Notes on §14.8: Lagrange Multipliers

14.8:9: Maximize/Minimize $f(x, y, z) = xyz$ subject to $x^2 + 2y^2 + 3z^2 = 6$.

Here we have $g(x, y, z) = x^2 + 2y^2 + 3z^2$. So, $\nabla f(x, y, z) = \langle yz, xz, yz\rangle$ and $\nabla g(x, y, z) = \langle 2x, 4y, 6z\rangle$. So, we wish to find all solutions (x, y, z, λ) of the system of equations

$$\begin{aligned} yz &= \lambda \cdot 2x \\ xz &= \lambda \cdot 4y \\ xy &= \lambda \cdot 6z \\ x^2 + 2y^2 + 3z^2 &= 6 \end{aligned}$$

Case I: $\lambda = 0$.

Here, the first three equations become $yz = 0$, $xz = 0$, and $xy = 0$; hence at least 2 of x, y, and z must be 0; in fact, it must be exactly 2, because if $x = y = z = 0$ then the last equation becomes $0 = 6$. So we either have $(x, 0, 0)$, $(0, y, 0)$, or $(0, 0, z)$; in the first case, we need $x^2 = 6$, so $x = \pm\sqrt{6}$. In the second, $2y^2 = 6$, so $y = \pm\sqrt{3}$. In the third, $3z^2 = 6$, so $z = \pm\sqrt{2}$. We have found 6 points that work:

$$(\sqrt{6}, 0, 0) \quad (-\sqrt{6}, 0, 0) \quad (0, \sqrt{3}, 0) \quad (0, -\sqrt{3}, 0) \quad (0, 0, \sqrt{2}) \quad (0, 0, -\sqrt{2})$$

For all of these points, we have $f(x, y, z) = 0$.

Case II: $\lambda \neq 0$.

We can divide by λ in this case. It will be more helpful to have x^2, y^2, and z^2 than x, y, and z, since our constraint is in terms of squares. Squaring the first three equation above yields

$$y^2z^2 = 4\lambda^2x^2 \quad x^2z^2 = 16\lambda^2y^2 \quad x^2y^2 = 36\lambda^2z^2$$

Solving the first for x^2, we get $x^2 = \frac{y^2z^2}{4\lambda^2}$. Plugging this in to the second, and third, we get

$$\left(\frac{y^2z^2}{4\lambda^2}\right) z^2 = 16\lambda^2y^2 \qquad \left(\frac{y^2z^2}{4\lambda^2}\right) y^2 = 36\lambda^2z^2$$

If $y = 0$, the first equation is true, and the second equation becomes $0 = 36\lambda z^2$, so either $z = 0$ or $\lambda = 0$. We have already covered both of these situations. If $y \neq 0$, rearranging the first equation yields $z^4 = 64\lambda^4$, or $z^2 = 8\lambda^2$.

For the right equation: if $z = 0$ it is true, and the first equation requires either $\lambda = 0$ or $y = 0$; we have already covered this as well. If $z \neq 0$, rearranging the second equation yields $y^4 = 144\lambda^4$, or $y^2 = 12\lambda^2$.

Using these two equations, we get

$$x^2 = \frac{y^2z^2}{4\lambda^2} = \frac{12\lambda^2 \cdot 8\lambda^2}{4\lambda^2} = 24\lambda^2$$

So, plugging these all in to the constraint $x^2 + 2y^2 + 3z^2 = 6$ yields $24\lambda^2 + 2 \cdot 12\lambda^2 + 3 \cdot 8\lambda^2 = 6$, or $72\lambda^2 = 6$. So $\lambda^2 = \frac{6}{72} = \frac{1}{12}$; hence $x^2 = \frac{24}{12} = 2$, $y^2 = \frac{12}{12} = 1$, and $z^2 = \frac{8}{12} = \frac{2}{3}$. This yields 8 possible solutions: $(x, y, z) = (\pm\sqrt{2}, \pm 1, \pm\sqrt{\frac{2}{3}})$. Plugging in these 8 points to $f(x, y, z)$ yields two different results: if we use an even number of negative terms, we get $\frac{2}{\sqrt{3}}$, while if we use an odd number of negative terms we get $-\frac{2}{\sqrt{3}}$.

