Math 366 Homework 6

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Thank you Soonsang Hong

4.3

9. *Proof [by Mathematical Induction]*:

The property is true for n = 0:

When $n = 0$, $7^0 - 1 = 0$. 0 is divisible by 6 because $0 = 4 \cdot 0$. Thus the property is true for $n = 0$.

For all integers $n \geq 0$, if the property is true for n = k then it is true for n = k + 1:

Let k be an integer with $k \geq 0$, and suppose the property is true for $n = k$. That is suppose $7^k - 1$ is divisible by 6. *[This is the inductive hypothesis]*.

Now, we want to show $7^{k+1} - 1$ is divisible by 6.

$$7^{k+1} - 1 = 7^k \cdot 7 - 1 = 7^k \cdot (6+1) - 1 = 7^k \cdot 6 + (7^k - 1)$$

By the inductive hypothesis, $7^k - 1 = 6r$ for some integer r. By substitution into equation,

$$7^{k+1} - 1 = 7^k \cdot 6 + 6r = 6(7^k + r)$$

But $7^k + r$ is an integer because sums and products of integers are integers. Therefore, $7^{k+1} - 1$ is divisible by 6 by the definition of divisibility.

18. *Proof [by Mathematical Induction]*:

The property is true for n = 2:

When $n = 2$, the property says that $5^2 + 9=34 < 36= 6^2$.

For all integers $n \geq 2$, if the property is true for n = k then it is true for n = k + 1:

Let k be an integer with $k \geq 2$, and suppose the property is true for $n = k$. That is, suppose $5^k + 9 < 6^k$. *[The inductive hypothesis]*

Now, we want to show $5^{k+1} + 9 < 6^{k+1}$ is true.

$6(5^k + 9) < 6 \cdot 6^k$ by the laws of exponents and the inductive hypothesis. - *

Since $k \geq 2$, $5 \cdot 5^k < 6 \cdot 5^k$ and $9 < 9 \cdot 6$

$5^{k+1} + 9 = 5 \cdot 5^k + 9 < 6 \cdot 5^k + 9 \cdot 6 = 6(5^k + 9) < 6 \cdot 6^k = 6^{k+1}$ by the inequality *.

Hence,

$$5^{k+1} + 9 < 6^{k+1}$$

25. *Proof [by Mathematical Induction]*:

The property is true for n = 0:

The left-hand side of the equation is $(b_0)^2 = 5^2 = 25$. The right-hand side of the equation is $16 \cdot 0^2 = 0$. Thus, the inequalities $25 > 0$ and the property is true for $n = 0$.

For all integers $n \geq 0$, if the property is true for $n = k$ then it is true for $n = k + 1$:

Let k be an integer with $k \geq 0$, and suppose the property is true for n = k. That is, suppose $(b_k)^2 > 16 \cdot k^2$. *[The inductive hypothesis]*

Now, we want to show $(b_{k+1})^2 > 16 \cdot (k+1)^2$ is true.

$b_{k+1} = 4 + b_k$ by definition of the sequence $b_1, b_2, b_3, \ldots$

Then, $(b_{k+1})^2 = (4+b_k)^2 = 16 + 8b_k + b_k^2 < 16+8(4k)+16k^2 = 16+32k+16k^2 = 16(1+2k+k^2) = 16(k+1)^2$ since (by inductive hypothesis) we have $(b_k)^2 > 16 \cdot k^2$ and (taking the square-root of both sides) $b_k > 4k$.

Hence, $(b_{k+1})^2 > 16 \cdot (k+1)^2$

4.4

2. *Proof [by Strong Mathematical Induction]:*

The property is true for n = 1 and n = 2:

Observe that $b_1 = 4$ and $b_2 = 12$, and both b_1 and b_2 are divisible by 4 because $b_1 = 4 \cdot 1$, $b_2 = 4 \cdot 3$. So, the property is true for n = 1 and n = 2.

For any integer k > 2, if the property is true for all integers i with $1 \leq i < k$, then it is true for k:

Let k > 2 be an integer, and suppose b_i is divisible by 4 for all integers i with $1 \leq i < k$. – *[The inductive hypothesis.]*

Now, we want to show b_k is divisible by 4.

