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A CONTRIBUTION TO THE STUDIES OF CONVERGENCE OF FOURIER SERIES[1]

As usual, let

$$S_n = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx),$$

$$\sigma_n = \frac{S_0 + S_1 + \cdots + S_{n-1}}{n}.$$

I. T h e o r e m. *If a sequence of integers* n_m $(m = 1, 2, \cdots)$ *satisfies*

$$n_{m+1}/n_m > \lambda > 1,$$

then for the Fourier series of any function the sequence S_{n_m} *converges to a given function almost everywhere.*

P r o o f. It is known that the sequence $\{\sigma_{n_m}\}$ converges almost everywhere to a given function, therefore, it is enough to prove that the sequence $\{S_{n_m} - \sigma_{n_m}\}$ converges almost everywhere to zero. However, it is easy to see that this is a resul of convergence of the following series:

$$\sum_{m=1}^{\infty} \int_0^{2\pi} (S_{n_m} - \sigma_{n_m})^2 dx. \tag{1}$$

Consider the p^{th} partial sum of the series (1). We have

$$\frac{1}{\pi} \sum_{m=1}^{p} \int_0^{2\pi} (S_{n_m} - \sigma_{n_m})^2 dx = \sum_{m=1}^{p} \frac{1}{n_m^2} \sum_{k=1}^{k=n_m} k^2 (a_k^2 + b_k^2) =$$

$$= \sum_{k=1}^{n_p} k^2 (a_k^2 + b_k^2) \left[\frac{1}{n_{m_k}^2} + \frac{1}{n_{m_k+1}^2} + \cdots + \frac{1}{n_p^2} \right], \tag{2}$$

where n_{m_k} is defined by the inequality

$$n_{m_k - 1} < k \leq n_{m_k}.$$

[1] Une contribution a l'étude de la convergence des séries de Fourier. — Fund. math., 1924, vol. 5, p. 96-97. Translation from French to Russian by I.A.Vinogradova.

Clearly,

$$\frac{1}{n_{m_k}^2}+\frac{1}{n_{m_k+1}^2}+\cdots+\frac{1}{n_p^2}\le\frac{1}{n_{m_k}^2}\left(1+\frac{1}{\lambda^2}+\cdots\frac{1}{\lambda^{2p}}\right)<$$

$$<\frac{1}{n_{m_k}^2}\frac{\lambda^2}{\lambda^2-1}\le\frac{1}{k^2}\frac{\lambda^2}{\lambda^2-1},$$

and, therefore, the sum (2) is no greater then

$$\frac{\lambda^2}{\lambda^2-1}\sum_{k=1}^{k=n_p}(a_k^2+b_k^2).$$

This leads to convergence of the series (1) and the proof is complete.

II. T h e o r e m. *If the only non-zero terms of a Fourier–Lebesgue have indices n_m (the sequence n_m satisfies the condition of theorem* **I***), then the series converges almost everywhere.*

P r o o f. If we only consider functions integrabe with a square, then theorem **II** follows immediately from theorem **I**; however the statement is true *for all integrable functions.* The sequence σ_{n_m} converges almost everywhere, therefore we only need to consider

$$|S_{n_m-1}-\sigma_{n_m-1}|\le\sum_{k=1}^{m-1}\frac{n_k}{n_m}(|a_{n_k}|+|b_{n_k}|). \tag{3}$$

Since $|a_{n_k}|+|b_{n_k}|$ approaches 0 as $k\to\infty$ and, on the other hand,

$$\sum_{k=1}^{m-1}\frac{n_k}{n_m}<\sum_{k=1}^{m-1}<\frac{1}{\lambda-1},$$

it can be seen that the difference (3) approaches 0 as $m\to\infty$.

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