On the Transformation of Infinite Series to Continued Fractions
by Leonhard Euler

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INTRODUCTION

The following is a translation of "De Transformatione Serium in Fractiones Continuas Ubi Simul Haec Theoria Non Mediocriter Amplificatur," an essay by Leonhard Euler.

Commentatio 593 indicis Enestroemiani

Opuscula analytica 2, 1785, p. 138-177

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1. Consider any continued fraction, which is

$$s = a + \cfrac{1}{b + \frac{1}{c + \frac{1}{d + \dots}}}$$

First, let us find simple fractions that continuously approach the value s. We construct the fractions as follows, so that

$$\frac{A}{\mathcal{A}} = a, \quad \frac{B}{\mathcal{B}} = a + \frac{1}{b}, \quad \frac{C}{\mathcal{C}} = a + \frac{1}{b+\frac{1}{c}}, \quad \frac{D}{\mathcal{D}} = a + \frac{1}{b+\frac{1}{c+\frac{1}{d}}}, \quad \text{etc...}$$

Therefore, the last of these fractions expresses the value of the proposed continued fraction. It will be shown immediately

$$\frac{A}{\mathcal{A}} = \frac{a}{1}, \quad \frac{B}{\mathcal{B}} = \frac{ab+1}{b}, \quad \frac{C}{\mathcal{C}} = \frac{abc+a+c}{bc+1}.$$

Let us investigate the way in which these fractions further proceed.

2. It is apparent that the second fraction arises from the first fraction, when in place of a,

$$a + \frac{1}{b}$$

is written. In the same way the third arises from the second, if in place of b,

$$b + \frac{1}{c}$$

is written, and the fourth from the third, if in place of c,

$$c+\frac{1}{d}$$

is written, and onward in this way. Here therefore, if the indefinite fraction $\frac{P}{\mathcal{P}}$ is formed from the indices a, b, c, d, ... , p and the pair $\frac{Q}{\mathcal{Q}}$ and $\frac{R}{\mathcal{R}}$ are the fractions which correspond to the indices a, b, c, d, ... , q and a, b, c, d, ... , r, it is clear from the fraction $\frac{P}{\mathcal{P}}$ the next one $\frac{Q}{\mathcal{Q}}$, is obtained if

$$p+\frac{1}{q}$$

is written in place of p, and from $\frac{Q}{\mathcal{Q}}$, $\frac{R}{\mathcal{R}}$ will next arise, if

$$q+\frac{1}{r}$$

is written in place of q.

Now, it is apparent that in the fraction $\frac{P}{\mathcal{P}}$ both the numerator P and the denominator $\mathcal{P}$ involve all the letters a, b, c, d, ... , p, so that none of them have powers greater than one. If all these indices a, b, c, d, e, etc... are considered as unequal, a square or higher power of one of them will never occur anywhere.

3. Because of this fact terms of two kinds occur in P and in $\mathcal{P}$, while some do not contain the index p, others do involve this factor; the numerator P will have the form of this kind $M+Np$, and in the same way the denominator $\mathcal{P}$ will have this form $\mathcal{M}+\mathcal{N}p$, so that it is

$$\tfrac{P}{\mathcal{P}}=\tfrac{M+Np}{\mathcal{M}+\mathcal{N}p}.$$

Therefore, we write this form in place of p

$$p+\tfrac{1}{q},$$

so that we obtain the fraction $\frac{Q}{\mathcal{Q}}$, which therefore, after we will multiply above and below by q, will be

$$\tfrac{Q}{\mathcal{Q}}=\tfrac{Mq+Npq+N}{\mathcal{M}q+\mathcal{N}pq+\mathcal{N}}=\tfrac{N+(M+Np)q}{\mathcal{N}+(\mathcal{M}+\mathcal{N}p)q}.$$

Now, so that we obtain the fraction $\frac{R}{\mathcal{R}}$, in place of q let us write $q + \frac{1}{r}$, and afterwards we will multiply above and below by r, producing

$$\frac{R}{\mathcal{R}} = \frac{Nr+(M+Np)qr+M+Np}{\mathcal{N}r+(\mathcal{M}+\mathcal{N}p)qr+\mathcal{M}+\mathcal{N}p},$$

or

$$\frac{R}{\mathcal{R}} = \frac{M+Np+(N+Mq+Npq)r}{\mathcal{M}+\mathcal{N}p+(\mathcal{N}+\mathcal{M}q+\mathcal{N}pq)r}.$$

Therefore, since $P = M + Np$ and $Q = N + (M + Np)q$, it will be

$$R = P + Qr.$$

In the same way, since $\mathcal{P} = \mathcal{M} + \mathcal{N}p$ and $\mathcal{Q} = \mathcal{N} + (\mathcal{M} + \mathcal{N}p)q$, it will be

$$\mathcal{R} = \mathcal{P} + \mathcal{Q}r.$$

So it is clear how any of our simple fractions are able to be formed easily from the two preceding fractions.