$b_k = b_{k-2} + b_{k-1}$ by definition of $b_1, b_2, b_3, \ldots$

By the inductive hypothesis, b_{k-2} and b_{k-1} are divisible by 4 and can be represented $b_{k-2} = 4r$ and $b_{k-1} = 4s$ for some integers r, s. Hence $b_k = 4r + 4s$ by substitution. $b_k = 4(r + s)$ by algebra. And r + s is an integer because sums of integers are integers. Therefore, by the definition of divisibility, b_k is divisible by 4.

5. *Proof [by Strong Mathematical Induction]:*

The property is true for n = 0, n = 1 and n = 2:

$e_0 = 1$, $e_1 = 2$ and $e_2 = 3$, and $1 \leq 1$, $2 \leq 3$ and $3 \leq 9$. So, the property is true for n = 0, 1 and 2.

For any integer k > 2, if the property is true for all integers i with $1 \leq i < k$, then it is true for k:

Let k > 2 be an integer, and suppose $e_i \leq 3^i$ for all integers i with $1 \leq i < k$. – *[The inductive hypothesis.]*

Now, we want to show $e_i \leq 3^i$.

$e_i = e_{i-1} + e_{i-2} + e_{i-3}$ by definition of $e_1, e_2, e_3, \ldots$

By the inductive hypothesis, $e_{i-1} + e_{i-2} + e_{i-3} \leq 3^{i-1} + 3^{i-2} + 3^{i-3}$. - *

$3^{i-1} + 3^{i-2} + 3^{i-3} = 3^{i-3}(3^2 + 3 + 1) < 3^i = 3^{i-3}(3^3)$. because $1 \leq i$ and $(3^2 + 3 + 1) = 13 < 3^3 = 27$. - **

Combining inequalities * and ** gives

$e_i = e_{i-1} + e_{i-2} + e_{i-3} \leq 3^i$

19. *Proof [by Strong Mathematical Induction on n]:*

The property is true for n = 2: 2 is prime hence the property is true for n=2.

For any integer k > 2, if the property is true for all integers i with $1 \le i < k$, then it is true for k:

Let n be an integer greater than 1. Consider the set $S = \{r \in Z \mid r > 1, r \mid n\}$. Since any integer greater than one has a prime divisor, $S \neq \phi$. Then by Well Ordering Principle, S has a smallest element p.

We claim that p is prime:

> Suppose not. Then there exists integers r,s >1 such that p = rs. Then r | p and r < p (since s=1 if r = p). But then (by transitivity of divisibility) r | p and p | n implies r | n. This means $r \in S$.
> But this is a contradiction to the assumption that p is smallest. Hence, p must be prime

Define m=n/p. Clearly, 0<m<n and there are two cases:

Case 1: m=1. Then n=p and n is prime.

Case 2: m>1. Then, by inductive hypothesis, m is prime or product of primes. Hence, n=m*p is product of primes.

In any case, n is prime or product of primes.*[as was to be shown]*

5.1

7. b. Yes, every element of C is in A.

8.

- e. Yes.
- g. Yes.
- h. No.

9.

- f. B-A = {6}
- g. $B \cup C = \{2, 3, 4, 6, 8, 9\}$
- h. $B \cap C = \{6\}$

11.

- a. $A \bigcup B = \{x \in R \mid -3 \le x < 2\}$
- b. $A \bigcap B = \{x \in R \mid -1 < x \le 0\}$
- c. $A^c = \{x \in R \mid x < -3 or x > 0\}$
- d. $A \bigcup C = \{x \in R \mid -3 \le x \le 0 or 6 < x \le 8\}$
- e. $A \bigcap C = \varnothing$
- f. $B^c = \{x \in R \mid x \le -1 or 2 \le x\}$
- g. $A^c \bigcap B^c = \{x \in R \mid x < -3 or 2 \le x\}$
- h. $A^c \bigcup B^c = \{x \in R \mid x \le -1 or x > 0\}$

i. $(A \bigcap B)^c = \{x \in R \mid x \leq -1 or x > 0\}$

j. $(A \bigcup B)^c = \{x \in R \mid x < -3 or 2 \leq x\}$

28.

a. $P(\varnothing) = \{\varnothing\}$

c. $P(P(P(\varnothing))) = \{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\}$

29.

b. B x A = {(a, x), (a, y), (a, z), (a, w), (b, x), (b, y), (b, z), (b, w)}

c. A x A = {(x, x), (x, y), (x, z), (x, w), (y, x), (y, y), (y, z), (y, w), (z, x), (z, y), (z, z), (z, w), (w, x), (w, y), (w, z), (w, w)}