

On the Movement of Water by Transport Pipes

Sections I–XV
by Leonhard Euler
translated by Seth Hulett

June 7, 2004

I. I propose here to develop by the principles of Hydrodynamics the case, where one uses pumps to push water into the transport pipes (exit pipes), which finally discharge it into a reservoir located at a certain height above the (original) level of water. The action of the pumps produces a double effect then; because while the piston is raised, water goes up by the suction pipe (intake), & fills the pump-housing while entering there by the valve which is at the bottom of the pump: then, when the piston is pushed back down, this valve is closed, & another, which had closed up till now the communication with the transport pipe (exit pipe), opens; & it is by this opening that water is driven from the pump housing into the transport pipe; which, becoming by this reiterated action finally full of water, discharges into the reservoir: & then the transport pipe provides to the reservoir as much water as the pumps attract by their suction pipes.

II. About the force, which acts upon the pumps and through the transport pipe, one will have to consider two questions, whose solution is of the utmost importance to Hydraulics. In the first, one asks about the quantity of water, that the pumps will be able to provide to the reservoir during a given time. The other question bears on the amount of force, that the pump-housings as well as the transport pipes have to withstand while the machine runs. The solution to each of these questions is absolutely necessary as soon as one has devised the plans for a machine of this nature: because it is always extremely important to know in advance, to what extent such a machine will be able to get water to the reservoir. Just in case it doesn't operate as planned, one can modify the plans, before spending money in building it.

III. The other question is not of lesser importance; because, as soon as one has settled upon a plan for the machine appropriate for the purposes

proposed, the amount of the forces should be known, that the machine and the transport pipes will have to support, so as to be able to determine the strength of the pumps & the pipes. Without this knowledge one is likely to give them either too much or too little strength; however either one is always an extreme defect. Too much strength only serves to uselessly increase the expense, even though it is not harmful to the overall project. But if the pumps or the transport pipes are made too weak, they will not be able to resist the forces which they have to support, and will burst. And the whole machine becomes useless. In order to prevent this great inconvenience, it is absolutely necessary that the forces to which the pumps as well the pipes are subjected to, be rather precisely known. So one is in the position to give them the correct strength that they need.

IV. If one looks at the great number of the Authors, who wrote on Hydraulics, & who praised themselves for the strength of their Theory, one should believe, that they had a thorough look into these questions, which contain the foundation of the Theory for transporting water. But if one more closely examines all that they wrote on this matter, one will be very surprised that one finds nothing to turn to which could be used to clear up such important questions. From where one stands, there is much to see about how defective the Theory of Hydraulics is, on which up to now one established the Practice. And they are not wrong, who usually reproach the Theory, and say that in practice it is almost of no use, & that one can not count on the success of a planned Machine, however much based it is in theory, until the machine actually works. It is clear from these reproaches that they, the authors, do not touch upon the true Theory, only superficial knowledge badly associated with this title.

V. For better understanding this defect of the ordinary Theory, two cases should be distinguished: one where water, which is the subject of a hydraulic Machine, is at rest, & the other case, when it is currently moving. The first belongs to Hydrostatics, whose principles are already so well known, that there does not remain there any more any doubt. According to these principles, if water is at rest in a rising pipe, one knows that at each place on this pipe carries the weight of a water column, whose height is equal to that of water in the pipe above this place; & if this water must be supported by a force, which operates the piston of the pump and forces the water out of this rising pipe, this force must equalize the weight of a water column, whose base is equal to that of the piston, & the height equal to that of water in the pipe above the base of the piston: & the cavity of the pump will subjected to this same force

VI. From there one concludes that as long as the force, which operates

the piston to push it downwards, is not higher than that which I have just indicated, it will only be able to maintain the water in balance, thus preventing it flowing back down: & it will be impossible for this force to move the water in the least, let alone push it higher in the pipe or even make it discharge into the tank. Therefore, so that water actually goes up and discharges into the tank, it is necessary that the force, which operates the piston is greater than in the preceding case, & one understands first of all that the more that the force is increased, the more it will be able to provide water to the tank: in this case it is obvious that the pump will support a greater strain, & consequently so also will the pipe, through which water is obliged to travel upwards.