4. Look, therefore it is a plain enough demonstration, and the well known methods for the conversion of continued fractions to simple fractions are clear, where both the numerators and the denominators are formed from the previous two by the same rule. Therefore, since for the first fraction $A = a$, $\mathcal{A} = 1$, then $B = ab + 1$ and $\mathcal{B} = b$, from these two fractions, all the rest are able to be formed by easy effort. So that it is clear, let us write the corresponding fractions in succession with the individual indices a, b, c, d, e, etc...

$$\begin{array}{cccccccl} a & b & c & d & e & f & g & \text{etc.} \\ \frac{A}{\mathcal{A}} & \frac{B}{\mathcal{B}} & \frac{C}{\mathcal{C}} & \frac{D}{\mathcal{D}} & \frac{E}{\mathcal{E}} & \frac{F}{\mathcal{F}} & \frac{G}{\mathcal{G}} & \text{etc.} \end{array}$$

and next both the numerators and the denominators will be determined in the same way from the two preceding by the rule.

For the numerators:	For the denominators:
$A = a$	$\mathcal{A} = 1$
$B = Ab + 1$	$\mathcal{B} = b$
$C = Bc + A$,	$\mathcal{C} = \mathcal{B}c + \mathcal{A}$
$D = Cd + B$,	$\mathcal{D} = \mathcal{C}d + \mathcal{B}$
$E = De + C$,	$\mathcal{E} = \mathcal{D}e + \mathcal{C}$
$F = Ef + D$,	$\mathcal{F} = \mathcal{E}f + \mathcal{D}$

From which it is clear in the sequence of numerators that the previous term of the progression should be 1 by the rule; however, in the series of denominators the previous term should be 0 by the first limit, so that the first preceding fraction is $\frac{1}{0}$.

5. Since it is clear that these fractions

$$\frac{A}{\mathcal{A}},\ \frac{B}{\mathcal{B}},\ \frac{C}{\mathcal{C}},\ \frac{D}{\mathcal{D}},\ \frac{E}{\mathcal{E}},\ \frac{F}{\mathcal{F}},\ \frac{G}{\mathcal{G}}, \text{ etc.}$$

continually approach closer to the truth and they draw close to the value of the continued fraction, it is necessary that the differences between two adjacent fractions continuously become less. We will disclose why these differences become smaller in succession. Therefore, first we will have

$$II - I = \frac{B\mathcal{A} - A\mathcal{B}}{\mathcal{A}\mathcal{B}}.$$

Now, in place of B and $\mathcal{B}$ the values from the table are substituted and it will produce the numerator $A\mathcal{A}b + \mathcal{A} - Ab$, which because $\mathcal{A} = 1$ reduces to 1, so that

$$\frac{B}{\mathcal{B}} - \frac{A}{\mathcal{A}} = \frac{1}{\mathcal{A}\mathcal{B}}.$$

Again it will be

$$III - II = \frac{C\mathcal{B} - B\mathcal{C}}{\mathcal{B}\mathcal{C}},$$

if the assigned values are written in place of C and $\mathcal{C}$, the numerator will be

$$\mathcal{B}(Bc + A) - B(\mathcal{B}c + \mathcal{A}) = A\mathcal{B} - B\mathcal{A}.$$

Moreover, we saw that $B\mathcal{A} - A\mathcal{B} = 1$, from which this numerator will be -1 and therefore

$$\frac{C}{\mathcal{C}} - \frac{B}{\mathcal{B}} = -\frac{1}{\mathcal{B}\mathcal{C}}.$$

Again

$$IV - III = \frac{D\mathcal{C} - C\mathcal{D}}{\mathcal{C}\mathcal{D}},$$

where, if the assigned values are written in place of D and $\mathcal{D}$,

$$\mathcal{C}D - C\mathcal{D} = \mathcal{C}(Cd + B) - C(\mathcal{C}d + \mathcal{B}) = B\mathcal{C} - C\mathcal{B}.$$

