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ON CONVERGENCE OF FOURIER SERIES[1]

Together with G.A.Seliverstrov

Hardy [1] proved the following theorem:

If the series

$$\sum_{n=1}^{\infty}(a_n^2+b_n^2)(\log n)^2$$

converges, then the series

$$\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx) \qquad (1)$$

converges almost everywhere (except on a set of measure zero).

From the work of D. E. Menchoff [2] we know that for general orthogonal series, the factor $(\log n)^2$ cannot be replaced by a function $w(n)$, satisfying $w(n)=o[(\log n)^2]$

In this article we will prove that in the case of trigonometric series the factor $(\log n)^2$ can be replaced with $(\log n)^{1+\varepsilon}$.

Lemma. *Let a trigonemetric sum*

$$S(x)=\sum_{k=1}^{n}(a_k\cos kx+b_k\sin kx) \qquad (n>1),$$

be given. Then we have

$$\int_0^{2\pi}S_{k(x)}(x)dx\le\sqrt{C\log n\sum_{p=1}^{n}(a_p^2+b_p^2)}.$$

In this expression, $k(x)$ is an arbitrary integer function, taking values from 1 *to n:*

$$S_{k(x)}(x)=\sum_{p=1}^{k(x)}(a_p\cos px+b_p\sin px)$$

while C is an absolute constant.

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[1] Sur la convergence des séries de Fourier. — C. r. Acad. sci. Paris, 1924, vol. 178, p. 303-306. Submitted by A. Lebegues. Translation from French to Russian by I.A.Vinogradova.

P r o o f. By using the Schwartz inequality we obtain

$$\int_0^{2\pi} S_{k(x)}(x)dx = \frac{1}{\pi}\int_0^{2\pi}\int_0^{2\pi} S(\alpha)\sum_{p=1}^{k(x)}\cos p(x-\alpha)d\alpha dx$$

$$\frac{1}{\pi}\int_0^{2\pi} S(\alpha)\int_0^{2\pi}\sum_{p=1}^{k(x)}\cos p(x-\alpha)dxd\alpha \leq$$

$$\leq \sqrt{\frac{1}{\pi}\int_0^{2\pi} S^2(\alpha)d\alpha\frac{1}{\pi}\int_0^{2\pi}\left[\int_0^{2\pi}\sum_{p=1}^{k(x)}\cos p(x-\alpha)dx\right]^2 d\alpha}.$$

Then,

$$\int_0^{2\pi}\left[\int_0^{2\pi}\sum_{p=1}^{k(x)}\cos p(x-\alpha)dx\right]^2 d\alpha =$$

$$= \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\sum_{p=1}^{k(x)}\cos p(x-\alpha)\sum_{p=1}^{k(y)}\cos p(y-\alpha)dydxd\alpha =$$

$$= \int_0^{2\pi}\int_0^{2\pi}\sum_{p=1}^{\min[k(x),k(y)]}\int_0^{2\pi}\cos p(x-\alpha)\cos p(y-\alpha)d\alpha dxdy =$$

$$= \int_0^{2\pi}\int_0^{2\pi}\cos p(x-y)dxdy.$$

It is easy to prove that the last expression is not greater then $C\log n$, where C is an absolute constant.

T h e o r e m. *If the following series converge,*

$$\sum_{n=1}^{\infty}\tau(n)(a_n^2+b_n^2) = A, \qquad (2)$$

$$\sum_{n=1}^{\infty}\frac{1}{n\tau(n)} = N \qquad (3)$$

and $\tau(n) < \tau(n+1)$, *then the series* (1) *converges almost everywhere.*

First prove that the partial sums of the series (1) are bounded almost everywhere.

Let

$$S_{p,l} = \sum_{q=2^{2^p}}^{l} (a_q \cos qx + b_q \sin qx),$$

$$p = 0, 1, 2, \cdots, 2^{2^p} \leq l < 2^{2^{p+1}},$$

and

$$A_p = \sum_{q=2^{2^p}}^{2^{2^{p+1}}-1} (a_q^2 + q_n^2).$$

By the lemma, we have

$$\int_0^{2\pi} S_{p,l(x)}(x)dx \leq \sqrt{C \log 2^{2^{p+1}} A_p} \leq C' \sqrt{2^p A_p},$$

where $l(x)$ is an arbitrary integer function satisfying the inequality for l. Let $\Phi(x)$ be the upper bound of the partial sums of series (1), then it follows that

$$\int_0^{2\pi} \Phi(x)dx \leq C' \sum_{p=0}^{\infty} \sqrt{2^p A_p} + 2\pi(|a_1| + |b_1|). \tag{4}$$

The series on the right converges. Indeed,

$$\sum_{p=0}^{\infty} A_p \tau(2^{2^p}) \leq \sum_{n=1}^{\infty} (a_n^2 + b_n^2)\tau(n) = A,$$

while by the Cauchy theorem,

$$\sum_{p=1}^{\infty} \frac{2^p}{\tau(2^{2^p})} < 2 \sum_{r=1}^{\infty} \frac{1}{\tau(2^r)} < 4 \sum_{n=1}^{\infty} \frac{1}{n\tau(n)} = 4N,$$

and therefore we have

$$\sum_{p=1}^{\infty} \sqrt{A_p \tau(2^{2^p}) \frac{2^p}{\tau(2^{2^p})}} \leq 2\sqrt{AN}.$$

By discarding the first terms of series (1) we can make the integral (4) as small as needed. This proves convergence of series (1).

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LITERATURE

1. *Hardy G. H.* On the summability of Fourier's series. — Proc. London Math. Soc., 1913, vol. 12, p.365—372.

2. *Menchoff D.* Sur les séries de functions orthogonales. — Fund. math., 1923, vol 4, p.82—105.