

Universality Limits for Random Matrices via Classical Complex Analysis

Doron S. Lubinsky,
Georgia Institute of Technology.

We present a new method for establishing universality limits in the bulk, based on the theory of entire functions of exponential type. Let μ be a measure on a compact subset of the real line. Assume that μ is absolutely continuous in a neighborhood of some point x in the support, and that μ' is bounded above and below near x , which is assumed to be a Lebesgue point of μ' . Let $\{p_j\}$ denote the orthonormal polynomials for μ , and K_n denote the n th reproducing kernel for μ , so that

$$K_n(x, y) = \sum_{j=0}^{n-1} p_j(x) p_j(y).$$

Then the following are equivalent:

(I) For all real a ,

$$\lim_{n \rightarrow \infty} \frac{K_n\left(x + \frac{a}{n}, x + \frac{a}{n}\right)}{K_n(x, x)} = 1.$$

(II) Uniformly for a, b in compact subsets of the plane,

$$\lim_{n \rightarrow \infty} \frac{K_n\left(x + \frac{a}{\tilde{K}_n(x, x)}, x + \frac{b}{\tilde{K}_n(x, x)}\right)}{K_n(x, x)} = \frac{\sin \pi(a - b)}{\pi(a - b)}.$$

Here K_n is the reproducing kernel for μ , while \tilde{K}_n is the normalized reproducing kernel. Thus universality is equivalent to universality "along the diagonal", or equivalently ratio asymptotics for Christoffel functions.

We emphasize that μ does not need to be a regular measure. In the case that μ is a regular measure, the condition (I) follows from results of Vili Totik, built on earlier results of Máté, Nevai and Totik. So for regular measures, we provide an alternative proof to recent results of Barry Simon, Vili Totik, and the author. The advantage of the method is that it does not need a "comparison measure" with similar support, for which universality is known.

The method also applies to exponential weights, and to the hard edge and soft edge of the spectrum, where the Bessel and Airy kernels arise. This has been studied by Eli Levin and the author.