

DEFINABILITY, CONSTRUCTIBILITY AND TRANSFER

HANS SCHOUTENS

1. ALGEBRAIC GEOMETRY VIA MODEL THEORY

Throughout, we will fix an algebraically closed field K . Classical algebraic geometry studies varieties defined over K , where in this proposal, a *variety* means a solution set of some polynomial equation system over K . Therefore, the study of a variety V is equivalent to the study of the maximal spectrum of its coordinate ring $A(V)$. GROTHENDIECK realized that a relative version of the concept of a variety, termed *scheme*, would allow for the infinitesimal study of varieties, and at the same time, greatly facilitate the study of algebraic families of varieties (if $f: X \rightarrow S$ is morphism of schemes, then the collection of all its fibers forms an *algebraic family* of varieties). To fully exploit this new point of view, the classical notion of point as a K -rational solution has to be replaced by allowing arbitrary L -rational solutions (up to some obvious congruence relation), where L is any extension field of K . Put differently, this leads to the study of all prime ideals of the coordinate ring, or, for that matter, of any ring. The main principle of modern algebraic geometry could be phrased as follows.

Principle 1. *If a collection \mathcal{E} of objects can be constructed in an algebraic-geometric way from a scheme S (and possibly some other data), then \mathcal{E} admits a description in terms of a single scheme X over S (that is to say, in terms of a morphism of schemes $X \rightarrow S$).*

I left the statement of this principle deliberately vague; all depends on what counts as an algebraic-geometric construction and what the precise nature of this description is. Nonetheless, if a certain collection satisfies this principle, then methods from algebraic geometry can be applied to the encoding scheme. For instance, the set of singular points on a scheme X is often just a closed subscheme $F \subset X$ and its dimension is a crude measure for the defect of a scheme to be non-singular; a continuously varying algebraic family parametrized by S arises as the fibers of a flat morphism $X \rightarrow S$ and the Hilbert polynomial is constant along these fibers, whence an invariant of the family; curves of a fixed genus form a moduli space and the irreducibility of this scheme is a deep fact. Unfortunately, this principle does not always apply, even in simple instances. Take for instance the corresponding variety S_{\max} , consisting of all closed (that is to say, K -rational) points of a scheme S ; this carries in no way the structure of a scheme nor can it be encoded in some other scheme. Put differently, S_{\max} has no *generic* members.

This shortcoming can often be removed when we take a model theoretic point of view. A variety, being a solution set in K^n of a polynomial system of equations, is an example of a definable set. Using parameters, we can obtain a relative version, encoding algebraic families. In other words, the model theory of the field K can be used to describe classical algebraic geometry over K . Let me explain how

we can also encompass the GROTHENDIECK version. An (affine) scheme X of finite type over K is determined by some polynomials f_1, \dots, f_s over K (so that X_{\max} is the corresponding variety given as the solution set of $f_1 = \dots = f_s = 0$). The tuple of coefficients \mathbf{a} of the f_i (listed in a once and for all fixed order) will be called a *code* for X . Although different tuples can encode isomorphic schemes, I will not discuss this issue here, but see Theorem 2. A point $x \in X$ then corresponds to a prime ideal \mathfrak{p} of the coordinate ring $A(X) = K[\xi]/(f_1, \dots, f_s)$. If $\mathfrak{p} = (g_1, \dots, g_t)$, for some polynomials g_i , then the tuple of their coefficients \mathbf{b} encodes the point x . However, not every tuple \mathbf{b} encodes a point: for this we need that the ideal encoded by \mathbf{b} , is prime. It follows from [26, Theorem 2.10] that the set of all tuples (of fixed length N) encoding a prime ideal in $A(X)$ is a definable subset of K^N . In other words, there exists an (exhaustive) filtration $\Omega_1 \subset \Omega_2 \subset \dots$ of X , such that for each N , we can find a formula φ_N (giving rise to a definable subset $\varphi_N(K)$ of K^N) with the property that every point in Ω_N is encoded by some code \mathbf{b} satisfying φ_N , and conversely, every tuple \mathbf{b} satisfying φ_N encodes a point of Ω_N . I will informally express this by saying that a scheme is a *local first order structure*. More to the point, I will call a subset Σ of X *first order definable* (in the local first order structure X), if for each N , there exists a formula ψ_N , such that the tuples satisfying ψ_N are precisely the codes of points in $\Omega_N \cap \Sigma$. In this terminology, the closed points X_{\max} of X is a first order definable subset.

1.1. Constructible Properties and Invariants. The previous discussion raises the question which naturally arising subsets Σ of a scheme X are first order definable. Moreover, to give geometric meaning, a criteria is needed for a first order definable subset Σ of a scheme X to be constructible (in the Zariski sense), that is to say, to be a finite union of locally closed subsets. One should be careful here: by Quantifier Elimination, a definable subset D of some variety $V \subset K^N$ is a (Zariski) constructible subset. However, if X is the scheme defined by V (so that $V = X_{\max}$), then not every first order definable subset is constructible ($X_{\max} \subset X$ is a trivial example). To provide a context, let $\mathcal{P}(R, M)$ stand for some (algebraic-geometric) property of a pair (R, M) , where R is a local ring (arising as the local ring of a point on a scheme) and M a finitely generated R -module. Let $f: X \rightarrow S$ be a morphism of finite type and \mathcal{F} a coherent \mathcal{O}_X -module. Let $\Sigma_{\mathcal{P}}$ be the set of points $x \in X$ for which $\mathcal{P}(\mathcal{O}_{f;x}, \mathcal{F}_{f;x})$ holds, where $\mathcal{O}_{f;x}$ denotes the local ring at the point x of the fibre $X_{f(x)} = f^{-1}(f(x))$ and where $\mathcal{F}_{f;x}$ is the stalk at x of the pull back $\mathcal{F}|_{X_{f(x)}}$. This notation will be fixed in the sequel of this discussion. To a large extent, this proposal is concerned with the following question.

