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A FOURIER–LEBESGUE SERIES, DIVERGENT ALMOST EVERYWHERE[1]

The purpose of this article is to give *an example of a summable function*[2], *whose Fourier series diverges almost everywhere* (i.e. everywhere, except on a set of measure zero).

The function given in this article is not summable with a square, and I know nothing about the magnitude of the coefficients of its Fourier series. The methods used here do not allow to construct a Fourier series divergent everywhere.

I. Below, I will prove existence of a sequence of functions $\varphi_1(x), \cdots, \varphi_n(x), \cdots$, defined for $0 \leq x \leq 2\pi$ satisfying the following conditions:

$$1^\circ.\ \varphi_n(x) \geq 0, \qquad \int_0^{2\pi} \varphi_n(x)dx = 2 \qquad (n = 1, 2, \cdots).$$

2°. Partial sums of the Fourier series of $\varphi_n(x)$ are bounded.

3°. For every function $\varphi_n(x)$ it is possible to find a positive number M_n, a set E_n, and an integer q_n such that:

$3a$). $\lim M_n = \infty$

$3b$). $\lim\limits_{n\to\infty} \text{mes } E_n = 2\pi$

$3c$). For every point of the set E_n there exists a partial sum of the Fourier series of $\varphi_n(x)$, with index less then or equal to q_n, whose absolute value is greater then M_n.

Assuming the functions $\varphi_n(x)$ have been constructed, it is easy to find an increasing sequence of integers $n_1, n_2, \cdots, n_k, \cdots$ such that:

A) $\dfrac{1}{\sqrt{M_{n_k}}} \leq \dfrac{1}{2^k}$ and consequently $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt{M_{n_k}}} \leq 1.$

B) The magnitude of $\frac{1}{2}\sqrt{M_{n_k}}$ is greater then the sum of maxima of absolute values of partial sums of the Fourier series of $(k-1)$ functions $\varphi_{n_1}, \cdots, \varphi_{n_{k-1}}$.

[1] Une série de Fourier—Lebesgue divergente pesque partout. — Fund. math., 1923, vol. 4, p.324–328. Translation from French to Russian by P. L. Ulianov.

[2] I.e. integrable in the Lebesgue sense

C) $q_{n_i} \leq \frac{1}{2^k}\sqrt{M_{n_k}}$ for all $i < k$.

If n_i are known for all values of i less then k, then it is possible to find n_k, satisfying the inequalities A), B), C).

Now let

$$\Phi(x) = \sum_{k=1}^{\infty} \frac{1}{\sqrt{M_{n_k}}}\varphi_{n_k}(x).$$

By 1° and A) this series converges[3] almost everywhere to a summable function, and the coefficients of the Fourier series of $\Phi(x)$ are equal to the sum of the Fourier coefficients of the functions

$$\frac{1}{\sqrt{M_{n_k}}}\varphi_{n_k}(x) \qquad (k = 1, 2, \cdots).$$

Consider the partial sum of the Fourier series of $\Phi(x)$, which is greater then M_{n_k} at any point of the set E_{n_k} by 3°.

a) For any term of the series $\varphi_{n_k}(x)/\sqrt{M_{n_k}}$ it is greater then $\sqrt{M_{n_k}}$.

b) For the sum of all terms with indices less then k, by the condition B), it is less then $\frac{1}{2}\sqrt{M_{n_k}}$.

c) For terms with indices $s > k$ it is less then $6/2^s$.

Indeed, the partial sum with index less then or equal to $q_{n_k} \leq \frac{1}{2^s}\sqrt{M_{n_s}}$, is, by C), less then $(2q_{n_k}+1)$ multiplied by the integral of the absolute value of the function, which in this case is equal to $2/\sqrt{M_{n_s}}$.

From a), b), c) it follws that the absolute value of the existing sum $\Phi(x)$ is greater then or equal to

$$\frac{1}{2}\sqrt{M_{n_k}} - \frac{6}{2^k}.$$

From here we conclude that the Fourier series of $\Phi(x)$ diverges at every point of the set $E = \overline{\lim_{k\to\infty}} E_{n_k}$, mes E $= 2\pi$.

II. Construction of the function $\varphi_n(x)$.

[3] See, for example: Fund. math., 1923, vol. 4,p. 211, the Fubini theorem

Let

$\lambda_1 = 1, \lambda_2, \cdots, \lambda_n$

be a finite increasing sequence of odd numbers such that the conditions given later are satisfied.

Define a sequence $m_1, m_2, \cdots, m_n$:

$$m_1 = n, \qquad 2m_k + 1 = \lambda_k(2n+1).$$

Let

$$A_k = k\tfrac{4\pi}{2n+1}, \qquad 1 \leq k \leq n, \qquad A_n = 2\pi - \tfrac{2\pi}{2n+1}$$

Finally, take $\varphi_n(x) = m_k^2/n$ on the segment

$$\Delta_k = \left[A_k - \tfrac{1}{m_k^2},\ A_k + \tfrac{1}{m_k^2}\right].$$

For every point outside the segments Δ_k, we definite $\varphi_n(x) = 0$.