Moreover, we saw that $C\mathcal{B} - B\mathcal{C} = -1$, from which it is concluded

$$\frac{D}{\mathcal{D}} - \frac{C}{\mathcal{C}} = +\frac{1}{\mathcal{CD}}.$$

And it is found for the following

$$\frac{E}{\mathcal{E}} - \frac{D}{\mathcal{D}} = -\frac{1}{\mathcal{DE}}, \quad \frac{F}{\mathcal{F}} - \frac{E}{\mathcal{E}} = +\frac{1}{\mathcal{EF}}, \quad \text{etc.}$$

6. Hence we are able to define our fractions one by one from the first one $\frac{A}{\mathcal{A}} = a$ alone and we will be able to define the script letters alone from the progressing fractions, since we will have

$$\frac{B}{\mathcal{B}} = a + \frac{1}{\mathcal{AB}},$$

$$\frac{C}{\mathcal{C}} = a + \frac{1}{\mathcal{AB}} - \frac{1}{\mathcal{BC}},$$

$$\frac{D}{\mathcal{D}} = a + \frac{1}{\mathcal{AB}} - \frac{1}{\mathcal{BC}} + \frac{1}{\mathcal{CD}},$$

$$\frac{E}{\mathcal{E}} = a + \frac{1}{\mathcal{AB}} - \frac{1}{\mathcal{BC}} + \frac{1}{\mathcal{CD}} - \frac{1}{\mathcal{DE}},$$

$$\frac{F}{\mathcal{F}} = a + \frac{1}{\mathcal{AB}} - \frac{1}{\mathcal{BC}} + \frac{1}{\mathcal{CD}} - \frac{1}{\mathcal{DE}} + \frac{1}{\mathcal{EF}},$$

etc.

7. Therefore, since the last or infinitesimal term of these fractions exhibits the true value, we have the proposed continued fraction designate with the letter s

$$s = a + \frac{1}{\mathcal{AB}} - \frac{1}{\mathcal{BC}} + \frac{1}{\mathcal{CD}} - \frac{1}{\mathcal{DE}} + \frac{1}{\mathcal{EF}} - \frac{1}{\mathcal{FG}} + etc.;$$

and so we reduced the continued fraction to an infinite series of fractions, all the numerators of which are alternately $+1$ or -1, and the denominators are in terms of script letters, so that it is not much work to determine the values of the letters A, B, C, etc..., but it suffices to disentangle the next formulas

$$\mathcal{A} = 1, \quad \mathcal{B} = b, \quad \mathcal{C} = \mathcal{B}c + \mathcal{A}, \quad \mathcal{D} = \mathcal{C}d + \mathcal{B}, \quad \mathcal{E} = \mathcal{D}e + \mathcal{C}, \quad \text{etc.}$$

8. Since every expression begins with the quantity a, for brevity it will be omitted from the calculation, because the script letters do not depend on it; from which, we come to this point so that the proposed continued fraction

$$s = \frac{1}{b + \frac{1}{c + \frac{1}{d + \ldots}}}$$

if script letters are defined by these indices b, c, d, e, etc., where indeed it remains $\mathcal{A} = 1$, it will always be

$$s = \frac{1}{\mathcal{AB}} - \frac{1}{\mathcal{BC}} + \frac{1}{\mathcal{CD}} - \frac{1}{\mathcal{DE}} + \frac{1}{\mathcal{EF}} - \frac{1}{\mathcal{FG}} + etc.;$$

the progression which continues infinitely. If the continued fraction is extended to infinity, it will correspond to the limiting value.

9. Therefore, since in this way we will transform the continued fraction into a regular series, it will not be difficult to convert any proposed series into a continued fraction. Therefore, let this be the proposed infinite series

$$s = \frac{1}{\alpha} - \frac{1}{\beta} + \frac{1}{\gamma} - \frac{1}{\delta} + etc.,$$

the numerators of which are all 1 endowed with alternating sign $+$ and $-$, the denominators constitute any progression, which are non-negative, all are contained in this form, not only α, β, γ, δ, but also the fractional numbers from the end of the series are not negative.

