Math 772
Problem Set 3

due Monday, April 14, 2008

11. Let L be the splitting field of $x^4 + 1$ over $\mathbb{Q}$.

(a) What is $Gal(L/\mathbb{Q})$? Describe each element of $Gal(L/\mathbb{Q})$ as a permutation of the four roots of $x^4 + 1$.

(b) For each proper subgroup H of G (i.e. $H \neq 1$ and $H \neq G$, find its fixed field L^H and write L^H in the form $\mathbb{Q}(\sqrt{d})$ where d is an integer.

12. Suppose that $[F(\alpha) : F]$ has odd degree, so that $F(\alpha^2) = F(\alpha)$, by Problem Set 1, #2. Let $f(x)$ be the minimal polynomial of α over F, and let $g(x)$ be the minimal polynomial of α^2 over F. Describe the factorization of $g(x^2)$ in $F[x]$ in terms of $f(x)$.

13. Let p be a prime. Determine the Galois group of $x^p - 2$, (this means: describe $Gal(K/\mathbb{Q})$, where K is the splitting field of $x^p - 2$) as follows:

(a) Show that $[K : \mathbb{Q}] = p(p-1)$.

(b) Show that there is an automorphism τ of K such that $\tau(\zeta_p) = \zeta_p$ and $\tau(\sqrt[p]{2}) = \zeta_p \sqrt[p]{2}$.

(c) Show that for each $t = 1, 2, 3, \ldots, p-1$, there is an automorphism σ_t such that $\sigma_t(\zeta_p) = \zeta_p^t$ and $\sigma_t(\sqrt[p]{2}) = \sqrt[p]{2}$.

(d) Describe G in terms of the automorphisms in (b) and (c) (list the elements of G and determine the multiplication rules).

14. Let $\alpha = \sqrt{2 + \sqrt{2}}$. Let $K = \mathbb{Q}(\alpha)$.

(a) Find the minimal polynomial of α over $\mathbb{Q}$.

(b) Show that all the roots of the minimal polynomial of α belong to $K = \mathbb{Q}(\alpha)$. (Hence K is the splitting field.)

(c) What is $Gal(K/\mathbb{Q})$?

15. Let $K = \mathbb{R}(t)$ be the field of rational functions (ratios of polynomials) with real coefficients. Let $\sigma(r(t)) = r(t+1)$ for $r(t) \in K$. σ is an automorphism of K. What is the fixed field of σ?