VII. But one does not find in all the books, who treat this matter, any rule, by which one can know, given the excess force pushing the piston, beyond the force required to maintain equilibrium the speed of the water that will be pushed through the pipe, nor a rule for the force, which is exerted on the pipe. The Authors almost do not touch at all on these questions, & when one looks at the last, it seems that they think the pressure on the pipe is always about the same, as if the water were in balance. It is on this quantity of the pressure they gauge the thickness necessary to give the pipes strength to sufficiently resist it. However, undoubtedly alerted by annoying experiments, that they carried out, they advise to give to the pipes considerably more thickness, than their rule requires. By doing so they acknowledge to themselves, how ill-founded their Theory is. And concerning the movement of water, one observes a deep silence everywhere, so strong that it seems that the Authors did not want to venture to state anything on this critical point.

VIII. But one will cease being surprised of this negligence of the Authors of Hydraulics, if one notices, that the principles of Hydrostatic can be explained by simple Geometry, with the help of the elementary Analysis; but that is not the case with the Principles of Hydraulics, where one treats actual movement of water. Because, to discover these principles, it is absolutely necessary to refer to Infinitesimal Analysis, & to make the application of these principles to quantify the case, even Infinitesimal Analysis, though it seems to have been cultivated already, it is not enough, and there must be more, to put to us in a position, to develop these cases. Therefore, if one considers, that without the help of the Infinitesimal Analysis, it is absolutely impossible to make the tiniest progress in Hydraulics, one will no longer be astonished any more, that one finds almost nothing in all the Authors, who treated this matter, which provides us with solid explanations.

IX. However, though it is a long time ago, that one made the discovery of

the Infinitesimal Analysis, it was necessary by its help to carry the Principles of Mechanics to a higher degree of clarity, before one was in a position to apply it successfully to Hydraulics. It is mainly through M. Daniel Bernoulli who we are indebted to these first insights in this Science and has so fortunately developed in an excellent Treatise on Hydrodynamics, in which he used with so much success the principle of Conservation of sharp forces. Then, this principle having been disputed by several Mathematicians, who did not perceive the strength of it, his late father treated this matter with much success with only the first principles of Mechanics, that no one could call it into question: & since then, M. Alembert produced a very particular method to arrive at the same goal. From where one sees, this solid knowledge, that we have already have a true Theory of Hydraulics, nothing less than general, & that it must be completely unknown to those, who usually treat this matter without the help of the Infinitesimal Analysis.

X. Since this Science has almost no limits, one will not be surprised, if many cases, which one meets in practice, are not yet sufficiently studied, & I can justly say this about the case which I propose to examine here. Because, though the Authors allege that they have already considered the movement of water by pipes of an arbitrary shape, one would however seek there in vain the application to the specific case, which I have in mind: at least the Experts (Engineers) did not know how to draw any help to direct their works. I will thus try to develop this matter into in such a way, that those also, who apply themselves only to the practice, can find explanations, which are necessary for them. But also I will employ a method different from those, which had been used up to now, but will not fail to facilitate the research, that on still has to undertake in this Science.

XI. (Refer to Fig. 1.) Let $ABCD$ be the pump housing where the piston fits. I suppose a cylindrical cavity made so that the piston must be of the same cylinder diameter to fill the cavity of the pump exactly. Attached to the pump at the bottom DE is the rising pipe $DEYZGH$ by which the water is drawn from the pump to be discharged above at FH through a gaping opening into Reservoir I . I suppose this pipe to be an unspecified shape, but of a sort that its perpendicular cross-sections are circular everywhere along its length, and whose diameters vary according by an arbitrary rule with respect to various places along this pipe. As this pipe is not supposed to be cylindrical, one will not be able to say that its circular sections are perpendicular to its sides; but they will be perpendicular rather with a line, that one conceives drawn by all the centers from these circles, & that we will name it the centric line. Not to muddle the figure too much, let DYG be this centric line, to which crosses DE , YZ , yz , GH , at right angles everywhere,

which express at the same time the pipes diameter at these points.

