with boundary over a ring* R *and if* $L = \partial X$*, then* Ω/Γ *is a closed* R*-homology* n*-manifold.*

(iii) If (X, L) *is a Poincaré pair and if* L *is a homology* $(n-1)$*- manifold, then* Ω/Γ *is a Poincaré space.*

Example 7.14. (The Bestvina-Brady example continued). This example is similar to Example 7.10. Let π be one of the Bestvina-Brady examples associated to a finite acyclic 2-complex. So there is a finite 2-complex Y and an epimorphism $\varphi : \pi_1(Y) \to \pi$ so that the induced covering space $\tilde{Y}$ is acyclic. Thicken Y to X, a compact manifold with boundary, (X can be of any dimension $n \geq 4$) and let L be a triangulation of ∂X as a flag complex. Since Ω is then an acyclic manifold and since $H^*(\Gamma; \mathbb{Z}\Gamma) = H_c^*(\Omega)$, we see that Γ is a PD^n-group. Since π is not finitely presented and since Γ retracts onto π, Γ is not finitely presented (Lemma 1.3 in [W1]). So, we have proved the following result (which answers Question F10 of [W3]).

Theorem 7.15. *([D4]). In each dimension* $n \geq 4$*, there are* PD^n*-groups which cannot be finitely presented.*

§8. Three more questions

Many open questions about aspherical manifolds make sense for Poincaré duality groups. Here are three such.

For any group Γ of type FP one can define its Euler characteristic $\chi(\Gamma)$, as in [Br2], [Br3]. If Γ is a PD^n-group and n is odd, then Poincaré duality implies that $\chi(\Gamma) = 0$. For n even, we have the following.

Question 8.1. *If Γ is a Poincaré duality group of dimension $2m$, then is*

$$(-1)^m \chi(\Gamma) \geq 0?$$

In the context of nonpositively curved Riemannian manifolds, the conjecture that this be so is due to Hopf. Thurston asked the question for aspherical manifolds. A discussion of this conjecture in the context of nonpositively curved polyhedral manifolds can be found in [CD2].

If Γ is orientable and of dimension divisible by 4, then its *signature* $\sigma(\Gamma)$ can be defined as the signature of the middle dimensional cup product pairing. Its absolute value is independent of the choice of orientation class.

Suppose that $B\Gamma$ is a finite complex and that $E\Gamma$ denotes its universal covering space. As in Section 8 of [G2], one can then define L_2-cochains on $E\Gamma$ and the corresponding cohomology groups $\ell_2 H^k(E\Gamma)$ (the so-called "reduced" L_2-cohomology). When nonzero, these Hilbert spaces will generally be infinite dimensional. However, following [A], there is a well-defined *von Neumann dimension* or *ℓ_2-Betti number* $h^k(\Gamma)$, which is a nonnegative real number and an invariant of $\ell_2 H^k(E\Gamma)$ with its unitary Γ-action. Atiyah proved in [A] that $\chi(\Gamma)$ is the alternating sum of the ℓ_2-Betti numbers. He also showed that when Γ is the fundamental group of a closed aspherical $4m$-dimensional manifold then its middle dimensional L_2-cohomology is a sum of two subspaces and $\sigma(\Gamma)$ is the difference of their von Neumann dimensions. Singer then observed that Question 8.1 would be answered affirmatively if the following question is answered affirmatively.

Question 8.2. *Suppose that Γ is a PD^{2m}-group of type F. Is it true that $\ell_2 H^i(E\Gamma) = 0$ for $i \neq m$? In other words, is $h^i(\Gamma) = 0$ for $i \neq m$?*

Similarly, in dimensions divisible by 4, an affirmative answer to this question implies an affirmative answer to the following stronger version of Question 8.1.

Question 8.3. *If Γ is an orientable Poincaré duality group of dimension $4m$, then is*

$$\chi(\Gamma) \geq |\sigma(\Gamma)|?$$

Further questions of this type can be found in Section 8 of [G2].

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