Problem 1. *For which properties \mathcal{P} is $\Sigma_{\mathcal{P}}$ constructible?*

For instance, if A is a finitely generated K -algebra and M a finitely generated A -module, then $\Sigma_{\mathcal{P}} \subset \text{Spec } A$ is simply the set of all points $\mathfrak{p} \in \text{Spec } A$ for which $\mathcal{P}(A_{\mathfrak{p}}, M_{\mathfrak{p}})$ holds (since the only fibre of $\text{Spec } A \rightarrow \text{Spec } K$ is $\text{Spec } A$ itself). For $\mathcal{P}(R)$ equal to the property that R is regular, this was first studied by Nagata. A systematic study of this question for other properties (such as being Cohen-Macaulay, Gorenstein, Complete Intersection, normal, reduced, irreducible, and many others) was initiated by GROTHENDIECK in [8] and below I will briefly discuss his methods. With notations as in 1, I say that a property \mathcal{P} is *first order definable*, if $\Sigma_{\mathcal{P}}$ is a first order definable subset of X . In particular, by Quantifier Elimination, the set $\Sigma_{\mathcal{P}} \cap X_{\max}$ is constructible in X_{\max} .

To breach the gap between first order definable and constructible, I have introduced the following notion. A property \mathcal{P} is said to be *saturated*, if for each prime ideal \mathfrak{p} of A , there exists a dense set of maximal ideals \mathfrak{m} containing \mathfrak{p} , such that the truth value of $\mathcal{P}(A_{\mathfrak{p}}, M_{\mathfrak{p}})$ is equal to the truth value of $\mathcal{P}(A_{\mathfrak{m}}, M_{\mathfrak{m}})$. In [28], I show that a first order definable and saturated property is constructible (and conversely). In slogan style, this principle reads

Principle 2. *First order definability plus saturatedness equals constructibility.*¹

The principle can be generalized to include the case of an *invariant* ω , that is to say, a map on pairs (R, M) , with values in some fixed set \mathbf{S} . With a *level set* of ω on X , I mean the collection $\Sigma_{\omega=s}$ of all points $x \in X$ for which $\omega(\mathcal{O}_{f;x}, \mathcal{F}_{f;x}) = s$, for some fixed $s \in \mathbf{S}$. I call an invariant *first order definable* if all its level sets $\Sigma_{\omega=s}$ are, and *saturated*, if for each $x \in X$, there is a dense collection of closed points $z \in \overline{\{x\}}$ (where $\overline{\{x\}}$ denotes the Zariski closure of the singleton $\{x\}$), with

$$(1) \quad \omega(\mathcal{O}_{f;x}, \mathcal{F}_{f;x}) = \omega(\mathcal{O}_{f;z}, \mathcal{F}_{f;z}).$$

The principle then also applies to this more general setup. In particular, a saturated first order definable invariant admits only finitely many non-empty level sets. The model theoretic version of Principle 1 can be formulated as follows.

Principle 3. *If a collection \mathcal{E} of objects can be constructed in an algebraic-geometric way from a scheme S (and possibly some other data), then \mathcal{E} is first order definable (in an appropriate local first order structure).*²

1.2. Uniform Bounds. A first step in proving that a property is first order definable, is showing the existence of some uniform bounds. Simply put, if an algebraic-geometric object is described by means of a code \mathbf{a} of length N , and if some algebraic process is applied to it to obtain a new algebraic-geometric object, then we say that this process is *uniformly bounded*, if there is an a priori bound N' on the length of a code for the newly obtained object. For instance, if $f_1T_1 + \dots + f_mT_m = 0$ is a linear equation with coefficients $f_i \in K[\xi]$ and indeterminates (T_1, \dots, T_m) , then the solution set in $K[\xi]^m$ is generated by solutions the degree of which is bounded by a number d only depending on m , the number of variables ξ and the ξ -degrees of the f_i .

To show the existence of uniform bounds, several techniques are available. Historically the first method to be used (by for instance HERMANN in [10] to prove the above mentioned result on linear equations), is classical elimination theory. An alternative is provided by the theory of Groebner basis. The key fact here is that the members of a (minimal) Groebner basis of an ideal in the polynomial ring $K[\xi]$ generated by polynomials of degree at most d , have degree at most d' , where d' only depends on d and the number of variables ξ . The advantage of these methods is that they often yield concrete estimates for the bounds. Recent work of KOLLAR (using local cohomology in [17]) and others have greatly improved the old bounds.

¹For an example of a first order definable property which is not saturated, let X be a singular irreducible curve and \mathcal{P} the property of a local ring to be of positive dimension and regular. Then the generic point of X does not belong to $\Sigma_{\mathcal{P}} \subset X$, since its local ring is the function field of X , which is regular but zero-dimensional. The only closed points lying above the generic point and not in $\Sigma_{\mathcal{P}}$ are the singular points of the curve. Since there are only finitely many, \mathcal{P} is not saturated.

²A partial link between this principle and Principle 1 is provided by (Uniform) Elimination of Imaginaries for algebraically closed fields.

The methods coming from model theory tend to give primitive recursive bounds, with unfortunately no more information. Therefore, these bounds are mainly of theoretical interest. A systematic use of model theory in the guise of non-standard arguments, was introduced by SCHMIDT and VAN DEN DRIES, in their germane paper [26]. This line of thought was continued by the second author in [25], to include also finite A -modules and the structure sheaf of $\text{Spec } A$ and $\text{Proj } A$. In [26], uniform bounds were established for many elementary algebraic operations and concepts. I used these bounds in [30] to investigate which homological properties and invariants (such as Bass and Betti numbers, intersection cycles) are uniformly bounded and, consequently, first order definable. In the last three years, I have also reported on these result at various conferences (Oberwolfach 1998 and 2000, New York Logic 1999, ASL 1999, AMS 1998, MSRI 1998) and seminars (Rutgers, CUNY, Ohio State, Louvain, Wesleyan). In combination with Principle 2, I also obtained in a preprint [28] the constructible nature of many properties and invariants (including those studied in [8]). MACINTYRE invited me to publish a survey paper on all these results in [29].

1.3. Transfer. Another instance in which it is useful to know whether a property is first order definable, is to transfer results from positive characteristic to zero characteristic, or vice versa. This follows from the fact that there is a (non-canonical) isomorphism between the field of complex numbers and the ultraproduct (with respect to a non-principal ultrafilter) of the algebraic closures $\mathbb{F}_p^{\text{alg}}$ of the p -element fields \mathbb{F}_p . I will refer to this kind of transfer as the Lefschetz Principle. Combining this principle with the previous remarks, yields the following principle.

Principle 4 (Lefschetz Principle).

$$\text{Uniform bounds} \implies \text{first order definability} \implies \text{transfer}.$$

2. CLASSICAL METHODS

Before giving the details of the proposal, I want to compare the present method with the classical methods from algebraic geometry.

