Ross Program 2010 Application Problems

This document is part of the application to the *Ross Mathematics Program*, and is posted at: www.math.ohio-state.edu/ross. This challenging eight-week residential program for high school students will run from June 21 to August 13, 2010.

The official application deadline is June 1, but most spaces are filled by early May.
For adequate consideration of your application, submit it by the end of April.

Applicants should work on each of the problems below. We are interested in seeing how you approach unfamiliar math problems, not whether you can find answers by searching through books or web sites.
Please submit your own work on *all* of these problems.

For each problem, explore the situation (with calculations, tables, pictures, etc), observe the patterns, make some guesses, test the truth of those conjectures, and describe the progress you have made. Where were you led by your experimenting?

Include your thoughts even though you may not have solved the whole problem.
If you've seen one of the problems before (e.g. in a class or online), please include a reference with your solution.

Write up your problem solutions on separate pages (one side only) and mail them, along with your completed Application Form, to the following address.

Ross Mathematics Program
Department of Mathematics
Ohio State University
231 W. 18th Ave.
Columbus, OH 43210
USA

Electronic submissions are discouraged.

Note:
Each *Ross Program* course concentrates deeply on one subject, unlike the problems here.
This Problem Set is an attempt to assess your general mathematical background and interests.

(1) The letters $a_1, a_2, a_3, a_4, a_5, a_6, a_7$ represent seven positive whole numbers. The letters $b_1, b_2, b_3, b_4, b_5, b_6, b_7$ represent the same numbers but in a different order. Will the product

$$(a_1 - b_1)(a_2 - b_2)(a_3 - b_3)(a_4 - b_4)(a_5 - b_5)(a_6 - b_6)(a_7 - b_7)$$

always be an even number? Explain your conclusion.

(Note: A number is "even" if it equals $2n$ for some integer n. For example, -2, 0, 2, and 4 are even.)

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(2) Call a number "nice" if it can be expressed as a sum of two or more consecutive positive integers. For example, 5 and 6 are nice numbers because 5 = 2+3 and 6 = 1+2+3.

(a) Which numbers from 1 to 50 are not nice? What's the pattern for sizes beyond 50?

(b) Explain why the pattern you observed holds true generally.

(c) Some integers are nice in several ways.
For instance, 15 is nice in three ways: 15 = 1+2+3+4+5 = 4+5+6 = 7+8.

List all the ways that 1000 can be expressed as a sum of consecutive positive integers.

Given a number n, is there a simple method to count the number of ways n is nice? Explain.

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(3) A set of numbers has "the triple-sum property" (or TSP) if there exist three numbers in the set whose sum is also in the set. [Repetitions are allowed.]

For example, the set $U = \{2, 3, 7\}$ has TSP since $2 + 2 + 3 = 7$, while $V = \{2, 3, 10\}$ fails to have TSP.

(a) Suppose the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ is separated into two parts, forming two subsets A and B.

Prove: Either A or B must have the triple-sum property.

[To begin the proof, suppose that statement is false and there are sets A and B as above, each without TSP. If 1 lies in A then $3 = 1 + 1 + 1$ must be in B. Complete the proof that this situation is impossible.]

(b) Is a similar result true when the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ is separated into two parts?

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(4) Suppose 100 dots are arranged in a square 10×10 array, and each dot is colored red or blue.

(a) Prove that this array must contain a "monochromatic" rectangle. That is, no matter how the red and blue colors are assigned, there must be either a set of four red dots that form a rectangle or else a set of four blue dots that form a rectangle.

[Don't consider colors of the dots inside that rectangle. Just the four corner points. Use only those rectangles having horizontal and vertical sides.]

(b) Does this result remain true for smaller rectangular arrays of dots?
To begin, find a 4×5 array that admits no monochromatic rectangle.
Must a monochromatic rectangle exist in a 5×5 array? In a 4×6 array?

(5) Suppose $a_1, a_2, \ldots, a_n$ is a list of n numbers with the following properties:

The sum of those n numbers is 500.
The sum of the smallest three of those numbers is 48.
The sum of the largest two of those numbers is 35.

(Note: There might be some repetitions among the numbers a_k in that list.)

(a) What are the possible values of n ? Explain your reasoning.

(b) Can each of those values of n occur if we require all the numbers a_k to be integers?

(6) Suppose p and q are odd integers.

(a) Show that the quadratic equation $x^2 + px + q = 0$ has no rational roots.

(A number α is a "root" of that equation if**:** $\alpha^2 + p\alpha + q = 0$. A number is *rational* if it is expressible as m/n for some integers m and n.)

(b) Does this result generalize to equations of the type $x^n + px + q = 0$?

(7) A "lattice point" has integer coordinates. Then, $A = (m, n)$ is a lattice point if both m and n are integers. Let's call a point $P = (x, y)$ "generic'' if all the distances from P to lattice points are different.

With some algebraic work, I checked that the point $S = (\sqrt{2}, \sqrt{3})$ is generic.

However, the point $T = (0, \pi)$ is not generic because it is equally distant from the lattice points $(1, 0)$ and $(-1, 0)$.

★ Is there some generic point with rational coordinates?

That is, if $Q = (r, s)$ for rational numbers r and s, must there exist two lattice points equidistant from Q?

As a first step, show that $R = (\frac{3}{4}, \frac{2}{5})$ is not generic. (Find lattice points A, B equidistant from R.)

Can you use those ideas to answer the general question?

(8) What numbers can be expressed as an alternating-sum of an increasing sequence of powers of 2 ?

To form such a sum, choose a subset of the sequence 1, 2, 4, 8, 16, 32, 64, . . . (these are the powers of 2). List the numbers in that subset in increasing order (no repetitions allowed), and combine them with alternating plus and minus signs. For example,

$1 = -1 + 2$;	$2 = -2 + 4$;	$3 = 1 - 2 + 4$;	
$4 = -4 + 8$;	$5 = 1 - 4 + 8$;	$6 = -2 + 8$;	etc.

(a) Is every positive integer expressible in this fashion? If so, give a convincing proof.

(b) There can be more than one expression of this type for a given number. For instance $5 = 1 - 4 + 8$ and $5 = -1 + 2 - 4 + 8$. Given a number n, how many different ways are there to write n in this way? Explain why your answer is correct.

(9) Which of the problems here did you enjoy the most? Why?

We hope you enjoyed working on these problems!

Further information about this summer mathematics program is available on the web at www.math.ohio-state.edu/ross or by email at ross@math.ohio-state.edu.