Let (Fig. 1) ABCD be the body of the pump in which the piston is moving, which I assume to have a cylindrical cavity; and, as is well known, the piston must be cylindrical with the same diameter in order to fill out exactly the cavity of the pump. Connected with the pump at the lower end DE is the tube DEYZGH leading upward, through which the water from the pump is pressed so that it flows out high up at GH through the open mouth into the reservoir J. I assume for this tube an arbitrary shape, but such that the perpendicular cross sections are everywhere circular along its length, with diameters varying in an arbitrary manner depending on the different positions along the tube... (M. Eckert)

XII. That the height BE represents the play (the range) of the piston, so that in the time of the intake it goes up from E in AB , & then when it returns, that it is pushed from AB in E . During the time of the intake, the communication between the pump & the rising pipe going up being closed by the means of a valve placed in DE , the action of the piston only fills up with water all the cavity of the pump up to AB ; which enters by a valve at the bottom of the pump CD . When the piston is pushed back down, this valve at the bottom CD is closed, & the other at DE opens, to give passage to the water driven out through the rising pipe, which finally discharges it into reservoir I . Of this movement it is clear, that during each descent of the piston, it must discharge into the tank precisely as much water as is necessary to fill the pump.

XIII. To make the application of the Theory to this case, let us make the following denominations:

Let the diameter of the cavity of the pump $AB = a$;

The length of the stroke of the piston height $BE = b$;

The diameter of the hole, by which the pipe communicates with the pump $DE = c$.

And to represent the figure of pipe $DEGH$, that one carries out a horizontal line DF to be used as axis;

& having taken an unspecified x -coordinate $DX = x$

let the height of the corresponding point y , $XY = y$,

& the corresponding arc of in the centric curve $DY = s$

where it is to be noticed that y will be a certain function of x ,

& that $ds = \sqrt{dx^2 + dy^2}$.

Moreover at the point Y , the pipe's diameter $YZ = z$

& for the end of the pipe GH let the horizontal distance $DF = f$

the vertical height $FG = g$

& the holes diameter $GH = h$.

These things having been set, I will now consider the following problems.

PROBLEM I.

XIV. *The piston being pushed down by a given force, find at every moment the water's movement & the pressure it exerts on all the points of the pipe.*

SOLUTION.

Let K be a weight equal to the force, which pushes the piston down; however it is appropriate for the convenience of calculation, to reduce this force to the weight of a water column, whose base is equal to that of the piston, whose the diameter AB is set equal to a . Let k be the height of this column, & setting the ratio of the diameter to the circumference = $1 : \pi$, the base of the piston = $\frac{1}{4}\pi aa$, & consequently the volume of this column will be $\frac{1}{4}\pi aak$; thus K will be the weight of the mass of water, of which volume = $\frac{1}{4}\pi aak$: & this force K will have the same effect, that if the piston were pressed down by the weight of a water column of height = k . Now I suppose at the beginning that the pump was filled with water up to AB , & the pipe up to GH , & that at the beginning, water was at rest everywhere, without loss of generality. Because, after the piston is raised until in AB , it needs to remain at rest for an instant, before it can begin the descent. After a time = t , the piston has been lowered to MN , having gone down by the height $AM = BN = r$: & it is clear during this time that the quantity of water pushed into the reservoir will be = $\frac{1}{4}\pi aar$. Further, let the speed at which the piston continues to go down at MN , = \sqrt{v} , where v marks the height, from which a body while falling acquires the same speed: therefore, since the piston will travel with this speed = \sqrt{v} during the differential in respect to time dt , the differential in respect to the space $Mm = Nn = dr$, we will have $dt = \frac{dr}{\sqrt{v}}$, & $dr = dt\sqrt{v}$.