2.1. Constructible Properties and Invariants. I will mainly refer to GROTHENDIECK's approach to the constructible nature of algebraic or geometric properties, amply described in [8, Chapitre IV §9]. To answer Problem 1, he reduces the general case to the case that X and S are schemes of finite type over \mathbb{Z} . This reduction is made possible by the fact that the properties under consideration commute with direct limits and admit some form of faithfully flat descent. The main objective of this reduction, is to reduce the question to affine Noetherian schemes. Therefore, we may as well assume that the scheme S is finitely generated over some field K , which I also will assume to be algebraically closed. Under these assumptions and with notation as before, GROTHENDIECK then argues that in order to prove that Σ is constructible, it suffices to show the following principle.

Principle 5. For each $x \in E$, where \overline{E} is either Σ or its complement, we can find some open U containing x , such that $\overline{\{x\}} \cap U \subset E$.

This last condition is very reminiscent of the notion of saturatedness given above, but in theory it means that we need to prove a stronger result. Indeed, if $x \in E$, then instead of showing that $E \cap \overline{\{x\}}$ contains a dense subset of closed points, we

need to show that x is an interior point of $E \cap \overline{\{x\}}$ in $\overline{\{x\}}$. Another advantage of saturatedness is the following principle (its proof is by a simple inductive argument).

Principle 6 (Devissage Principle). *For an invariant ω to be saturated, it suffices that for each h and for each $x \in X$ of codimension h (that is to say, $\mathcal{O}_{X,x}$ is h -dimensional), the set of points $z \in \overline{\{x\}}$ of codimension $h + 1$ for which (1) holds, is dense in $\overline{\{x\}}$.*

The corresponding devissage for Principle 5 would give a weaker criteria than 5 which is insufficient to conclude constructibility. The advantage of the Devissage Principle, apart for allowing inductive arguments, is that we often can reduce the problem to the case that R is discrete valuation ring, by restricting to $\{x\}$ and using that its regular locus is open.

Let me give an example, for which I do not know how to apply GROTHENDIECK's technique.³ The *Poincaré series invariant* $\zeta(R, M)$ is defined as the power series in T for which the n -th coefficient equals the n -th Betti number $\dim \operatorname{Tor}_n^R(M, \kappa)$ of M (where κ is the residue field of R). In order to better study the constructible nature of this invariant, it is convenient to normalize this series by dividing it by $(1 + T)^h$, where h is the dimension of M . It was conjectured for a while that this series would always be rational, but ANICK et al., gave counterexamples, even in the case that M is just the residue field of R . Therefore, one might not hope in general for this invariant to be first order definable. However, it is first order *type* definable, that is to say, definable by an infinite conjunction of formulae. What can be said in this situation about the locus $\Sigma_{\zeta=P}$ of all points x on a scheme X for which the residue field $\kappa(x)$ has prescribed Poincaré series $P(T) \in \mathbb{Z}[[T]]$? By an argument similar to the one in the first order definable case, $X_{\zeta=P} \cap X_{\max}$ is *pro-constructible* (=an arbitrary intersection of constructible sets) in X_{\max} . I show, using the Devissage Principle, that ζ is saturated and, consequently, the level sets $\Sigma_{\zeta=P}$ are pro-constructible. The following Corollary is a purely geometric consequence of this.

Corollary 1. *If the Poincaré series of each closed point on a scheme X is rational, then the Poincaré series of an arbitrary point on X is rational.*

In fact, it is reasonable to ask the following question.

Problem 2. *In the situation of the Corollary, does it follow that the level sets are constructible (so that there are only finitely many)? More generally, if P is a rational power series, does it follow that the level set $\Sigma_{\zeta=P}$ is constructible?*

2.2. Transfer. Using positive characteristic methods (such as tight closure, see 3.3 below) in order to prove results in characteristic zero, was an idea, first successfully introduced into commutative algebra in the dissertation [21] of PESKINE and SZPIRO, and later by HOCHSTER, to settle several cohomological conjectures. The most explicit form is perhaps HOCHSTER's Finiteness Theorem in [11] (see also the more model theoretic approach by VAN DEN DRIES in [36]). However, these latter applications all use the following transfer principle, which is weaker than the Lefschetz Principle.

³In [8, Théorème 9.2.2], he gives as criteria for Σ to be pro-constructible, that whenever $x \notin \Sigma$, then x has to be interior in $\overline{\{x\}} - \Sigma$. I was not able to apply this to the present problem.

Principle 7 (Reduction Principle). *A polynomial system of equations which has a solution over a finitely generated \mathbb{Z} -algebra, has also a solution over a finitely generated algebra over a finite field, simply by reduction.*

This works perfectly well if the condition (or its negation) in question is Zariski closed. In contrast, the Lefschetz Principle allows also inequations, at the price⁴ that one has to work over an algebraically closed field. Therefore the model theoretic approach has potentially more power than the HOCHSTER approach, as many algebraic-geometric conditions are constructible rather than Zariski closed (or open). In fact, due to Quantifier Elimination, quantifiers are allowed in the formulae ψ_N on the codes \mathbf{a} : these conditions become quantifier free over an algebraically closed field.

Using first order definability, I succeeded in [30, Theorem 5.6] to replace the transfer argument in the work of PESKINE and SZPIRO (that is to say, HOCHSTER's Finiteness Result based on Artin Approximation and the Reduction Principle), to derive the Bass Conjecture⁵ in characteristic zero from its validity in positive characteristic (where it can be proved using a tight closure argument, that is to say, by controlling the behavior of the action of Frobenius on local cohomology). Of course, this gives no new result, and in fact, only solves the Conjecture for local rings essentially of finite type over an algebraically closed field. Nonetheless, the model theoretic method is, in principal, more versatile, although it is not yet clear whether the transfer part from positive to zero characteristic, is any stronger than HUNEKE's and HOCHSTER's tight closure theory in characteristic zero. I will return to this issue below in 3.3. However, tight closure theory cannot say anything about transfer from zero to positive characteristic, and the following result is an example in which the model theoretic method is successfully used and for which I do not know an alternative proof. The Zariski-Lipman Conjecture for a local ring R over a field K states that if the module of K -invariant derivations $\text{Der}_K(R)$ is free, then R is regular. LIPMAN has shown this with 'regular' replaced by 'normal', provided K has characteristic zero. In fact, he gives counterexamples to both the Conjecture and his result in characteristic $p > 0$, but the degree of these counterexamples grows with p . In [24], SCHEJA and STORCH show the validity of the Conjecture for R the local ring of a hypersurface (and K of characteristic zero). In [30, Corollary 6.4], I showed that the property \mathcal{P}_{Lip} is first order definable, where $\mathcal{P}_{\text{Lip}}(R)$ holds if, and only if, the Zariski-Lipman Conjecture is true for R . As a corollary of this together with the Lefschetz Principle, I obtain the following positive characteristic form of the Zariski-Lipman Conjecture (a similar statement can be derived from LIPMAN's original normality result).