10. Let us determine the continued fraction that is equal to this series. First let us make

$$\mathcal{AB} = \alpha, \quad \mathcal{BC} = \beta, \quad \mathcal{CD} = \gamma,$$

and thus again, we will find the next values from $\mathcal{A} = 1$:

$$\begin{array}{ll} \mathcal{B} = \alpha & \mathcal{C} = \frac{\beta}{\alpha} \\ \mathcal{D} = \frac{\alpha\gamma}{\beta} & \mathcal{E} = \frac{\beta\delta}{\alpha\gamma} \\ \mathcal{F} = \frac{\alpha\gamma\epsilon}{\beta\delta} & \mathcal{G} = \frac{\beta\delta\zeta}{\alpha\gamma\epsilon} \\ \mathcal{H} = \frac{\alpha\gamma\epsilon\nu}{\beta\delta\zeta} & \mathcal{I} = \frac{\beta\delta\zeta\theta}{\alpha\gamma\epsilon\nu} \end{array}$$

etc.

Therefore, now all that remains is to elicit from the values of script letters the indices b, c, d, e, etc. for the continued fraction.

11. From the formula, the script letters above are determined by the indices of the continued fractions, in turn, let us define from these same letters the indices b, c, d, e, f, etc. and we will observe

$$b = \mathcal{B}, \quad c = \tfrac{\mathcal{C}-\mathcal{A}}{\mathcal{B}}, \quad d = \tfrac{\mathcal{D}-\mathcal{B}}{\mathcal{C}}, \quad e = \tfrac{\mathcal{E}-\mathcal{C}}{\mathcal{D}}, \quad f = \tfrac{\mathcal{F}-\mathcal{D}}{\mathcal{E}}, \quad \text{etc.}$$

Therefore, we unroll these values in succession, while we substitute the previously determined formulas in place of the letters $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, etc.

12. First $\mathcal{B} = \alpha$, from which $b = \alpha$; next

$$\mathcal{C} - \mathcal{A} = \frac{\beta - \alpha}{\alpha},$$

from which

$$c = \frac{\beta - \alpha}{\alpha^2}.$$

Again it will be

$$\mathcal{D} - \mathcal{B} = \frac{\alpha(\gamma - \beta)}{\beta},$$

from which

$$d = \frac{\alpha^2(\gamma - \beta)}{\beta^2}.$$

Then we will have

$$\mathcal{D} - \mathcal{C} = \frac{\beta(\delta - \gamma)}{\alpha\gamma},$$

and here

$$e = \frac{\beta^2(\delta - \gamma)}{\alpha^2\gamma^2}.$$

And in the same way from

$$\mathcal{F} - \mathcal{D} = \frac{\alpha\gamma(\epsilon - \delta)}{\beta\delta},$$

it will be

$$f = \frac{\alpha^2\gamma^2(\epsilon - \delta)}{\beta^2\delta^2}.$$

In the same way from

$$\mathcal{G} - \mathcal{E} = \frac{\beta\delta(\zeta - \epsilon)}{\alpha\gamma\epsilon},$$

it will be

$$g = \frac{\beta^2\delta^2(\zeta - \epsilon)}{\alpha^2\gamma^2\epsilon^2}.$$

$$etc.$$

Therefore, by this reasoning, the indices of the continued fraction, which we are seeking, are expressed in the following way:

$$\begin{array}{ll}
b = \alpha, & c = \dfrac{\beta - \alpha}{\alpha^2}, \\
d = \dfrac{\alpha^2(\gamma - \beta)}{\beta^2}, & e = \dfrac{\beta^2(\delta - \gamma)}{\alpha^2\gamma^2}, \\
f = \dfrac{\alpha^2\gamma^2(\epsilon - \delta)}{\beta^2\delta^2}, & g = \dfrac{\beta^2\delta^2(\zeta - \epsilon)}{\alpha^2\gamma^2\epsilon^2}, \\
h = \dfrac{\alpha^2\gamma^2\epsilon^2(\eta - \zeta)}{\beta^2\delta^2\zeta^2}, & i = \dfrac{\beta^2\delta^2\zeta^2(\theta - \eta)}{\alpha^2\gamma^2\epsilon^2\eta^2}
\end{array}$$

etc.