If we consider the water's speed at MN as known, we will be able to determine its speed everywhere, as well as in the pump and in the pipe DG : because since the pump has the same width down to E , the speed of water will be also the same = \sqrt{v} , & at each point of the pipe Y , speed will be just as much larger or smaller, as the width of the pipe at Y , which is expressed by = $\frac{1}{4}\pi zz$, is smaller or larger than the width of the pump = $\frac{1}{4}\pi aa$. Hence, the water's speed at YZ will be = $\frac{aa}{zz}\sqrt{v}$ the speed of the discharged water of GH will be = $\frac{aa}{hh}\sqrt{v}$.

The water's speed at the section YZ being = $\frac{aa}{zz}\sqrt{v}$, during the time dt it will travel the space of the pipe $YY' = \frac{aa}{zz}dt\sqrt{v} = \frac{aa}{zz}dr$ because $dr = dt * \sqrt{v}$: thus while the piston descends through space $Mm = Nn = dr$, the section

of water ZY will travel the space YY' & will arrive at $Y'Z'$.

With the piston having arrived at mn , its speed will be equal to

$$\sqrt{v + dv} = \sqrt{v} + \frac{dv}{2\sqrt{v}} :$$

& then to find the speed of the section YZ it is necessary to take account of its width, which depends on the length of the part of the pipe $DY = s$. Since z can be looked like a function of s , let $dz = Sds$. Therefore, after the section YZ , water will have traveled the space ds , its diameter will become $= z + Sds$; but in our case, the space traveled is $= \frac{aa}{zz} dr$, thus the width of the diameter at $Y'Z'$ will be $= z + \frac{aa}{zz} Sdr$: & from this speed at $Y'Z'$ will be

$$= \frac{aa}{(z + \frac{aa}{zz} Sdr)^2} \sqrt{v + dv},$$

& the height due to this speed

$$= \frac{a^4}{(z + \frac{aa}{zz} Sdr)^4} (v + dv)$$

. However it is:

$$\frac{1}{(z + \frac{aa}{zz} Sdr)^4} = \frac{1}{z^4} - \frac{4aa}{z^7} Sdr :$$

thus this height will be

$$= \frac{a^4}{z^4} - \frac{4a^6}{z^7} * (v + dv),$$

& consequently that will exceed at YZ which is $= a^4/z^4 v$ of the particle

$$\frac{a^4}{z^4} dv - \frac{4a^6}{z^7} Svd r.$$

This particle $\frac{a^4}{z^4} dv = \frac{4a^6}{z^7} Svd r$ will be thus the effect of force of acceleration, of which the section YZ is computed according to $YY' = \frac{aa}{zz} dr$. Because, if we set this force of acceleration equal to V , one knows by the Principles of Mechanics, that the increment in height due to this velocity, which is

$$\frac{a^4}{z^4} dv - \frac{4a^6}{z^7} Svd r$$

, is equal to the product of the accelerating force, V , multiplied by the space transversed $\frac{aa}{zz} dr$, which gives us this equation:

$$\frac{aa}{zz}Vdr = \frac{a^4}{z^4}dv - \frac{4a^6}{z^7}Svdr,$$

which simplifies to

$$V = \frac{aa}{zz} * \frac{dv}{dr} - \frac{4a^4}{z^5}Sv.$$

Now it is a question of finding the force of acceleration which acts on the section YZ : however for this effect it is necessary to give to this section an infinitely small thickness like Yy , to have the layer of water $YZzy$. Since we named $DX = x$, $XY = y$, & $DY = s$: then $Xx = dx$; $xy = y + dy$ & $Yy = ds$: from which we derive the mass of this layer will be $= \frac{1}{4}\pi zz ds$.