Corollary 2 (Low Degree Zariski-Lipman Conjecture ([30, Corollary 6.6])). *For each d , there is a bound d' , such that if R is the local ring of a point on a hypersurface given by $f = 0$, with f a polynomial of degree at most d , and if the characteristic of K is greater than d' , then the Zariski-Lipman Conjecture holds for R .*

⁴Often this price is not too high, as one can obtain results for arbitrary fields from the corresponding results for algebraically closed fields using faithfully flat descent. See for instance [28, §7].

⁵The Bass Conjecture states that a Noetherian local ring admitting a non-zero finitely generated module of finite injective dimension is Cohen-Macaulay.

In fact, by reversing the transfer, we obtain that if one succeeds to prove the corollary for a larger first order definable class of local rings, then the Zariski-Lipman Conjecture holds for all local rings of characteristic zero in this class. It should be noted, that the characteristic zero proofs of LIPMAN and of SCHEJA and STORCH cannot be transferred to positive characteristic, as they are analytical in nature.

3. RESEARCH PROPOSAL

3.1. Artin Approximation. Let \mathfrak{R} denote the ring of algebraic power series in the variables ξ over K . As \mathfrak{R} is the Henselization of $K[\xi]$ at the maximal ideal (ξ_1, \dots, ξ_n) , every element $\omega_1 \in \mathfrak{R}$ can be enlarged to a tuple $\omega = (\omega_1, \dots, \omega_N)$ over \mathfrak{R} which satisfies an *etale system of equations* $(F_1, \dots, F_N) = 0$ in the variables (X_1, \dots, X_N) , with $F_i \in K[\xi, X]$. With this we mean that

$$(2) \quad F_1(\omega) = \dots = F_N(\omega) = 0$$

and the Jacobian matrix

$$(3) \quad \begin{pmatrix} \partial F_1 / \partial X_1 & \partial F_1 / \partial X_2 & \dots & \partial F_1 / \partial X_N \\ \partial F_2 / \partial X_1 & \partial F_2 / \partial X_2 & \dots & \partial F_2 / \partial X_N \\ \vdots & \vdots & \ddots & \vdots \\ \partial F_N / \partial X_1 & \partial F_N / \partial X_2 & \dots & \partial F_N / \partial X_N \end{pmatrix}$$

is invertible at ω . Moreover, it follows from Taylor expansion that the solution ω is uniquely determined by the initial condition $\omega(0) = \mathbf{u}$. Let us say that the *etale degree* of ω_1 is at most d , if we can choose the F_i of total degree at most d with also $N \leq d$. It follows that \mathfrak{R} is a local first order structure, by taking as code for an element, the tuple of coefficients of an etale system of equations F_i together with the entries of the initial condition \mathbf{u} . We then have the following uniform version of Artin Approximation, the proof of which is almost identical to the uniform version of Strong Artin Approximation in [1] (using an ultraproduct construction and ARTIN's theorem; see also below).

Theorem 1 ([27, Theorem 2.6]). *For each $d \in \mathbb{N}$, there exists a bound d' with the following property. Let $f_i \in \mathfrak{R}[X_1, \dots, X_N]$ of X -degree at most d and each coefficient of etale degree at most d . If the system of equations $f_1(X) = \dots = f_s(X) = 0$ in the variables X has a solution in $K[[\xi]]$, then there exists a solution tuple ω in \mathfrak{R} with each entry of etale degree at most d' .*

It should be pointed out that the above statement is false if we replace \mathfrak{R} and $K[[\xi]]$ by $K[\xi]$ and replace etale degree by polynomial degree. This follows from an observation made by SCHMIDT and VAN DEN DRIES in their paper [26]. In other words, solvability of a system of equations over $K[\xi]$ is not uniformly bounded, but it is so over \mathfrak{R} .⁶ I do not know whether the above uniformity still holds when we add some inequations $g \neq 0$, but this seems unlikely.⁷

Let me indicate two examples how Theorem 1 can be put to use.

⁶In [27], I describe an algorithm to calculate such a solution.

⁷In fact, the main difficulty in proving the decidability of the existential theory of $\mathbb{F}_p[[t]]$ in [5] lies in the lack of uniform bounds in case there are inequalities. To repair this we need to give an argument relying on (the still conjectural) Resolution of Singularities in positive characteristic.

Theorem 2 ([27, Theorem 5.6]). *The set of pairs (A, B) of finitely generated K -algebras which are isomorphic locally in the étale topology (that is to say, after a finite étale covering—see also Problem 7) is first order definable.*

This theorem gives a partial answer to a question raised by EKLOF in [7]. The next theorem shows that the corresponding statement for finitely generated modules over local rings is even valid in the Zariski topology, by using some form of étale descent.

Theorem 3 ([29, Theorem 10.1]). *Let R be the local ring of some point on a scheme of finite type over K . The isomorphism relation for finitely generated R -modules is first order definable. In other words, for each d , we can find a d' , such that if two finitely generated R -modules M and N can be described by means of polynomials of degree at most d and are isomorphic, then there exists an isomorphism given by polynomials of degree at most d' .*

Note that the statement is false for non-local rings, for then being free would be first order definable, by taking for one of the modules a free module. This, however, is shown to be false in general in [25, Remark 1.10 (b)]. From Theorem 2 it follows that for finite K -algebras, the (ring theoretic) isomorphism relation is first order definable. This raises the following question.⁸

Problem 3. *Given a finite morphism $Y \rightarrow X$ of schemes of finite type over K . Are there only finitely many different isomorphism types among its fibers (up to an appropriate base change over an algebraically closed extension field containing all residue fields of X)⁹*

To answer the problem in the affirmative, it suffices to prove that the isomorphism invariant on finite K -algebras is constructible (it is first order definable by the previous argument, so it remains to verify that it is saturated using the Devissage Principle).

