Notes on §14.5: Chain Rule

Recall the formula given for the chain rule: if $f(x, y)$ is a differentiable function, and $x = x(t)$ and $y = y(t)$ are differentiable, then

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

This expands to more than two variables in the obvious way: if we have $f(x, y, z)$ and $x = x(t)$, $y = y(t)$, $z = z(t)$, then

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial f}{\partial z} \cdot \frac{dz}{dt}$$

We can get the formula for the $f(x, y)$ with $x = x(s, t)$ and $y = y(s, t)$ case the same way we handle partial derivatives: just apply the above formula to the variable in question!

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s}$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t}$$

14.5:5: Find $\frac{dw}{dt}$ for $w = xe^{y/z}$, $x = t^2$, $y = 1 - t$, $z = 1 + 2t$.

The formula is

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial w}{\partial z} \cdot \frac{dz}{dt}$$

We compute

$$\frac{\partial w}{\partial x} = e^{y/z} \qquad \frac{\partial w}{\partial y} = \frac{x}{z}e^{y/z} \qquad \frac{\partial w}{\partial z} = xe^{y/z} \cdot \frac{-1}{z^2} = -\frac{x}{z^2}e^{y/z}$$

Further,

$$\frac{dx}{dt} = 2t \qquad \frac{dy}{dt} = -1 \qquad \frac{dz}{dt} = 2$$

So, we have

$$\frac{dw}{dt} = 2te^{y/z} - \frac{x}{z}e^{y/z} - 2\frac{x}{z^2}e^{y/z} = e^{y/z}\left[2t - \frac{x}{z} - 2\frac{x}{z^2}\right]$$

Plugging in the appropriate expressions for x, y, and z yields

$$\frac{dw}{dt} = 2te^{(1-t)/(1+2t)}\left[2t - \frac{t^2}{1+2t} - 2\frac{t^2}{(1+2t)^2}\right]$$

14.5:9: Find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ for $z = \arctan(2x + y)$, $x = s^2 t$, $y = s \cdot \ln t$.

First,

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}$$

We compute

$$\frac{\partial z}{\partial x} = \frac{2}{1 + (2x + y)^2} \qquad \frac{\partial z}{\partial y} = \frac{1}{1 + (2x + y)^2}$$

Further, $\frac{\partial x}{\partial s} = 2st$ and $\frac{\partial y}{\partial s} = \ln t$. So,

$$\frac{\partial z}{\partial s} = \frac{2}{1 + (2x + y)^2} \cdot 2st + \frac{1}{1 + (2x + y)^2} \cdot \ln t = \frac{2st + \ln t}{1 + (2s^2 t + s \ln t)^2}$$

Now, $\frac{\partial x}{\partial t} = s^2$ and $\frac{\partial y}{\partial t} = \frac{s}{t}$; so, the corresponding formula for t yields

$$\frac{\partial z}{\partial t} = \frac{2}{1 + (2x + y)^2} \cdot s^2 + \frac{1}{1 + (2x + y)^2} \cdot \frac{s}{t} = \frac{2s^2 + \frac{s}{t}}{1 + (2s^2 t + s \ln t)^2}$$

Implicit Differentiation Revisited

This section also gives us a new means of Implicit Differentiation. Suppose we are given a relation between x and y of the form $F(x, y) = 0$ (if not, rewrite it this way). Then

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$$

Similarly, if we are given a relation between x, y, and z of the form $F(x, y, z) = 0$, we have

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} \qquad \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$$

14.5:27: Find $\frac{dy}{dx}$ if $\sqrt{xy} = 1 + x^2 y$.

First, we rewrite this: if $\sqrt{xy} = 1 + x^2 y$, then $1 + x^2 y - \sqrt{xy} = 0$. So, here we have $F(x, y) = 1 + x^2 y - \sqrt{xy}$. So,

$$\frac{\partial F}{\partial x} = 2xy - \frac{y}{2\sqrt{xy}} = \frac{4x^{3/2}y^{3/2} - y}{2\sqrt{xy}}$$

$$\frac{\partial F}{\partial y} = x^2 - \frac{x}{2\sqrt{xy}} = \frac{2x^{5/2}\sqrt{y} - x}{2\sqrt{xy}}$$

(The last algebraic step is so that things will cancel when we divide!) Hence we have

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{4x^{3/2}y^{3/2} - y}{2x^{5/2}\sqrt{y} - x}$$

14.5:31: Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $x^2 + y^2 + z^2 = 3xyz$.

If $x^2 + y^2 + z^2 = 3xyz$, then $x^2 + y^z + z^2 - 3xyz = 0$; so here our $F(x, y, z) = x^2 + y^2 + z^2 - 3xyz$. We have

$$\frac{\partial F}{\partial x} = 2x - 3yz \qquad \frac{\partial F}{\partial y} = 2y - 3xz \qquad \frac{\partial F}{\partial z} = 2z - 3xy$$

So,

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = -\frac{2x - 3yz}{2z - 3xy}$$

$$\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = -\frac{2y - 3xz}{2z - 3xy}$$