

NOTES ON STEKLOV'S CONJECTURE IN L^p AND ON DIVERGENCE OF LAGRANGE INTERPOLATION IN L^p

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ABSTRACT. Given a compact interval Δ , it is shown that for E. A. Rakhmanov's weight w on Δ which is bounded from below by the Chebyshev weight v on Δ [Math. USSR Sbornik **42** (1982), p. 263] the corresponding orthonormal polynomials are unbounded in every L_v^p (and L_w^p) with $p > 2$, and that the Lagrange interpolation process based on their zeros diverges in every L_v^p with $p > 2$ for some continuous f . This yields an affirmative answer to Conjecture 2.9 in "Research problems in orthogonal polynomials" ["Approximation Theory VI: Vol. 2", C. K. Chui, L. L. Schumaker, and J. D. Ward, Eds., Academic Press, New York, 1989, p. 454], a positive answer to Problem 8, and a negative answer to Problem 10 of P. Turán [J. Approx. Theory **29** (1980), pp. 32-33].

Throughout this note, let Δ be a fixed compact interval of positive length, say $\Delta \stackrel{\text{def}}{=} [a, b]$ with $-\infty < a < b < \infty$. The function w is called a weight in Δ if $w(\geq 0) \in L^1(\Delta)$ and $\int_{\Delta} w > 0$. Given a weight w in Δ , let $\{p_n(w)\}_{n=0}^{\infty}$ ($\deg(p_n) = n$) denote the system of orthonormal polynomials with respect to w . Given $n \in \mathbb{N}$ and $f \in C(\Delta)$, the Lagrange interpolating polynomial $L_n(w, f)$ based on the zeros of p_n is defined as the unique polynomial of degree $n - 1$ that takes the values of f at the zeros of $p_n(w)$.

In what follows, we will use the notation

$$\|g\|_p \stackrel{\text{def}}{=} \left\{ \int_{\Delta} |g|^p \right\}^{1/p}, \quad \|g\|_{u,p} \stackrel{\text{def}}{=} \left\{ \int_{\Delta} |g|^p u \right\}^{1/p}, \quad \|g\|_{\infty} \stackrel{\text{def}}{=} \max_{\Delta} |g|,$$

and

$$\|L_n(w)\|_{u,p} \stackrel{\text{def}}{=} \sup_{\|f\|_{\infty}=1} \|L_n(w, f)\|_{u,p}$$

where $u \geq 0$ in Δ and $0 < p < \infty$. The Chebyshev weight corresponding to Δ is denoted by v , that is,

$$v(x) \stackrel{\text{def}}{=} \frac{1}{\pi \sqrt{(b-x)(x-a)}}, \quad x \in \Delta = [a, b].$$

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For mean convergence of Lagrange interpolation the first two significant results, one of them due to Paul Erdős and Paul Turán [2], and the other one due to Erdős and Ervin Feldheim [1], are as follows.¹

Theorem A. *Given a weight w in Δ , $\lim_{n \rightarrow \infty} \|f - L_n(w, f)\|_{w,2} = 0$ holds for all $f \in C(\Delta)$.*

Theorem B. *If $0 < p < \infty$ then $\lim_{n \rightarrow \infty} \|f - L_n(v, f)\|_{v,p} = 0$ holds for all $f \in C(\Delta)$.*

In 1975, Turán restated two of his favorite problems in this direction in [9, p. 32–33].

Problem 8. *Are there a weight w in Δ and a function $f \in C(\Delta)$ such that we have $\limsup_{n \rightarrow \infty} \|f - L_n(w, f)\|_{w,p} = \infty$ for every $p > 2$?*

Problem 10. (Erdős-Feldheim) *Let $w \geq v$ in Δ . Is it true that if $p > 0$ and $f \in C(\Delta)$ then $\lim_{n \rightarrow \infty} \|f - L_n(w, f)\|_p = 0$?*

Problem 8 was first solved in [3, Corollary 10.18, p. 181] and [4, Theorem, p. 190], and then improved in [8, Corollary 14, p. 326] as follows.