Surrounding Artin Approximation, it is an interesting question to understand the uniformity of it in greater detail. I already mentioned the uniform version of Strong Artin Approximation from [1], stating that for each d , there exists a d' , such that whenever we have a system of polynomial equations $f_1(\xi, X) = \dots = f_s(\xi, X) = 0$ of total degree at most d and a tuple of polynomials $\mathbf{p}(\xi)$ with $f_i(\xi, \mathbf{p}(\xi)) \equiv 0 \pmod{\mathfrak{M}^{d'}}$, where $\mathfrak{M} = (\xi_1, \dots, \xi_n)\mathfrak{R}$ is the maximal ideal of \mathfrak{R} , then we have a tuple $\boldsymbol{\omega}$ over \mathfrak{R} , such that $f_1(\xi, \boldsymbol{\omega}) = \dots = f_s(\xi, \boldsymbol{\omega}) = 0$. In the joint paper [5] with DENEFF, the following relative version of this result is shown.

Theorem 4 ([5, Theorem 3.1]). *For each d , there exists a bound d' with the following property. Let \mathbf{u} be some n -tuple from \mathfrak{R} . Let $f_1(\xi, X), \dots, f_s(\xi, X) \in K[\xi, X]$ be of total degree at most d . Suppose there is some tuple of polynomials $\mathbf{p}(\xi)$ for which all $f_i(\mathbf{u}, \mathbf{p}(\xi)) \equiv 0 \pmod{\mathfrak{M}^{d'}}$. Then there exists a tuple $\boldsymbol{\omega}$ over \mathfrak{R} , such that $f_1(\mathbf{u}, \boldsymbol{\omega}) = \dots = f_s(\mathbf{u}, \boldsymbol{\omega}) = 0$.*

⁸I have asked already several experts in algebraic geometry, but so far no one could provide me with a conclusive answer.

⁹For instance, if X is the affine line with parameter t and Y is the planar curve with equation $\xi^2 - t$, then the fibers have affine coordinate ring $K[\xi]/(\xi^2 - a)$, $K[\xi]/(\xi^2)$ and $K(t)[\xi]/(\xi^2 - t)$ corresponding to the closed points $a \neq 0$, $a = 0$ and to the generic point of X respectively. If L is an algebraically closed field containing $K(t)$, then the base change of each of these fibers over the respective residue fields, are all isomorphic to L^2 , except for the middle one corresponding to $a = 0$, which is $L[\xi]/(\xi^2)$.

This is proved using the general solution to the Artin Conjecture by [35] or [22]. A weaker version, requiring substantially more work but only using ARTIN's original result was proven in [4]. One would like to prove a similar theorem in mixed characteristic. However, we cannot expect Theorem 4 to hold in the above form.¹⁰ A similar problem arises when we deal with arbitrary excellent Henselian local rings (for which Artin Approximation is now known to hold), so that the very formulation of uniform Strong Artin Approximation is still problematic. In a future collaboration with DENEFF, we intend to return to this problem, which I here will state in the following vague manner.

Problem 4. *Let R be an excellent Henselian local ring. Find some complexity notion on an equation system $f_1(X) = \dots = f_s(X) = 0$ over $R[X]$ for it to satisfy Strong Artin Approximation (that is to say, in the statement of Theorem 4, the bound d' is only to depend on the complexity, not on the particular system of equations).*

In equicharacteristic, we may reduce the problem, using Artin Approximation and COHEN's structure theorem for equicharacteristic complete regular local rings, to the case that R is \mathfrak{R} or $K[[\xi]]$. In mixed characteristic, apart from problems caused by ramification, we can reduce at best to the case that R is $\mathcal{O}[[\xi]]$, where \mathcal{O} is a complete discrete valuation ring. Moreover, in this reduction, we have to keep track of the parameters as well.

3.2. Set-Theoretic Complete Intersections. While I was a visitor at the MSRI three years ago, I got interested in the following outstanding open problem.

Conjecture 1. *Is every curve in affine 3-space over \mathbb{C} a set-theoretic complete intersection? In other words, can we find for every prime ideal \mathfrak{p} in $\mathbb{C}[X, Y, Z]$ defining a curve, two polynomials f, g , such that \mathfrak{p} is generated by f and g up to radical (that is to say, such that \mathfrak{p} is equal to the radical of the ideal (f, g))?*

It was already observed by VAN DEN DRIES in [38] that, if there would be a bound on the degree of such polynomials f, g depending only on the degree of the polynomials generating \mathfrak{p} , then the Conjecture would hold. Indeed, by Principle 4, this would follow from the fact that the Conjecture holds in positive characteristic by [3].

An easier question would be to ask whether the minimal number of generators of an ideal \mathfrak{a} in a finitely generated K -algebra A is first order definable. However, in this generality, the answer is negative already in dimension 1, as an example [25, Remark 1.10 (a)] of SCHMIDT shows. Namely, take for A the coordinate ring of some elliptic curve. There exists d , such that for each d' , we can find two (images of) polynomials $f, g \in A$ of degree at most d , such that the ideal (f, g) is principal, but every generator of this ideal must have degree at least d' . On the other hand, if A is local, then the answer is yes, by an easy application of Nakayama's Lemma. To link the local with the global case, one can use the FORSTER-SWAN theorem [20, Theorem 5.8]. Using this together with some non-standard argument, I was able to prove that the answer is affirmative for $A = K[\xi]$ and \mathfrak{a} the prime ideal of a curve. Hence it makes sense to ask the previous question for certain special classes

¹⁰If d' would be the bound for linear equations, then the equation $\pi^{d'}(\pi X - 1)$, with π some non-zero element in \mathfrak{M} , has trivially a solution modulo $\mathfrak{M}^{d'}$, but no solution in R . So some control on the coefficients is required.

of ideals. The failure in the case of an elliptic curve is also exhibited by the fact that to be a free module is not first order definable.¹¹ By the solution of Serre's Conjecture, to be free is equivalent with being projective for A a polynomial ring over a field. Hence it still makes sense to ask the previous question in this special case. In summary, the following general problem could be posed (including the original one for solving the Conjecture).

Problem 5. *For which finitely generated K -algebras A and for which classes of ideals, is the minimal number of generators first order definable? When is it constructible?*

Same question for the minimal number of generators up to radical.

