Solution to problem 13, Sec. 17.5

$y = x^a$ so, of course $, y' = a x^{a-1}, y'' = a(a-1) x^{a-2}$ therefore

$$k = \frac{a(a-1)x^{a-2}}{(1 + a^2 x^{2a-2})^{3/2}} \quad , x > 0 \quad , a > 0$$

Reminder $\lim_{x \to 0+} x^p = \begin{cases} 0 & \text{if } p > 0 \\ 1 & \text{if } p = 0 \\ +\infty & \text{if } p < 0 \end{cases}$

therefore let us first distinguish among 3 cases

$(2a-2>0, \; 2a-2=0, \; 2a-2<0)$

Case I: $a > 1$ therefore $2a-2>0$ and $\lim_{x \to 0+} x^{2a-2} = 0$

then the denominator of k goes to 1 and therefore

$$\lim_{x\to 0+} k = \lim_{x \to 0+} a(a-1)x^{a-2} = \begin{cases} 0 & \text{if } a > 2 \\ a(a-1) = 2(2-1) & \text{if } a = 2 \\ +\infty & \text{if } a < 2 \end{cases}$$

} finite limit for $a \geq 2$

Case II $a = 1$ then $k = 0 \longrightarrow$ finite limit

Case III $0 < a < 1$. Then both x^{2a-2} and x^{a-2} go to $+\infty$

Intuitively, $1 \ll x^{2a-2}$ so we can neglect 1 in the limit:

$$\lim_{x\to 0+} k = \lim_{x \to 0+} \frac{a(a-1)x^{a-2}}{(a^2 x^{2a-2})^{3/2}} = \lim_{x\to 0+} \frac{a(a-1)}{a^3} \frac{x^{a-2}}{x^{3a-3}}$$

$$= \lim_{x \to 0+} \frac{a-1}{a^2} \cdot x^{1-2a} = \begin{cases} 0 & \text{if } 1-2a > 0 \text{ so } a < \frac{1}{2} \\ \frac{a-1}{a^2} & \text{if } 1-2a = 0 \text{ so } a = \frac{1}{2} \\ +\infty & \text{if } a > \frac{1}{2} \end{cases}$$

] finite limit

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