Math 254.01, Quiz 3 SOLUTIONS

October 22nd, 2009

Instructions: Show all of your work and justify all steps. Name theorems where appropriate. Correct answers are not worth points without the accompanying work. Write legibly. **No decimal answers will be accepted unless there are decimals in the problem.** You have 20 minutes.

[10 points] Find the directional derivative of $f(x, y) = e^{xy} + x^2$ at the point $P(2, 3)$ in the direction of the point $Q(3, 4)$.

First, we compute $\nabla f(2, 3)$: note that

$$\nabla f(x, y) = \langle ye^{xy} + 2x, xe^{xy} \rangle$$

So, plugging in $(2, 3)$ yields $\nabla f(2, 3) = \langle 3e^6 + 4, 2e^6 \rangle$.

Next, for our direction. The vector which goes from $(2, 3)$ to $(3, 4)$ is $\langle 1, 1 \rangle$; hence $\vec{u}$ is the unit vector in this direction. So

$$\vec{u} = \frac{\langle 1, 1 \rangle}{|\langle 1, 1 \rangle|} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

So, we have

$$f_{\vec{u}}(2, 3) = \langle 3e^6 + 4, 2e^6 \rangle \cdot \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle = \frac{3e^6 + 4}{\sqrt{2}} + \frac{2e^6}{\sqrt{2}} = \frac{5e^6 + 4}{\sqrt{2}}$$

[10 points] Find and classify all local maximum and minimum points and saddle points of the following function.

$$f(x, y) = x^2 + y^2 + \frac{1}{x^2y^2}$$

Computing derivatives, we get

$$\nabla f(x, y) = \left\langle 2x - \frac{2}{x^3y^2}, 2y - \frac{2}{x^2y^3} \right\rangle = \left\langle \frac{2x^4y^2 - 2}{x^3y^2}, \frac{2x^2y^4 - 2}{x^2y^3} \right\rangle$$

This is undefined whenever $x = 0$ or $y = 0$, but these aren't in the domain. Both of our derivatives are 0 when $2x^4y^2 - 2 = 0$ and $2x^2y^4 - 2 = 0$, i.e. when $x^4y^2 = 1$ and $x^2y^4 = 1$.

Since $x \neq 0$, the first condition is equivalent to $y^2 = \frac{1}{x^4}$; plugging this in to the other equation, we must have

$$1 = x^2y^4 = x^2(y^2)^2 = \frac{x^2}{x^8} = \frac{1}{x^6}$$

Hence $x^6 = 1$, which happens only when $x = \pm 1$. Plugging either of these x-values in to $x^4y^2 = 1$ yields $y^2 = 1$, and so $y = \pm 1$ as well. We get four critical points: $(1, 1)$, $(1, -1)$, $(-1, 1)$, and $(-1, -1)$.

Now, for higher derivatives. We computed f_x and f_y above; using those, we have

$$f_{xx}(x, y) = 2 + \frac{6}{x^4y^2} \qquad f_{xy}(x, y) = \frac{4}{x^3y^3} \qquad f_{yy}(x, y) = 2 + \frac{6}{x^2y^4}$$

For all of these points, we get $f_{xx}(x, y) = 8$ and $f_{yy}(x, y) = 8$. The value of $f_{xy}(x, y)$ depends on how many negatives we have. In any case, we get

$$\begin{aligned}
(1, 1) :& \quad D = 8 \cdot 8 - (4)^2 = 48 \\
(1, -1) :& \quad D = 8 \cdot 8 - (-4)^2 = 48 \\
(-1, 1) :& \quad D = 8 \cdot 8 - (-4)^2 = 48 \\
(-1, -1) :& \quad D = 8 \cdot 8 - (4)^2 = 48
\end{aligned}$$

All points have positive discriminant and positive f_{xx}, so all four points are local minima.