13. Therefore, it is enough to substitute these values in place of the indices b, c, d, e, f, etc in the continued fraction.

$$s = a + \cfrac{1}{b + \frac{1}{c + \frac{1}{d + \dots}}}$$

Since these values are fractions, we extract a form from the continued fraction quite easily. First let us multiply by the denominators of the discovered values, and it will be

$$\begin{array}{ll} b = \alpha, & \alpha^2 c = \beta - \alpha, \\ \beta^2 d = \alpha^2(\gamma - \beta), & \alpha^2\gamma^2 e = \beta^2(\delta - \gamma), \\ \beta^2\delta^2 f = \alpha^2\gamma^2(\epsilon - \delta), & \alpha^2\gamma^2\epsilon^2 g = \beta^2\delta^2(\zeta - \epsilon), \\ \beta^2\delta^2\zeta^2 h = \alpha^2\gamma^2\epsilon^2(\eta - \zeta), & \alpha^2\gamma^2\epsilon^2\eta^2 i = \beta^2\delta^2\zeta^2(\theta - \eta) \end{array}$$

etc.

14. Now let us transform the continued fraction in this way, so that the the appropriate formulae occur in place of the indices, the values of which we assigned here. Next, of course, let us multiply above and below by α^2; third, by β^2; fourth, by $\alpha^2\gamma^2$; fifth, by $\beta^2\delta^2$; sixth, by $\alpha^2\gamma^2\epsilon^2$, etc. so that the result is this form

$$s = \cfrac{1}{b + \cfrac{\alpha^2}{\alpha^2 c + \cfrac{\alpha^2\beta^2}{\beta^2 d + \cfrac{\alpha^2\beta^2\gamma^2}{\alpha^2\gamma^2 e + \cfrac{\alpha^2\beta^2\gamma^2\delta^2}{\beta^2\delta^2 f + etc.}}}}}$$

15. But if we substitute the above determined values in place of the new indices $\alpha^2 c$, $\beta^2 d$, $\alpha^2\gamma^2 e$, etc., this continued fraction will spring forth

$$s = \cfrac{1}{\alpha + \cfrac{\alpha^2}{\beta - \alpha + \cfrac{\alpha^2\beta^2}{\alpha^2(\gamma - \beta) + \cfrac{\alpha^2\beta^2\gamma^2}{\beta^2(\delta - \gamma) + \cfrac{\alpha^2\beta^2\gamma^2\delta^2}{\alpha^2\gamma^2(\epsilon - \delta) + etc.}}}}}$$

If we consider this form carefully, we recognize that the third fraction is able to be factored above and below by α^2, then the fourth by β^2, the fifth by γ^2, the sixth by δ^2, etc.; this continued fraction springs forth

$$s = \cfrac{1}{\alpha + \cfrac{\alpha^2}{\beta - \alpha + \cfrac{\beta^2}{\gamma - \beta + \cfrac{\gamma^2}{\delta - \gamma + \cfrac{\delta^2}{\epsilon - \delta + etc.}}}}}$$

Therefore, let us next establish

THEOREM 1

16. If such a infinite series will have been proposed

$$s = \frac{1}{\alpha} - \frac{1}{\beta} + \frac{1}{\gamma} - \frac{1}{\delta} + \frac{1}{\epsilon} - etc.,$$

a continued fraction of the form

$$\frac{1}{s} = \alpha + \cfrac{\alpha^2}{\beta - \alpha + \cfrac{\beta^2}{\gamma - \beta + \cfrac{\gamma^2}{\delta - \gamma + etc.}}}$$

can always be formed from it.

17. We elicited this reduction from the consideration of continued fractions by many round-about means. We certainly satisfied our proposition, since we transformed any series into a continued fraction. A direct method is desired, by which the continued fraction equal to a given series can be immediately derived, without these ambiguities. Therefore, such a method, will be illustrated by the theory of continued fractions extraordinarily, which I will explain here.

PROBLEM 1

18. Transform the proposed infinite series

$$s = \frac{1}{\alpha} - \frac{1}{\beta} + \frac{1}{\gamma} - \frac{1}{\delta} + \frac{1}{\epsilon} - etc.,$$

into a continued fraction.