Theorem C. *Let $2 \leq p_0 < \infty$, and let w and u be weights in Δ . If $\int_{\Delta} [w/v]^{-p/2} u = \infty$ holds for every $p > p_0$, then there exists a function $f \in C(\Delta)$ such that $\limsup_{n \rightarrow \infty} \|f - L_n(w, f)\|_{u,p} = \infty$ holds for every $p > p_0$.*

Unfortunately, this theorem gives answers neither to Problem 8 with the additional condition $w \geq v$ in Δ nor to Problem 10, since if $w \geq v$ in Δ then $\int_{\Delta} (w/v)^{-p/2} u \leq \int_{\Delta} u < \infty$.

Closely related to these problems is a conjecture given in [5, Conjecture 2.9, p. 454] which is related to Steklov's conjecture (see, for instance, [9, Problems 68 and 69, p. 60] and [6, p. 549]). Since Evgenii A. Rakhmanov disproved the original conjecture of Steklov in [6, Theorem 2, p. 566]), it is natural to expect that its L_v^p variant with all $p > 2$ fails as well. Hence we have the following.

Conjecture 2.9. *Given $p > 2$, there exists a weight w in Δ such that $w \geq v$ in Δ and the sequence $\{p_n(w)\}_{n=0}^{\infty}$ is unbounded in L_w^p .*

The purpose of this note is to give a positive answer to Problem 8 with the additional condition $w \geq v$ in Δ (Theorem 1), a negative answer to Problem 10 (Theorem 1), and an affirmative answer to Conjecture 2.9 (Theorem 2).

Theorem 1. *There exists a weight w in Δ and a function $f \in C(\Delta)$ such that $w \geq v$ in Δ and*

$$\limsup_{n \rightarrow \infty} \|f - L_n(w, f)\|_{v,p} = \infty, \quad \forall p > 2. \quad (1)$$

In particular,

$$\limsup_{n \rightarrow \infty} \|f - L_n(w, f)\|_{w,p} = \infty, \quad \forall p > 2, \quad (2)$$

and

$$\limsup_{n \rightarrow \infty} \|f - L_n(w, f)\|_p = \infty, \quad \forall p > 4. \quad (3)$$

¹Most of the subsequent problems, conjectures, and results were originally formulated for the case when $\Delta = [-1, 1]$, but, of course, they are equivalent to the case when Δ is an arbitrary compact interval.

Theorem 2. *There exists a weight w in Δ such that $w \geq v$ in Δ and*

$$\limsup_{n \rightarrow \infty} \|p_n(w)\|_{v,p} = \infty \quad (4)$$

for every $p > 2$.

Note that Theorem 2 delivers more than what Conjecture 2.9 says since $\{p_n(w)\}_{n=0}^{\infty}$ turns out to be unbounded in L_v^p not just in L_w^p .

Theorem 3 gives a useful relationship between mean boundedness of orthogonal polynomials and mean convergence of the corresponding Lagrange interpolation process.

Theorem 3. *Let $0 < p < \infty$, and let w and u be weights in Δ . Suppose that $\lim_{n \rightarrow \infty} \|f - L_n(w, f)\|_{u,p} = 0$ holds for all $f \in C(\Delta)$. Then*

$$\sup_{n \in \mathbb{N}} n^{-2/p} \|p_n(w)\|_{\infty} < \infty$$

if $u \geq 1$ in Δ , and

$$\sup_{n \in \mathbb{N}} n^{-1/p} \|p_n(w)\|_{\infty} < \infty$$

if $u \geq v$ in Δ , respectively.

The proofs of these theorems are based on the following lemmas.

Lemma 1. [8, Theorem 12, p. 324] *Let $0 < p < \infty$, and let w and u be weights in Δ . Then*

$$\sup_{n \in \mathbb{N}} \frac{\|p_n(w)\|_{u,p}}{\|L_n(w)\|_{u,p}} < \infty .$$

The following Nikol'skiĭ type inequalities are well known. They are a special case of [3, Theorem 6.3.13, p. 113] where the case of more general Jacobi weights is dealt with.

Lemma 2. *Let $0 < p < \infty$. Given $n \in \mathbb{N}$, let Q be a polynomial of degree n . Then there is a constant $c > 0$ depending on p only such that*

$$\|Q\|_{\infty} \leq c n^{2/p} \|Q\|_{1,p} \quad \text{and} \quad \|Q\|_{\infty} \leq c n^{1/p} \|Q\|_{v,p} . \quad (5)$$

The following two extensions of the *uniform boundedness principle* proved to be quite useful when constructing universal examples for the divergence of some approximation processes.