In fact, if the second question has a negative solution for prime ideals in $\mathbb{C}[\xi_1, \xi_2, \xi_3]$ defining curves (which is a first order definable collection), then one would be able to construct a counterexample to Conjecture 1 by an ultraproduct argument. A good candidate for such a counterexample might be the curve in projective space (where the problem of set-theoretic complete intersections is harder) given by the parametrization

$$(4) \quad \xi_0 = tu^3, \quad \xi_1 = t^3u, \quad \xi_2 = t^4, \quad \xi_3 = u^4.$$

HARTSHORNE has shown in [9] that it is a set-theoretic complete intersection in each positive characteristic p , but it is not known whether this is a set-theoretic complete intersection over \mathbb{C} .¹²

To discuss the next problem, I need to introduce some more terminology. Let \mathfrak{a} be an ideal in the ring A . We say that an ideal I is an *embedded infinitesimal extension* of \mathfrak{a} , if we have inclusions $\mathfrak{a}^t \subset I \subset \mathfrak{a}$, for some $t \in \mathbb{N}$. We call t the *exponent* of this extension. This geometric terminology comes from the fact that this data gives rise to an inclusion of closed subschemes $\text{Spec } A/\mathfrak{a} \subset \text{Spec } A/I$ of $\text{Spec } A$, such that the underlying sets of these two closed subschemes are the same. In [2], BORATYŃSKI asks—in the graded case, but that does not matter—the question when such an embedded infinitesimal extension exists with the extra property that A/I be Gorenstein. Note that since, by a theorem of SERRE, Gorenstein in codimension two is the same as complete intersection, the question for affine 3-space becomes then simply when is A/\mathfrak{a} a set-theoretic complete intersection. In this light, I want to study the following problem.

Problem 6. *Given a map $f: Y \rightarrow X$ of finite type between schemes over K . Fix some $t \in \mathbb{N}$. For which properties \mathcal{P} (Cohen-Macaulay, Gorenstein, complete intersection, ...) is the set X_t a constructible set, where X_t is the subset of all points x on X , for which the fiber $f^{-1}(x)$ admits an embedded infinitesimal extension of exponent t which has moreover property \mathcal{P} ? Are all but finitely many of the X_t empty?*

Note that if both questions have a positive answer, then even the collection of all points for which no embedded infinitesimal extension of type \mathcal{P} exists, is

¹¹In contrast, the property for a finitely generated module to be projective is first order definable. This is proven in [25, Theorem 1.2] (and independently by myself in [30, Proposition 6.3], where in fact it is proven that the property of having a prescribed projective dimension is first order definable). Observe that a projective module is free if, and only if, its rank (a first order definable invariant) equals its minimal number of generators.

¹²Since the degree of the two hypersurfaces cutting out this curve grow with p , the ultraproduct argument is not available.

also constructible. This would entail the first order definability of the notion of set-theoretic complete intersection and thus would solve Conjecture 1.¹³

There are some indications that Conjecture 1 might hold at least locally in the étale topology, in the form below. This would be a consequence of a positive solution to Problem 3 and an application of Theorem 4, using the Devissage Principle.

Problem 7. *Is every curve C in affine 3-space over \mathbb{C} locally isomorphic in the étale topology¹⁴ to a set-theoretic complete intersection? That is to say, do there exist a set-theoretic complete intersection C' in $\mathbb{A}_{\mathbb{C}}^3$, a curve D and surjective étale maps $D \rightarrow C$ and $D \rightarrow C'$?*

As a corollary one would get a positive answer to Conjecture 1, provided we replace $K[S]$ by $K[[S]]$; even this local version of the Conjecture is not known, as HARTSHORNE has pointed out to me.

3.3. Non-standard Tight Closure. Inspired by the work [18] of KUNZ on regularity in terms of the Frobenius and the work of PESKINE and SZPIRO in [21] on cohomological conjectures using Frobenius, HOCHSTER and HUNEKE introduced, in the late 80's, the notion of *tight closure*. In the mean time, this has become a flourishing field, with surprising strength and scope (among its corollaries, are the Strong Kodaira-Vanishing Theorem, the Direct Summand Theorem, the New Intersection Theorem, Fujita's Freeness Theorem and Uniform Artin-Rees). The key idea is, in positive characteristic, to apply iterates of the Frobenius, and in zero characteristic, using Artin Approximation as in HOCHSTER's Finiteness Theorem, to reduce to positive characteristic. Let me give some more details. In the sequel, for the sake of exposition, A will be a domain, finitely generated over K and $\mathfrak{a} = (f_1, \dots, f_s)$ an arbitrary ideal. Assume first that K has characteristic $p > 0$. An element $x \in A$ lies in the *tight closure* of \mathfrak{a} , if there exists some non-zero element $c \in A$, such that for all large $q = p^e$, we have that

$$(5) \quad cx^q \in (f_1^q, \dots, f_s^q).$$

The tight closure of I is again an ideal, denoted by I^* . We call I *tightly closed*, if $I = I^*$. To understand the definition of tight closure, let A^{1/p^∞} denote the overring of A consisting of all p^n -th roots of elements of A . In this ring, equation (5) becomes

$$(6) \quad c^{1/q}x \in \mathfrak{a}A^{1/p^\infty}.$$

One might say that as q grows, then $c^{1/q}$ converges to 1, so that 'in the limit' x belongs to \mathfrak{a} in this overring. However, it is still an open question whether $\mathfrak{a}^* = \mathfrak{a}\tilde{A} \cap A$, where \tilde{A} is the integral closure of A in the algebraic closure of its fraction field (so that in particular, $A^{1/p^\infty} \subset \tilde{A}$). For parameter ideals this is shown by SMITH in [31].

Tight closure is a closure operation on ideals satisfying the following five 'axioms' (see for instance [34] or [16]).

Theorem 5 (Tight Closure Properties). I. *If A is regular, then all ideals are tightly closed.*

II. *If $A \hookrightarrow B$ is an integral extension, then $\mathfrak{a}B \cap A \subset \mathfrak{a}^*$.*

¹³One also has to use the result [37] of SZPIRO that any curve in affine 3-space which is locally a complete intersection, is a set-theoretic complete intersection.

¹⁴This is a rather weak notion of isomorphism; it follows from COHEN's Structure Theorem for local rings that any two smooth curves are locally isomorphic in the étale topology.

- III. If A is local with maximal ideal \mathfrak{m} minimally generated by (x_1, \dots, x_d) , then $(x_1, \dots, x_i) : x_{i+1}$ is contained in $(x_1, \dots, x_i)^*$.
- IV. If \mathfrak{a} is generated by d elements, then we have the following inclusions of ideals

$$(7) \quad \widetilde{(\mathfrak{a}^d)} \subset \mathfrak{a}^* \subset \tilde{\mathfrak{a}} \subset \text{rad } \mathfrak{a},$$

where for any ideal I , we denote its integral closure by \tilde{I} .

- V. If $A \rightarrow B$ is a ring homomorphism, then \mathfrak{a}^*B is contained in the tight closure of $\mathfrak{a}B$.

