Notes on §14.3: Partial Derivatives

I: Implicit Differentiation

Previously, you used implicit differentiation to find derivatives of one variable in terms of another. In the case of three or more variables, we can do something similar; there's just one caveat.

Suppose you're given an equation in terms of x, y, and z, and you are asked to find $\frac{\partial z}{\partial x}$ using implicit differentiation. The idea here is that you have to treat x as an independent variable, y as a constant, and z as a function of x; y is *NOT* treated as a function of x, and so you should not get any $\frac{\partial y}{\partial x}$ terms out of the deal.

14.3:43: Use implicit differentiation to find $\partial z/\partial x$ and $\partial z/\partial y$ for the equation $x - z = \arctan(yz)$.

First, for $\partial z/\partial x$. We treat x as a variable, z as a function of x, and y as a constant. Taking the partial derivative of the left side of the equation, we get

$$\frac{\partial}{\partial x}(x - z) = \frac{\partial}{\partial x}(x) - \frac{\partial}{\partial x}z = 1 - \frac{\partial z}{\partial x}$$

Taking the partial derivative of the right side using the chain rule, we get

$$\frac{\partial}{\partial x}[\arctan(yz)] = \frac{1}{1+(yz)^2} \cdot \frac{\partial}{\partial x}(yz) = \frac{1}{1+y^2z^2} \cdot y \cdot \frac{\partial z}{\partial x}$$

(Notice that we treated y as a constant, but did not treat z as a constant!) So, we have that

$$1 - \frac{\partial z}{\partial x} = \frac{y}{1+y^2z^2} \cdot \frac{\partial z}{\partial x}$$

Moving the terms with $\partial z/\partial x$ to one side and everything else to the other, we have

$$1 = \left[1 + \frac{y}{1+y^2z^2}\right] \cdot \frac{\partial z}{\partial x} = \frac{1+y^2z^2+y}{1+y^2z^2} \cdot \frac{\partial z}{\partial x}$$

So, finally, solving yields

$$\frac{\partial z}{\partial x} = \frac{1+y^2z^2}{1+y^2z^2+y}$$

Next, for $\partial z/\partial y$. Here we treat x as a constant, y as an independent variable, and z as a function of y. So, for the left side, we get

$$\frac{\partial}{\partial y}(x - z) = -\frac{\partial z}{\partial y}$$

(Again, note that the derivative of x is 0, as we think of x as constant!) On the right side, we get

$$\frac{\partial}{\partial y}[\arctan(yz)] = \frac{1}{1+(yz)^2} \cdot \frac{\partial}{\partial y}(yz) = \frac{1}{1+y^2z^2} \cdot \left(z + y \cdot \frac{\partial z}{\partial y}\right) = \frac{z}{1+y^2z^2} + \frac{y}{1+y^2z^2} \cdot \frac{\partial z}{\partial y}$$

(Note the use of the product rule here!) Setting the two sides equal yields

$$-\frac{\partial z}{\partial y} = \frac{z}{1+y^2z^2} + \frac{y}{1+y^2z^2} \cdot \frac{\partial z}{\partial y}$$

Moving terms involving $\partial z/\partial y$ to the right and everything else to the left, we get

$$-\frac{z}{1+y^2z^2} = \left[1 + \frac{y}{1+y^2z^2}\right] \cdot \frac{\partial z}{\partial y} = \frac{1+y^2z^2+y}{1+y^2z^2} \cdot \frac{\partial z}{\partial y}$$

So, solving, we get

$$\frac{\partial z}{\partial y} = -\frac{z}{1+y^2z^2} \cdot \frac{1+y^2z^2}{1+y^2z^2+y} = -\frac{z}{1+y^2z^2+y}$$

II: Higher Order Partial Derivatives

These are much less tricky problems than implicit differentiation; here, the variable with respect to which you are differentiating is considered a true variable, and the other variable (or variables) are treated as constants. The only difficulty here is with the notation.

Note that

$$f_{xyz} = \frac{\partial}{\partial z}\left(\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\left(f(x,y,z)\right)\right)\right)$$

In other words, you differentiate with respect to x first, then differentiate the result with respect to y, then differentiate *that* result with respect to z. **The order of differentiation in the f_{xyz} notation is left-to-right - differentiate using the left-most variable first.** A lot of times, the order will not matter; but sometimes it will, so always go in order.

14.3:57: $f(x, y, z) = 3xy^4 + x^3y^2$; find f_{xxy} and f_{yyy}.

First, for f_{xxy}. We compute

$$\begin{aligned} f_x(x,y,z) &= 3y^4 + 3x^2y^2 \\ f_{xx}(x,y,z) &= 6xy^2 \\ f_{xxy}(x,y,z) &= 12xy \end{aligned}$$

Next, for f_{yyy}. We have

$$\begin{aligned} f_y(x,y,z) &= 12xy^3 + 2x^3y \\ f_{yy}(x,y,z) &= 36xy^2 + 2x^3 \\ f_{yyy}(x,y,z) &= 72xy \end{aligned}$$