So the maximum of this function is $\frac{2}{\sqrt{3}}$, which is achieved at (for instance) $(x, y, z) = (\sqrt{2}, 1, \sqrt{\frac{2}{3}})$; the minimum of this function is $-\frac{2}{\sqrt{3}}$, which is achieved at (for instance) $(x, y, z) = (-\sqrt{2}, -1, -\sqrt{\frac{2}{3}})$.

14.8:12: Maximize/minimize $f(x, y, z) = x^4 + y^4 + z^4$ subject to $x^2 + y^2 + z^2 = 1$.

Here $g(x, y, z) = x^2 + y^2 + z^2$; we have $\nabla f(x, y, z) = \langle 4x^3, 4y^3, 4z^3 \rangle$ and $\nabla g(x, y, z) = \langle 2x, 2y, 2z \rangle$. So, our system of constraints is

$$\begin{aligned} 4x^3 &= \lambda \cdot 2x \\ 4y^3 &= \lambda \cdot 2y \\ 4z^3 &= \lambda \cdot 2z \\ x^2 + y^2 + z^2 &= 1 \end{aligned}$$

Note that $\lambda \neq 0$: if $\lambda = 0$, then the first three equations yields $x = y = z = 0$, but this does not satisfy the constraint. Further, note that all three variables are involved in these equations and in f exactly the same way; so, we can get away with some symmetry. Break in to cases based on the number of variables which are 0.

Case I: None of x, y, and z are 0.

Then we can solve all three of the gradient equations for λ; we get $\lambda = 2x^2 = 2y^2 = 2z^2$, or in other words $x^2 = y^2 = z^2 = \frac{\lambda}{2}$. Plugging these in to the constraint equation yields $\frac{1}{2}\lambda + \frac{1}{2}\lambda + \frac{1}{2}\lambda = 1$, or $\lambda = \frac{2}{3}$. So $x^2 = y^2 = z^2 = \frac{1}{3}$, and we get 8 possible points: $(x, y, z) = (\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}})$. For all of 8 points, we have

$$x^4 + y^4 + z^4 = \frac{1}{9} + \frac{1}{9} + \frac{1}{9} = \frac{1}{3}$$

Case II: Exactly one of x, y, and z is 0.

Suppose $x = 0$, and y and z are both non-zero. Then the first equation above disappears, and we are left with $4y^3 = 2\lambda y$, $4z^3 = 2\lambda z$, and $y^2 + z^2 = 1$. Because y and z are nonzero, we can solve the first two equations to get $2y^z = 2z^2 = \lambda$, or $y^2 = z^2 = \frac{\lambda}{2}$. Plugging these in to $y^2 + z^2 = 1$ yields $\frac{\lambda}{2} + \frac{\lambda}{2} = 1$, or $\lambda = 1$. So $y^2 = z^2 = \frac{1}{2}$, and we get four points: $(x, y, z) = (0, \pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}})$. These points all have the same value:

$$x^4 + y^4 + z^4 = 0 + \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

By symmetry, we get results of $(x, y, z) = (\pm\frac{1}{\sqrt{2}}, 0, \pm\frac{1}{\sqrt{2}})$ for the case of only $y = 0$, and $(x, y, z) = (\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}, 0)$ for the case of only $z = 0$; these yield the same f-values.

Case III: Exactly two of x, y, and z are 0.

Suppose $y = 0$ and $z = 0$, but $x \neq 0$. Then the first gradient equation can be solved to get $\lambda = 2x^2$; both of the other gradient equations disappear. Our constraint equation becomes $x^2 = 1$. So, we get $x = \pm 1$, in which case $\lambda = 2$. So we get two points out of this: $(x, y, z) = (\pm 1, 0, 0)$. For each of these points, we have

$$x^4 + y^4 + z^4 = 1 + 0 + 0 = 1$$

By symmetry, we get the points $(x, y, z) = (0, \pm 1, 0)$ for $x = z = 0$ and $y \neq 0$, and for the case $x = y = 0$ and $z \neq 0$ we get $(x, y, z) = (0, 0, \pm 1)$. These points all have the same f-values.

Case IV: $x = y = z = 0$.

This is not possible, as $(0, 0, 0)$ does not satisfy the constraint.

So, overall, $f(x, y, z)$ has a minimum of $\frac{1}{3}$, achieved at (for instancce) $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$. It has a maximum of 1, achieved at (for instance) $(1, 0, 0)$.