Solutions to Graded Problems in 14.7, 14.8, and 15.1

14.7:17: Find and classify all critical points of the function $f(x, y) = (x^2 + y^2)e^{y^2-x^2}$.

First, we find our partial derivatives:

$$f_x(x, y) = (x^2 + y^2)e^{y^2-x^2}(-2x) + (2x)e^{y^2-x^2} = 2xe^{y^2-x^2}(1 - x^2 - y^2)$$
$$f_y(x, y) = (x^2 + y^2)e^{y^2-x^2}(2y) + (2y)e^{y^2-x^2} = 2ye^{y^2-x^2}(1 + x^2 + y^2)$$

Now, if $f_y = 0$, then either $y = 0$, $e^{y^2-x^2} = 0$, or $1 + x^2 + y^2 = 0$; but, the only one of these that is possible is $y = 0$. So, $y = 0$ at all critical points. Plugging $y = 0$ in to f_x, we have $f_x(x, 0) = 2xe^{-x^2}(1 - x^2)$, which is 0 when $x = 0$ or $1 - x^2 = 0$, i.e. $x = \pm 1$. So, we have three critical points: $(1, 0)$, $(0, 0)$, and $(-1, 0)$.

Now, we need our second-order partials; we'll compute them numerically rather than simplifying.

$$f_{xx}(x, y) = 2e^{y^2-x^2}(1 - x^2 - y^2) + 2x\left[e^{y^2-x^2}(-2x)(1 - x^2 - y^2) + e^{y^2-x^2}(-2x)\right]$$
$$f_{xx}(1, 0) = \frac{2}{e}(0) + 2\left[\frac{1}{e}(-2)(0) + \frac{1}{e}(-2)\right] = -\frac{4}{e}$$
$$f_{xx}(0, 0) = 2e^0(1) = 2$$
$$f_{xx}(-1, 0) = \frac{2}{e}(0) - 2\left[\frac{1}{e}(-2)(0) + \frac{1}{e}(2)\right] = -\frac{4}{e}$$
$$f_{xy}(x, y) = 2x\left[e^{y^2-x^2}(2y)(1 - x^2 - y^2) + e^{y^2-x^2}(-2y)\right]$$
$$f_{xy}(1, 0) = f_{xy}(0, 0) = f_{xy}(-1, 0) = 0$$
$$f_{yy}(x, y) = 2e^{y^2-x^2}(1 + x^2 + y^2) + 2y\left[e^{y^2-x^2}(2y)(1 + x^2 + y^2) + e^{y^2-x^2}(2y)\right]$$
$$f_{yy}(1, 0) = \frac{2}{e}(2) + 0 = \frac{4}{e}$$
$$f_{yy}(0, 0) = 2e^0(1) + 0 = 2$$
$$f_{yy}(-1, 0) = \frac{2}{e}(2) + 0 = \frac{4}{e}$$

So, we have

$$D(1, 0) = -\frac{4}{e} \cdot \frac{4}{e} - 0^2 = -\frac{16}{e^2} < 0$$
$$D(0, 0) = 2 \cdot 2 - 0^2 = 4 > 0 \text{ and } f_{xx}(0, 0) = 2 > 0$$
$$D(-1, 0) = -\frac{4}{e} \cdot \frac{4}{e} - 0^2 = -\frac{16}{e^2} < 0$$

Hence we have saddle points at $(1, 0)$ and $(-1, 0)$, and a local minimum at $(0, 0)$.

14.8:19: Find the max/min of $f(x,y) = e^{-xy}$ when $x^2 + 4y^2 \leq 1$.

Any max or min either occurs at a critical point or on the boundary. To find critical points, note that $f_x(x,y) = -ye^{-xy} = 0$ only when $y = 0$, and $f_y(x,y) = -xe^{-xy} = 0$ only when $x = 0$; hence our only critical point is at $(0,0)$, and $f(0,0) = e^0 = 1$.

To check the boundary, we use Lagrange multipliers. As above, $\nabla f(x,y) = \langle -ye^{-xy}, -xe^{-xy} \rangle$. If our constraint is $g(x,y) = x^2 + 4y^2 = 1$, then $\nabla g(x,y) = \langle 2x, 8y \rangle$. So, our system of equations is

$$-ye^{-xy} = \lambda \cdot 2x \qquad -xe^{-xy} = \lambda \cdot 8y \qquad x^2 + 4y^2 = 1$$

Note that neither x nor y is 0: if $x = 0$, then $-ye^0 = 0$, and so $y = 0$; but then $x^2 + 4y^2 = 0$. Similarly, if $y = 0$, then $x = 0$, and we can't have that. So, we can solve the first two equations to get

$$-\frac{ye^{-xy}}{2x} = \lambda = -\frac{xe^{-xy}}{8y}$$

Cross-multiplying and cancelling, this means $-2x^2 = -8y^2$, or $x^2 = 4y^2$. Plugging this in to the contraint yields $x^2 + 4y^2 = x^2 + x^2 = 2x^2 = 1$, or $x^2 = \frac{1}{2}$; hence $x = \pm\frac{1}{\sqrt{2}}$. Then we need $4y^2 = \frac{1}{2}$, or $y = \pm\frac{1}{\sqrt{8}}$. So, we get four special points: $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{8}})$, $(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{8}})$, $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{8}})$, and $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{8}})$. So, our list of points and values is

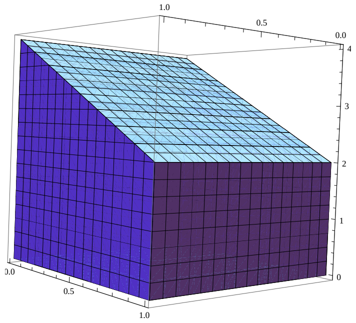
$$f(0,0) = 1 \qquad f(\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{8}}) = f(-\tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{8}}) = e^{-1/4} \qquad f(\tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{8}}) = f(-\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{8}}) = e^{1/4}$$

So, our max is $e^{1/4}$ at (for instance) $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{8}})$, and our min is $e^{-1/4}$ at (for instance) $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{8}})$.

15.1:13: Evaluate the following integral by identifying it as a volume:

$$\iint\limits_R (4 - 2y)\, dA \qquad R = [0,1] \times [0,1]$$

This integral is the volume of the following solid:



It is composed of two parts: a rectangular prism with height 2, base 1, and width 1, and a triangular prism with height 2, base 1, and width 1. The rectangular prism then has volume 2 and the triangular prism has volume $\frac{1}{2} \cdot 2 \cdot 1 \cdot 1 = 1$, so we must have

$$\iint\limits_R (4 - 2y)\, dA = 2 + 1 = 3$$