Clearly,

$$\varphi_n(x) \geq 0, \qquad \int_0^{2\pi} \varphi_n(x)dx = 2 \qquad \text{(condition } 1°\text{)}$$

and $\varphi_n(x)$ has a bounded displacement, therefore condition 2° is also satisfied.

Consider the partial sum of the Fourier series of $\varphi_n(x)$ with index m_k

$$\frac{1}{\pi}\int_0^{2\pi} \varphi_n(\alpha)\frac{\sin\frac{2m_k+1}{2}(\alpha - x)}{2\sin\frac{1}{2}(\alpha - x)}d\alpha. \qquad (4)$$

Assume that the point x lies inside the segment

$$\sigma_k = \left[A_{k-1} + \tfrac{2}{n^2}, A_k - \tfrac{2}{n^2}\right]. \qquad (5)$$

If λ_i for $i < k$ are defined and accordingly the function $\varphi_n(x)$ is defined on segments Δ_i, then it is possible to take λ_k so large that the integral (4), computed on all the segments $\Delta_i (i < k)$, would be as small as desired for any x in σ_k. I assume it to be less then 1 in absolute value.

Now consider the integral (4) on a segment $\Delta_s (s \geq k)$

$$\frac{1}{\pi}\int_{\Delta_s} \frac{m_s^2 \sin\frac{2m_k+1}{2}(\alpha - x)}{n \times 2\sin\frac{1}{2}(\alpha - x)}d\alpha = \frac{1}{\pi}\int_{\Delta_s} \frac{m_s^2}{n}\frac{\sin\frac{2m_k+1}{2}(A_s - x)}{2\sin\frac{1}{2}(A_s - x)}d\alpha +$$

$$+\frac{1}{\pi}\int_{\Delta_s}\frac{m_s^2}{n}\left[\frac{\sin\frac{2m_k+1}{2}(\alpha-x)}{2\sin\frac{1}{2}(\alpha-x)}-\frac{\sin\frac{2m_k+1}{2}(A_s-x)}{2\sin((A_s-x)/2)}\right]d\alpha. \tag{6}$$

Taking into account that $|\alpha - A_s| \le q/m_s^2$ one can see that the difference inside the brackets is less then

$$\frac{1}{m_s^2}\max\left|\frac{d}{d\alpha}\frac{\sin\frac{2m_k+1}{2}\alpha}{2\sin(\alpha/2)}\right| \le \frac{4m_k^2}{m_s^2} \le 4.$$

And, since the length of Δ_s is equal to $2/m_s^2$, the second term in (6) is less then $4/n$.

Taking out the constant in the second integral we obtain that expression (6) is equal to

$$\frac{2}{\pi n}\frac{\sin\frac{2m_k+1}{2}(A_s-x)}{2\sin\frac{1}{2}(A_s-x)}+\frac{\tau}{n}, \qquad |\tau|\le 4. \tag{7}$$

The sum of terms with τ for $s = k, k+1, \cdots, n$ is in absolute value less then 4. Noticing that

$$\frac{2m_k+1}{2}(A_s - A_k) = (s-k)\lambda_k \times 2\pi,$$

$$\sin\frac{2m_k+1}{2}(A_s-x) = \sin\frac{2m_k+1}{2}(A_k-x),$$

$$A_s - x < A_s - A_{k-1} = (s-k+1)\frac{4\pi}{2n+1} < (s-k+1)\frac{2\pi}{n},$$

$$\frac{1}{\sin\frac{1}{2}(A_s-x)} > \frac{n}{\pi(s-k+1)},$$

we see that the absolute value of the sum of the first terms of (7) for $s = k, k+1, \cdots, n$ is equal to

$$\frac{1}{\pi n}\left|\sin\frac{2m_k+1}{2}(A_k-x)\right|\sum_{s=k}^{n}\frac{1}{\sin\frac{1}{2}(A_s-x)} >$$

$$> \frac{1}{\pi^2}\left|\sin\frac{2m_k+1}{2}(A_s-x)\right|\sum_{r=1}^{n-k}\frac{1}{r} - 5. \tag{8}$$

So, for every point x from σ_k the absolute value of the integral (4) is greater then

$$\frac{1}{\pi^2}\left|\sin\frac{2m_k+1}{2}(A_s-x)\right|\sum_{r=1}^{n-k}\frac{1}{r}-5.$$

Let E_n be the set of all points x, located in the segments σ_k for $n-k>\sqrt{n}$, such that the following condition is true:

$$\frac{1}{\pi^2}\left|\sin\frac{2m_k+1}{2}(A_s-x)\right|>\frac{1}{\sqrt{\sum_{r=1}^{[\sqrt{n}]}\frac{1}{r}}}=\frac{1}{N_m}.$$

One can see, that for every point in E_n, belonging to σ_k, the partial sum of the Fourier series of $\varphi_n(x)$ with index m_k is greater then $N_n-5=M_n$.

It can be easily shown that

$$\lim_{n\to\infty}\operatorname{mes}E_n=2\pi.$$

Moscow, June 2nd, 1922.