Solution

Since,

$$s = \frac{1}{\alpha} - \frac{1}{\beta} + \frac{1}{\gamma} - \frac{1}{\delta} + \frac{1}{\epsilon} - etc.,$$

let us set

$$t = \frac{1}{\beta} - \frac{1}{\gamma} + \frac{1}{\delta} - \frac{1}{\epsilon} + etc.,$$

and

$$u = \frac{1}{\gamma} - \frac{1}{\delta} + \frac{1}{\epsilon} - etc.,$$

etc...
Therefore,

$$s = \frac{1}{\alpha} - t = \frac{1 - \alpha t}{\alpha}$$

from which it becomes

$$\frac{1}{s} = \frac{\alpha}{1 - \alpha t} = \alpha + \frac{\alpha^2 t}{1 - \alpha t}.$$

Then, it is

$$\frac{\alpha^2 t}{1 - \alpha t} = \frac{\alpha^2}{-\alpha + \frac{1}{t}},$$

from which it becomes

$$\frac{1}{s} = \alpha + \frac{\alpha^2}{-\alpha + \frac{1}{t}}.$$

Therefore, in the same way

$$\frac{1}{t} = \beta + \frac{\beta^2}{-\beta + \frac{1}{u}}$$

and

$$\frac{1}{u} = \gamma + \frac{\gamma^2}{-\gamma + \frac{1}{v}}$$

etc.,
from these substituted values, the next continued fraction will be obtained

$$\frac{1}{s} = \alpha + \cfrac{\alpha^2}{\beta - \alpha + \cfrac{\beta^2}{\gamma - \beta + \cfrac{\gamma^2}{\delta - \gamma + etc.}}}$$

which is the very form given in the theorem.

19. If the proposed series is

$$s = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + etc. = \log 2.$$

Since

$$\alpha = 1, \quad \beta = 2, \quad \gamma = 3, \quad \delta = 4, \quad \text{etc.}$$

it will be

$$\frac{1}{\log 2} = 1 + \cfrac{1 \cdot 1}{1 + \cfrac{2 \cdot 2}{1 + \cfrac{3 \cdot 3}{1 + etc.}}}.$$

However, if we take this series

$$s = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - etc. = \frac{\pi}{4}.$$

Since

$$\alpha = 1, \quad \beta = 3, \quad \gamma = 5, \quad \delta = 7, \quad \text{etc.}$$

then

$$\frac{4}{\pi} = 1 + \cfrac{1 \cdot 1}{2 + \cfrac{3 \cdot 3}{2 + \cfrac{5 \cdot 5}{2 + etc.}}}$$

which is the very continued fraction once mentioned by Brouncker.

20. Let us take

$$s=\int\frac{x^{m-1}dx}{1+x^n},$$

and after integrating, let us set $x=1$; because of this the same value of s will be expressed by the following series

$$s=\frac{1}{m}-\frac{1}{m+n}+\frac{1}{m+2n}-\frac{1}{m+3n}+etc.,$$

so that

$$\alpha=m,\quad \beta=m+n,\quad \gamma=m+2n,\quad \delta=m+3n, \text{ etc.};$$

therefore, the following continued fraction emerges here

$$\frac{1}{s}=m+\cfrac{m^2}{n+\cfrac{(m+n)^2}{n+\cfrac{(m+2n)^2}{n+etc.}}}$$

which was already given by [some reference].

21. However, if the proposed series is

$$s=\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta}+\frac{1}{\epsilon}+etc.,$$

for which all the terms are positive, it is enough work that in the above continued fraction in place of the letters β, δ, η, θ, etc. the letters $-\beta$, $-\delta$, $-\eta$, and $-\theta$, etc. are written; then it will become

$$\frac{1}{s}=\alpha+\cfrac{\alpha^2}{-\beta-\alpha+\cfrac{\beta^2}{\gamma+\beta+\cfrac{\gamma^2}{-\delta-\gamma+\cfrac{\delta^2}{\epsilon+\delta+etc.}}}}$$

which is easily transformed into this form

$$\frac{1}{s} = \alpha - \cfrac{\alpha^2}{\alpha + \beta - \cfrac{\beta^2}{\beta + \gamma - \cfrac{\gamma^2}{\gamma + \delta + etc.}}}.$$

22. The proposed series itself is able to be transformed by many methods, from which many other continued fractions are obtained directly. Let us assess carefully some forms of this kind here. Let

$$\alpha = ab, \quad \beta = bc, \quad \gamma = cd, \quad \delta = de, \text{ etc.,}$$

so that this series is obtained

$$s = \frac{1}{ab} - \frac{1}{bc} + \frac{1}{cd} - \frac{1}{de} + etc.,$$

and here this continued fractions will be formed

$$\frac{1}{s} = ab + \cfrac{a^2b^2}{b(c-a) + \cfrac{c^2d^2}{d(e-c) + etc.}}$$

which is easily reduced to the next form

$$\frac{1}{s} = ab + \cfrac{a^2b}{c - a + \cfrac{bc}{d - b + \cfrac{cd}{e - c + etc.}}}$$

or

$$\frac{1}{as} = b + \cfrac{ab}{c - a + \cfrac{bc}{d - b + \cfrac{cd}{e - c + etc.}}}$$

which is supplied by our next Theorem.