Lemma 3. [3, Theorem 10.19, p. 182] *Let $0 < p < \infty$, and let $\mathbb{N}_0 \subseteq \mathbb{N}$ with $\text{card}(\mathbb{N}_0) = \infty$. Let w and u be weights in Δ . If $\lim_{n \in \mathbb{N}_0} \|f - L_n(w, f)\|_{u,p} = 0$ for all $f \in C(\Delta)$, then $\sup_{n \in \mathbb{N}_0} \|L_n(w)\|_{u,p} < \infty$.*

Lemma 4. [4, Lemma, p. 191] *Let $0 < s_0 < \infty$. Let D be a Banach space with norm $\|\cdot\|$ and let $\{B_s\}_{s_0 < s < \infty}$ be a collection of Banach spaces B_s with norms $\|\cdot\|_s$ such that $B_s \subseteq B_t$ for $s > t$, and $\|b\|_t \leq \|b\|_s$ if $t < s$ and $b \in B_s$. Let $\{L_m : D \rightarrow B_\infty\}_{m \in \mathbb{N}}$ be a sequence of bounded linear operators such that $\lim_{m \rightarrow \infty} \sup_{\|f\| \leq 1} \|L_m(f)\|_s = \infty$ for every $s_0 < s \leq \infty$. Then there exists $f \in D$ such that $\limsup_{m \rightarrow \infty} \|L_m(f)\|_s = \infty$ for every $s_0 < s \leq \infty$.*

The following result was formulated by Rakhmanov in [7, p. 263] where he points out that it can be proved in the same way that [7, Theorem 1', p. 261] is deduced from [7, Theorem 2', p. 258].²

Lemma 5. *Let τ be one of the endpoints of Δ , and let $\{\delta_n\}_{n \in \mathbb{N}}$ be a sequence of positive numbers such that $\lim_{n \rightarrow \infty} \delta_n = 0$. Then there exists a weight w in Δ and $\mathbb{N}_1 \subseteq \mathbb{N}$ with $\text{card}(\mathbb{N}_1) = \infty$ such that $w \geq v$ in Δ and $|p_n(w, \tau)| \geq \delta_n n^{1/2} (\log n)^{-3/2}$ for $n \in \mathbb{N}_1$.*

Proof of Theorem 3. By Lemma 3, we have $\sup_{n \in \mathbb{N}} \|L_n(w)\|_{u,p} < \infty$. Hence, by Lemmas 1 and 2 the theorem follows. \square

Proof of Theorem 2. Let w be chosen as in Lemma 5 with $\delta_n = (1 + \log n)^{-1}$. By (5), $\|p_n(w)\|_\infty \leq c n^{1/p} \|p_n(w)\|_{v,p}$ for $n \in \mathbb{N}$. Hence, by Lemma 3, formula (4) holds for every $p > 2$. \square

Proof of Theorem 1. By Theorem 2, there exists w such that $w \geq v$ in Δ and (4) holds for every $p > 2$. By Lemma 1, $\limsup_{n \rightarrow \infty} \|L_n(w)\|_{v,p} = \infty$ for every $p > 2$. Hence, by Lemma 4, there is a function $f \in C(\Delta)$ such that (1) holds for every $p > 2$. Formula (2) follows from (1). To prove (3), given $p > 4$, choose $\epsilon > 0$ such that $p(1 - \epsilon) > 4$. By Hölder's inequality

$$\int_{\Delta} |f - L_n(w, f)|^{p(1-\epsilon)/2} v \leq \left\{ \int_{\Delta} |f - L_n(w, f)|^p \right\}^{(1-\epsilon)/2} \left\{ \int_{\Delta} v^{2/(1+\epsilon)} \right\}^{(1+\epsilon)/2}.$$

Note that $2/(1 + \epsilon) < 2$ so that $v^{2/(1+\epsilon)} \in L^1(\Delta)$. Since $p(1 - \epsilon)/2 > 2$, (3) follows from (1). \square

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²More specifically, 'he means to point out' since what is actually written is a little different. In addition, although it is not stated explicitly in [7, Theorem 2', p. 258], Rakhmanov's weight is symmetric with respect to \mathbb{R} so that the transition from the circle to Δ is straightforward (cf. [7, p. 261, lines 2 and 3 from the top]).

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