Math 772
Problem Set 1

due Monday, March 31, 2008

1. (Lang V, Ex. 1) Let $E = \mathbb{Q}(\alpha)$, where α satisfies $\alpha^3 + \alpha^2 + \alpha + 2 = 0$. Express $(\alpha^2+\alpha+1)(\alpha^2+\alpha)$ and $(\alpha-1)^{-1}$ in the form $a\alpha^2+b\alpha+c$, with a, b, c rational.

2. (Lang V, Ex. 2) Suppose that $E = F(\alpha)$ is a finite extension of fields, of odd degree. Show that $E = F(\alpha^2)$.

3. Let $q(x) = x^3 + x + 1 \in \mathbb{F}_2[x]$.

(a) Show that $q(x)$ is irreducible in $\mathbb{F}_2[x]$.

(b) Let θ be a root of $q(x)$ (in some field containing $\mathbb{F}_2$; but $q(\theta) = 0$ is all you need to know about θ). Compute the powers of θ in $\mathbb{F}_2(\theta)$ by writing each power θ^k in the form $a + b\theta + c\theta^2$, where a, b, c belong to $\mathbb{F}_2$.

4. Let $f(x) = x^3 + ax + b$, where a, b belong to some field F. Suppose that $f(x)$ factors into linear factors in $F[x]$: $f(x) = (x-\alpha)(x-\beta)(x-\gamma)$, where $\alpha, \beta, \gamma \in F$.

(a) Let $g(x) = (x-\alpha^2)(x-\beta^2)(x-\gamma^2)$: find the coefficients of g in terms of a, b. (Hint: $-f(-x) = x^3 + ax - b$).

(b) Find a formula for $\alpha^2 + \beta^2 + \gamma^2$ in terms of a and b.

(c) Find a formula for $(\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\alpha)^2$ in terms of a and b.

5. (a) Show that $\mathbb{Q}(\sqrt{2}+\sqrt{3}) = \mathbb{Q}(\sqrt{2},\sqrt{3})$. (Obviously $\mathbb{Q}(\sqrt{2}+\sqrt{3}) \subseteq \mathbb{Q}(\sqrt{2},\sqrt{3})$, so what you really have to show here is that $\mathbb{Q}(\sqrt{2},\sqrt{3}) \subseteq \mathbb{Q}(\sqrt{2}+\sqrt{3})$. Be explicit: let $\theta = \sqrt{2}+\sqrt{3}$, and write $\sqrt{2}$ and $\sqrt{3}$ as polynomials in θ.)

(b) Find the minimal polynomial of θ over $\mathbb{Q}$.

(c) Find the minimal polynomial of θ over $\mathbb{Q}(\sqrt{2})$.