2

ON THE ORDER OF MAGNITUDE OF COEFFICIENTS OF THE FOURIER–LEBESGUE SERIES[1]

It is known that the coefficients of the Fourier series of a summable function tend to zero. In this article I will prove the following proposition concerning the series by cosines.

I. *For every sequence* $\{a_n\}_{n=1}^{\infty}$*, converging to zero, a sequence* $\{a_n^{'}\}_1^{\infty}$ *can be found, such that*

1) $|a_n| < a_n^{'}$;

2) $\sum\limits_{n=1}^{\infty} a_n^{'} \cos nx$

is a Fourier series of a summable function.

Consider the series

$$a_0/2 + a_1 \cos x + \cdots + a_n \cos nx + \cdots \tag{1}$$

Define

$$\Delta_n = a_n - a_{n+1}, \qquad \Delta_n^{'} = \Delta_n - \Delta_{n+1}$$

Applying the Abel transformation twice, we obtain two series

$$\frac{1}{2}\Delta_0 + \sum_{n=1}^{\infty} \Delta_n \frac{\sin((2n+1)x/2)}{2\sin(x/2)}, \tag{2}$$

$$\frac{1}{2}\Delta_0^{'} + \sum_{n=1}^{\infty} \Delta_n^{'} \frac{1}{2}\left(\frac{\sin((2n+1)x/2)}{2\sin(x/2)}\right)^2. \tag{3}$$

If the conditions

$$\lim_{n\to\infty} a_n \frac{\sin((2n+1)x/2)}{2\sin(x/2)} = \lim_{n\to\infty} \Delta_n \left(\frac{\sin((2n+1)x/2)}{2\sin(x/2)}\right)^2 = 0$$

---

[1] Sur l'ordre de grandeur des coéfficients de la série doe Fourier–Lebegue. — Bull. Acad. pol. sci. A, 1923, p. 83–86. Submitted by V. Serpinski. Translation from French to Russian by P. L. Ulianov.

are satisfied, the series (1) converges together with the series (3). These conditions are satisfied everywhere, except for points $x \equiv 0 (\mathrm{mod}\ 2\pi)$, if $a_n \to 0$ and, consequently, $\Delta_n \to 0$.

If the series $\sum\limits_{n=1}^{\infty} |\Delta_n|$ converges, the series (3) and (1) converge to some function $f(x)$everywhere, except for points $x \equiv 0(\mathrm{mod}\ 2\pi)$. Noticing that

$$\frac{1}{2}\int_0^{2\pi} |\Delta_n^{'}| \left(\frac{\sin((2n+1)x/2)}{2\sin(x/2)}\right)^2 dx = \pi(n+1)|\Delta_n^{'}|,$$

we obtain the assertion:

II. *If the series* $\sum\limits_{n=1}^{\infty}(n+1)|\Delta_n^{'}|$ *converges and the coefficinets* a_n *tend to zero, then the series* (3) *converges to a summable function. So, the series* (1)*, converging to the summable function* $f(x)$*, excluding points* $x \equiv 0(\mathrm{mod}\ 2\pi)$*, is a Fourier-Lebesgue series.*

In particular, if all $\Delta_n^{'}$ are positive, then

$$\sum_{n=0}^{\infty}(n+1)|\Delta_n^{'}| = \sum_{n=0}^{\infty}(n+1)\Delta_n^{'} = a_0.$$

Therefore,

III. *If the coefficients of a series by cosines tend to zero and their second differences are positive, then this series is a Fourier-Lebesgue series.*

Since for every sequence $\{a_n\}$ converging to zero there exists another sequence $\{a_n^{'}\}$ converging to zero, such that $a_n^{'} > |a_n|$ and the second differences are positive, the suggestion **I** is proven.

Remark 1. Suppose the series

$$\sum_{n=1}^{\infty} a_n \cos nx = f(x)$$

satisfies the conditions in suggestion **II**. Let $x = my + \pi/2$. Then

$$f(my+\frac{\pi}{2}) = \sum_{n=1}^{\infty}(-a_{4n-3}\sin(4n-3)my + a_{4n-1}\sin(4n-1)my) +$$
$$+\sum_{n=1}^{\infty}(-a_{4n-2}\cos(4n-2)my + a_{4n}\cos 4nmy).$$

The first sum is the Fourier-Lebesgue series for

$$\frac{f(my+\pi/2)-f(-my+\pi/2)}{2},$$

in particular, if $m=1$, we see that the seeries

$-a_1\sin y + a_3\sin 3y - a_5\sin 5y + a_7\sin 7y\cdots$

is a Fourier-Lebesgue series.

Remark 2. The remainder of the series (1) is equal

$$R_n = \frac{1}{2}\sum_{k=n+1}^{\infty}\Delta_k^{'}\left(\frac{\sin((k+1)x/2)}{\sin(x/2)}\right)^2 + \frac{1}{2}\Delta_n\left(\frac{\sin((n+1)x/2)}{\sin(x/2)}\right)^2 +$$

$$+a_n\frac{\sin((2n+1)x/2)}{2\sin(x/2)}$$

If the condition in suggestion **II** is satisfied, we have

$$\lim_{n\to\infty}\int_0^{2\pi}\left|\sum_{k=n+1}^{\infty}\Delta_k^{'}\left(\frac{\sin((k+1)x/2)^2}{\sin(x/2)}\right)^2\right|dx = \lim_{n\to\infty}\sum_{k=n+1}^{\infty}\pi(k+1)|\Delta_k^{'}| = 0,$$

$$\lim_{n\to\infty}\int_0^{2\pi}\left|\Delta_n\left(\frac{\sin((n+1)x/2)}{\sin(x/2)}\right)^2\right|dx = \lim_{n\to\infty}\pi(n+1)|\Delta_n| =$$

$$= \lim_{n\to\infty}\pi(n+1)\left|\sum_{k=n}^{\infty}\Delta_k^{'}\right| \le \lim_{n\to\infty}\pi\sum_{k=n}^{\infty}(k+1)|\Delta_k^{'}| = 0.$$

In this case $\int_0^{2\pi}|R_n|dx$ converges to zero together with

$$|a_n|\int_0^{2\pi}\left|\frac{\sin((2n+1)x/2}{2\sin(x/2)}\right|dx$$

and, consequently, together with $|a_n|\log n$.

IV. *If* $\sum_{n=1}^{\infty} n|\Delta_n^{'}|$ *converges, then the condition*

$$\lim_{n\to\infty} a_n\log n = 0$$

is necessary and sufficient for convergence of the series (1) *on average* [by metric L]

So, out of the two Fourier-Lesbegue series

$$\sum_{n=2}^{\infty} \frac{\cos nx}{\log n}, \qquad \sum_{n=2}^{\infty} \frac{\cos nx}{(\log n)^{1+\varepsilon}}$$

the second converges on average, while the first does not (see [1]).

December 3, 1922.

LITERATURE

1. *Banach S., Steinhaus H.* Sur la convergence en moyenne. — Bull. Acad. sci. Cracovie, 1918.