The forces which currently act on this layer are, first its own weight expressed by its mass $= \frac{1}{2}\pi zz ds$, whose direction is vertical YX : from there one will derive the force which acts in the direction y , $= \frac{1}{4}\pi zz dy$, which is opposite to the acceleration that has been found.

Besides this the layer finds itself solicited on the side YZ by the pressure of the water following, and on the side yz by the pressure of the preceding water; & if these two pressures were equal, one would destroy the effect of the other, & then no acceleration nor deceleration would result from it. Let the height p expresses the pressure of water on surface YZ , & p being of function of x or s , the pressure on surface yz will be expressed by the height $p + dp$. To explain this manner of expressing the water pressure, it should be noticed that p expresses the height at which water would gush out, if one made an infinitely small hole in Z . Or, what amounts to same, the water pressure against the walls of the pipe in YZ will be equivalent to a water column whose height $= p$: from one will see, that when we find the value of p , it will mark for us at the same time the strains of the forces, that the pipe will have to support at each one of its points. This water column of height p acting on the base YZ , which is $= \frac{1}{4}\pi zz$, will give a force equal to the weight of a volume of water $= \frac{1}{2}\pi zz p$: whose layer will be pushed in direction YY' ; however on other side it will be pushed back by the force, which is worth a volume of water $= \frac{1}{4}\pi zz(p + dp)$: where it is necessary to notice, that though the base yz perhaps is not equal to base YZ , one should not think of this difference; because, though the bases are unequal, it is known that pressures expressed by equal heights are always in balance: thus the pressure on yz exceeds that on YZ only to the extent that $p + dp$ is larger than p , as large, that is to say, as the difference of bases YZ & yz .

Thus from these two forces, a driving force will result, which pushes the layer $YZzy$ from behind, which will be $= \frac{1}{4}\pi zz dp$: which adding the force,

resulting from the weight of the layer, which was $= \frac{1}{4}\pi zz dy$, this layer will be co-jointly pushed by the motive force $\frac{1}{4}\pi zz(dp + dy)$ which by being evenly divided by the mass layer $= \frac{1}{4}\pi zz ds$, gives the force of acceleration

$$= \frac{dp + dy}{ds},$$

which being against the direction YY' , it is necessary to equalize V with

$$-\frac{dp - dy}{ds}.$$

From there we will obtain this equation:

$$dp + dy = -\frac{aads}{zz} * \frac{dv}{dr} + \frac{4a^4 S ds}{z^5} v$$

which will be used to determine the true value of the pressure p . For this effect it is necessary to look at the movement of the element of water in the pump $Mm = Nn = dr$ with the quantities r, v, dr & dv , which depend on it, like constants, to vary only the quantities when one looks at the position of the point Y . And the integral of our equation taken under these conditions will give us the waters pressure on the pipe of the moment, that the piston is in MN , & that there is so much speed, the acceleration, that we have just supposed.

Since we let $dz = S ds$, our equation will take this form:

$$dp = -dy - \frac{aadv}{dr} * \frac{ds}{zz} + 4a^4 v \frac{dz}{z^5}$$

of these factors $\frac{aadv}{dr}$ & $4a^4 v$ regarding the as constants, the integral will be

$$p = C - y - \frac{aadv}{dr} \int \frac{ds}{zz} - a^4 v \frac{1}{z^4}$$

To determine the value of this constant C , let us take the value of the term $\int \frac{ds}{zz}$ surface MN from the integral term $\int \frac{ds}{zz}$, where the waters pressure start: because one can look at the part of pump $MNCD$ as continuing from the pipe, & since it has $EN = b - r$, & the diameter of the pump $= a$, the value of $\int (ds/zz)$ for this part of the pump will be $= (b-r)/(aa)$; & because $\int (\frac{ds}{zz})$ marks the value of this integral from the beginning of the pipe DE until section YZ , where the pressure is sought: thus it will be

$$p = C - y - \frac{aa dv}{dr} - \frac{b-r}{aa} - \frac{aadv}{dr} \int \frac{ds}{zz} - \frac{a^4 v}{z^4}$$