Just from these five properties alone, many interesting results can be proven. To just give two, *the ring of invariants of a linearly reductive group acting linearly on a regular ring is Cohen-Macaulay* (this was first proved by different means by HOCHSTER-ROBERTS) and *the ring \tilde{A} is a (big) Cohen-Macaulay A -module*. As these are formal consequences of Theorem 5, any closure operation admitting these five properties would yield similar results. Therefore, as tight closure makes only sense in positive characteristic, one wants a closure operation in characteristic zero having the five properties of Theorem 5. This was worked out, under the name of tight closure in characteristic zero, by HOCHSTER and HUNEKE, using Principle 7 and Artin Approximation (or, rather a general Neron desingularization due to ARTIN and ROTHAUSS; see for instance the appendix of [16]). However, apart from having a much more involved definition than the neat formulation (5), some non-canonical choices have to be made that yield a priori different closure operations (see for instance [14]). A substantial part in this reduction process is to show, given a map $X \rightarrow \text{Spec } A$, with A a finitely generated domain over \mathbb{Z} and given a certain property \mathcal{P} , that if the generic fiber satisfies \mathcal{P} , then a dense collection of closed fibers also satisfy \mathcal{P} . However, such a result will be an immediate consequence of the saturatedness of \mathcal{P} whence could alternatively be established via the Devissage Principle 6.

One cannot hope to apply the Lefschetz Principle (4) in this situation, for, most likely, tight closure will not be first order definable, in view of the infinitely many conditions in (5). Nonetheless, tight closure is uniformly bounded, since by (IV), tight closure is contained in the radical, and the latter is uniformly bounded, and even first order definable. Therefore, a more direct approach has to be taken, in order to apply non-standard arguments. Let Frob_p denote the Frobenius homomorphism $x \mapsto x^p$ in characteristic $p > 0$. Let A_p be some finitely generated domain over some field K_p of characteristic p , so that Frob_p acts on A_p . Then the ultraproduct Frob_∞ of the Frob_p acts on the ultraproduct A_∞ of these domains.¹⁵ Unfortunately, the ring A_∞ is no longer finitely generated over the ultraproduct K_∞ of the K_p (note that K_∞ is now a field of characteristic zero). Nonetheless, if we bound the length of the respective codes \mathfrak{a}_p of the algebras A_p , then the ultraproduct \mathfrak{a}_∞ of these codes defines a finitely generated K_∞ -algebra A inside A_∞ . By [26, Theorem 1.1], the extension $A \subset A_\infty$ is faithfully flat, indicating some plausible descent among these two rings. Notwithstanding that Frob_∞ is no longer an endomorphism of this subring A , I propose the following definition of non-standard tight closure on A (note that any finitely generated K_∞ -algebra can be obtained by the previous process). An element $x \in A$ lies in the *non-standard*

¹⁵It follows from the twisted Lang-Weil Estimates by HRUSHOVSKI [15] or MACINTYRE [19] that Frob_∞ is a *generic* automorphism of the ultraproduct K_∞ .

tight closure of the ideal $\mathfrak{a} = (f_1, \dots, f_s)$, if there exists a non-zero $c \in A$, such that for all sufficiently big n , we have that

$$(8) \quad c \cdot \text{Frob}_\infty^n(x) \in (\text{Frob}_\infty^n(f_1), \dots, \text{Frob}_\infty^n(f_s))A_\infty.$$

Given this definition, one can pose the following two questions.

Problem 8. *Does non-standard tight closure satisfy the five properties of tight closure in Theorem 5?*

Problem 9. *How does non-standard tight closure relate to any of the tight closure operations in characteristic zero of HOCHSTER and HUNEKE?*

In a preliminary study, I was able to partially verify the validity of the first problem and I obtained strong indications that it holds without any restriction. For instance, using KUNZ's Theorem, I can prove the first property in Theorem 5, so that non-standard tight closure is contained in *regular* closure. An answer to the second problem will shed light on the interdependence of the various tight closures in characteristic zero. HOCHSTER has strongly encouraged me to pursue these issues, as this might lead to a more transparent presentation of the subject.

However, there is another motivation for considering this line of thought. We call A *F-regular* (any characteristic), if every ideal is tightly closed. If this only holds for parameter ideals (an ideal is a parameter ideal if it can be generated by as many elements as its height), then we call A *F-rational*. If A has characteristic zero, and if $A_0 \subset A$ is finitely generated over \mathbb{Z} so that A_0 generates A over K , then we say that A has *F-rational type* (respectively, *F-regular type*), if there is a dense set of primes p , such that A_0/pA_0 is F-rational (respectively F-regular); this is independent from a sufficiently general choice of A_0 . It is shown by SMITH in [32, Theorem 4.3] that being of F-rational type is equivalent with having rational singularities. It is an open question whether F-rational is the same as F-rational type.¹⁶ In this light, I propose to study the following problem.

Problem 10. *Suppose A has characteristic zero. If every parameter ideal (respectively, every ideal) of A is equal to its non-standard tight closure, does A have F-rational type (respectively, F-regular type)?*

Another topic in tight closure theory which is not yet well understood is the *Hilbert-Kunz function* $\text{HK}_\mathfrak{a}$ of a primary ideal $\mathfrak{a} = (f_1, \dots, f_s)$ in a d -dimensional local K -algebra A , where $\text{HK}_\mathfrak{a}(e)$ is defined as the K -vector space dimension of $A/(\text{Frob}_p^e(f_1), \dots, \text{Frob}_p^e(f_s))$. MONSKY proved that the limit of the quotient $\text{HK}_\mathfrak{a}(e)/p^{de}$, for e going to ∞ , exists. This limit value is called (in analogy with REES's characterization of multiplicity through Hilbert polynomials) the *Hilbert-Kunz multiplicity* of \mathfrak{a} . For instance, if \mathfrak{a} is the maximal ideal in

$$(9) \quad \mathbb{F}_5[[X_1, \dots, X_4]]/(X_1^2 + \dots + X_4^2)$$

then $\text{HK}_\mathfrak{a}(e)$ is equal to $(168/61)5^{3e} - (107/61)3^e$, so that its Hilbert-Kunz multiplicity is $168/61$. It is a mystery where these coefficients come from, and how this fits in a more general pattern. It is also not known whether the Hilbert-Kunz multiplicity is always rational. It is bounded by the usual multiplicity of the ideal \mathfrak{a} , and therefore it is a uniformly bounded invariant. This raises the following problem.

¹⁶In a private conversation with SMITH, she told me that a positive solution of this result for F-regular instead of F-rational would give more vanishing results on rings of invariants of group actions on Fano varieties; see [33].

