Math 772
Problem Set 6

due Monday, May 5, 2008

26. Let p be a prime, F a field of characteristic p, let $a \in F$, and let $f(x) = x^p - x - a$. Let K be a splitting field for $f(x)$ and let α be a root of $x^p - x - a$ in K.

(a) Show that α, $\alpha + 1$, $\ldots \alpha + (p-1)$ are all roots of $f(x)$ in K, and so $f(x) = (x - \alpha)(x - (\alpha + 1)) \cdots (x - (\alpha + (p-1)))$. Hence $K = F(\alpha)$ is the splitting field of $f(x)$.

(b) Show that $f(x)$ is either irreducible or factors completely in $F[x]$.

27. (same notations as in #26) Suppose that $f(x)$ is irreducible in $F[x]$. Show that $x^p - x - a\alpha^{p-1}$ is irreducible in $K[x]$. *Hint: If β is a root of $x^p - x - a\alpha^{p-1}$ in K, write*

$$\beta = a_0 + a_1\alpha + \cdots + a_{p-1}\alpha^{p-1}, \text{ with } a_i \in F.$$

Raise to the p^{th} power, and use the fact that $1, \alpha, \ldots, \alpha^{p-1}$ are a basis for K/F.

28. Let $K = \mathbb{R}(x)$, and let $\sigma, \tau : K \to K$ be defined by

$$\sigma(r(x)) = r(1 - x), \qquad \tau(r(x)) = r(1/x),$$

for $r(x) \in K$. Find the group G of automorphisms of K generated by σ and τ, and determine the fixed field of G. (#17 (a)(b) will help.)

29. Let $K = \mathbb{R}(x)(\sqrt{1 - x^2})$. Show that $K = \mathbb{R}(t)$ for some element $t \in K$.

30. Let $K = \mathbb{R}(x)(\sqrt{1 - x^3})$. Show that K cannot be written in the form $\mathbb{R}(t)$ for any element $t \in K$.