Thus setting this integral $\int \frac{ds}{zz} = 0$, one will find the pressure at the place DE , where it becomes $y = 0$ & $z = c$; thus the pressure in DE will be

$$= C - \frac{aadv}{dr} \frac{b-r}{aa} - \frac{a^4v}{c^4} :$$

but if we rise it just up to the surface MN , the height becomes $y = b-r$, the amplitude $z = a$, & for this case the whole integral $\frac{b-r}{aa} + \int \frac{ds}{zz}$ disappears: for the surface pressure MN results in $C - b + r - v$. However this pressure must be precisely the same one, whose water is pressed by the piston in MN ; thus this pressure being represented by the height k , we will have $k = C - b - r + v$, thus constant C is determined $C = k + b - r + v$.

Therefore in general, the water pressure of any random YZ section, of the rising pipe, will be

$$p = k + b - r + v - y - \frac{(b-r)dv}{dr} - \frac{aadv}{dr} \int \frac{ds}{zz} - \frac{a^4v}{z^4}$$

or

$$p = k - y + \frac{b-r}{1 - \frac{dv}{dr}} - \frac{aadv}{dr} \int \frac{ds}{zz} + v \frac{1}{a^4/z^4}$$

from where one will be able to determine the water pressure along all the points of the pipe, provided that speed is known, from which the piston descends in MN & its acceleration. Because the pipes diameter z depends on the pipes length $DY = s$, and since the shape of the pipe is supposed to be known, z will be given by S , & from that one will have integral $\int \frac{ds}{zz}$.

To find the pressure at the very end of the pipe, GH , it is necessary to take the value of the integral $\int \frac{ds}{zz}$ for whole length of the pipe from DE to GH ; this value reduces to a constant quantity, let H be this quantity, so that $\int \frac{ds}{zz} = H$ at the time which one transports the indefinite point Y until G . Then one will have moreover $y = g$ & $z = h$. From there thus the pressure of water at end GH of the pipe will be expressed by the height =

$$k - g + (b-r)(1 - \frac{dv}{dr}) - \frac{Haadv}{dr} + v(1 - \frac{a^4}{z^4})$$

However since the water discharges at GH , there could not there be any more of pressure, therefore it is necessary that this last expression is = 0; which leads us to an equation, by which one will be able to determine the actual speed of the piston, when it has been pushed to MN . Thus this equation will be

$$k - g + b - r(1 - \frac{dv}{dr}) - \frac{Haadv}{dr} + v(1 - \frac{a^4}{z^4}) = 0.$$

We thus arrived at an equation, by which the water's movement, or that of the piston is given. This equation is reduced to this form:

$$dc(b + Haa - r) - vdr(1 - \frac{a^4}{h^4}) = (k - g + b - r)dr$$

or

$$dv + \frac{vdr(\frac{a^4}{h^4} - 1)}{b + Haa - r} = \frac{(k - g + b - r)dr}{b + Haa - r}$$

whose resolution does not have any difficulty, since the variable v only rises in one dimension. Thus I will not stop with the evolution of this integration; since it is now easy to determine the speed of the piston for each moment of its movement.

XV. Though this last equation is absolutely integrable, it is however of very little importance to know the integral of it, since the expression becomes so complicated, that one could not know how to draw any conclusions from it, at least if one is not applying it to a specific case of it: because in general the consequences, that one could conclude about it, would be too complicated by all the circumstances, who enter the calculation, so that they could be employed to the aid of the Practice (Engineering or Practical Applications). But as one is not usually useful raise water in this way, unless the tank is quite high above the top of the level of water, from where the pump draws: it will be mainly with this case, which I to which will apply the found solution. To this end I will suppose, that the tank is incomparably higher, that the height of the pump, & then the formulas to which one arrives, will not become only much simpler, but they will also provide the general rules, which will be of the utmost importance in the provision & the control of the works of this type.