Problem 11. *Is the Hilbert-Kunz multiplicity a first order definable or a first order type definable invariant? Is it saturated, pro-constructible or constructible?*

The connection with tight closure is simply that \mathfrak{a}^* is the largest ideal containing \mathfrak{a} with the same Hilbert-Kunz multiplicity.

One of the great challenges for tight closure theory is to find a generalization to mixed characteristic (for which most cohomological conjectures remain open). A first attempt by HOCHSTER is given by solid closure. We call an A -algebra C *solid*, if there exists a non-zero A -module homomorphism $h: C \rightarrow A$. We say that $x \in A$ lies in the *solid closure* of \mathfrak{a} , if there exists a solid A -algebra C , for which $x \in \mathfrak{a}C$. Applying the non-zero homomorphism h to equation (5), one shows that every element in the solid closure lies in the tight closure of \mathfrak{a} , whenever A has positive characteristic. However, ROBERTS gave an example in [23] showing that solid closure is bigger than tight closure in characteristic zero. Nonetheless, even in mixed characteristic (when there is no field present), the definition of solid closure makes sense and the solid closure of an ideal is always contained in its integral closure (whence in its radical). A first open problem (in dimension at least 3) is whether every ideal in a mixed characteristic regular local ring is equal to its own solid closure; this would imply the Direct Summand Conjecture in mixed characteristic. In fact, a negative answer to the following open question would essentially prove this in the unramified case for dimension 3.

Problem 12. *Does $p^2\xi_1^2\xi_2^2$ lie in the solid closure of (p^3, ξ_1^3, ξ_2^3) in $\mathbb{Z}_p[[\xi_1, \xi_2]]$?*

If in the above one replaces the ring of p -adic integers by the discrete valuation rings $\mathbb{F}_p[[t]]$ and p by a single variable t , then, in view of Property I of Theorem 5, the answer is negative, since then tight closure and solid closure coincide. This also suggest that perhaps the following model theoretic Transfer Principle might shed some light on the situation.

Principle 8 (Ax-Kochen-Ershov Principle). *The ultraproduct \mathbb{Z}_∞ of all \mathbb{Z}_p is isomorphic to the ultraproduct of all $\mathbb{F}_p[[t]]$.*

Moreover, the ultraproducts of all $\mathbb{Z}_p[[\xi]]$ and of all $\mathbb{F}_p[[t, \xi]]$ (with ξ a finite set of variables), have isomorphic Haussdorffifications (the quotient modulo the intersection of all powers of the maximal ideal), to know $\mathbb{Z}_\infty[[\xi]]$.¹⁷

One can rephrase solid closure in terms of power series ([12]). In the case of interest, the criteria is as follows. Consider the power series

$$(10) \quad f = p^2\xi_1^2\xi_2^2X_0X_1X_2 - p^3X_1X_2 - \xi_1^3X_0X_2 - \xi_2^3X_0X_1$$

in the variables X_0, X_1, X_2 . If one can show that no non-zero multiple cf is special (a power series is *special* in the X -variables, if it none of its monomials is divisible by $X_0X_1X_2$), then the answer to Problem 12 is negative. Using the Ax-Kochen-Ershov Principle, I succeeded in proving that for each d , there are only finitely many prime characteristics p , for which there is some $c \notin \mathfrak{M}^d$ with cf special. In how far can this be improved, that is to say, how can we better control possible multipliers c ?

¹⁷In fact, in a certain sense, $\mathbb{Z}_\infty[[\xi]]$ is the subring of *standard* elements of these ultraproducts.

4. CONCLUDING REMARKS

I will briefly discuss the benefits of giving a systematic treatment of algebraic geometry and commutative algebra along the lines of this proposal. For instance, the proof of the New Intersection Theorem and the Bass Conjecture, for an algebra A of finite type over an algebraically closed field of characteristic zero, follows immediately from their positive characteristic counterparts by first order definability and the Lefschetz Principle. Here is another such example. In [6], EIN, LAZARSFELD and SMITH prove the following result for A as above. Suppose A is regular and let us denote by $h = h_{\text{Ass}}(\mathfrak{a})$ the maximal height of an associated prime of an ideal \mathfrak{a} . Then for all n , and for all radical ideals \mathfrak{a} , we have that

$$(11) \quad \mathfrak{a}^{(hn)} \subset \mathfrak{a}^n$$

where in general $I^{(m)}$ is the m -th *symbolic power* of an ideal I , that is to say, the contraction in A of the expansion of I^m to the localization of A with respect to all non-zero divisors modulo I . In [13], HOCHSTER and HUNEKE generalize this to positive characteristic as well and \mathfrak{a} an arbitrary ideal. In fact, they give a tight closure proof of this (which is extremely easy if \mathfrak{a} is assumed to be radical), so that the result in characteristic zero follows from the result in positive characteristic by means of their reduction technique. However, using the first order definability results from [30], this lifting to characteristic zero (thus reproving the original result from [6]), can be done in a few lines. Indeed, in that paper I showed that to be regular is first order definable, that the associated primes are first order definable (and their number is uniformly bounded) and that height is first order definable, and consequently, the function h_{Ass} . Therefore the statement $\theta_n(\mathfrak{a}, \mathfrak{i})$ on a pair of codes $(\mathfrak{a}, \mathfrak{i})$ expressing that, if the algebra A encoded by \mathfrak{a} is regular, then the ideal \mathfrak{a} encoded by \mathfrak{i} with h_{Ass} -value h satisfies equation (11) for n , is first order. Since these formulae θ_n are true in algebraically closed fields of positive characteristic they also hold in characteristic zero, as required.

In conclusion, in continuing the systematic study of first order definability of properties from commutative algebra and algebraic geometry, I will develop a unified and simplified treatment of those results obtained by transfer, tight closure, constructibility, and other methods described here. This would lead also to a better presentation of the subject and enable a more streamlined teaching on a graduate level. For instance, at Wesleyan, I taught both a model theory class, a commutative algebra class and an algebraic geometry class, and I found incorporating ideas from model theory in the two other classes very useful. It facilitated drawing similarities between the fields and enabled quicker applications of one field in the other. For instance, the use of homological algebra in the study of singularities is easier understood if its constructible nature is made explicit, as is carried out in [30]. Moreover, the key paradigm of algebraic geometry, Principle 1, can be made more transparent by means of its model theoretic counterpart, Principle 3. In carrying out the above program, I will continue paying attention to these educational issues.

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