THEOREM II

23. If the proposed series is of this form

$$s = \frac{1}{ab} - \frac{1}{bc} + \frac{1}{cd} - \frac{1}{de} + \frac{1}{ef} - etc.,$$

from this, the following continued fraction springs forth

$$\frac{1}{as} = b + \cfrac{ab}{c - a + \cfrac{bc}{d - b + \cfrac{cd}{e - c + \cfrac{de}{f - d + etc.}}}}$$

24. This form, although easily derived from its predecessor, for that reason it is worth noting because it allows a continued fraction of quite a different form, from which it will be worth the trouble to adapt examples given above to this form. Since it was given

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - etc.,$$

it will be

$$\log 2 - 1 = -\frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + etc.$$

and adding these series gives

$$2\log 2 - 1 = \frac{1}{1 \cdot 2} - \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} - \frac{1}{4 \cdot 5} + \frac{1}{5 \cdot 6} - etc.$$

Therefore, here it is

$$s = 2\log 2 - 1$$

and

$$a = 1, \quad b = 2, \quad c = 3, \quad d = 4, \quad \text{etc.};$$

Therefore this continued fraction is formed

$$\frac{1}{2\log 2-1}=2+\cfrac{1\cdot 2}{2+\cfrac{2\cdot 3}{2+\cfrac{3\cdot 4}{2+etc.}}}.$$

25. In the same way, because

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-etc.,$$

it will be

$$\frac{\pi}{4}-1=\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-etc.,$$

and the sum of that series gives

$$\frac{\pi}{2}-\frac{1}{2}=\frac{2}{1\cdot 3}-\frac{2}{3\cdot 5}+\frac{2}{5\cdot 7}-\frac{2}{7\cdot 9}+etc.$$

Therefore, here it will be

$$s=\frac{\pi}{4}-\frac{1}{2},$$

then

$$a=1,\quad b=3,\quad c=5,\quad d=7,\quad \text{etc.}$$

from which the continued fraction will arise

$$\frac{4}{\pi-2}=3+\cfrac{1\cdot 3}{4+\cfrac{3\cdot 5}{4+\cfrac{5\cdot 7}{4+\cfrac{7\cdot 9}{4+etc}}}}.$$

26. Now let us consider this even more general transformation. Let the value of the integral

$$\int\frac{x^{m-1}dx}{1+x^n}$$

be denoted by Δ, after integrating set $x = 1$, and since it was as we saw above in §20,

$$\Delta = \frac{1}{m} - \frac{1}{m+n} + \frac{1}{m+2n} - etc.,$$

it will be

$$\Delta - \frac{1}{m} = -\frac{1}{m+n} + \frac{1}{m+2n} - etc.,$$

adding the series it becomes

$$2\Delta - \frac{1}{m} = \frac{n}{m(m+n)} - \frac{n}{(m+n)(m+2n)} + \frac{n}{(m+2n)(m+3n)} - etc.;$$

dividing by n it will be

$$\frac{2m\Delta - 1}{mn} = \frac{1}{m(m+n)} - \frac{1}{(m+n)(m+2n)} + \frac{1}{(m+2n)(m+3n)} - etc.$$

Therefore, here we have

$$s = \frac{2m\Delta - 1}{mn},$$

then

$$a = m, \quad b = m+n, \quad c = m+2n, \quad d = m+3n, \quad \text{etc.}$$

because of which the continued fraction will be in this form

$$\frac{n}{2m\Delta - 1} = m + n + \cfrac{m(m+n)}{2n + \cfrac{(m+n)(m+2n)}{2n + \cfrac{(m+2n)(m+3n)}{2n + \cfrac{(m+3n)(m+4n)}{2n + etc.}}}}$$

which does not have a simpler form.

27. Let us now grant to the first continued fraction

$$\frac{1}{\alpha} - \frac{1}{\beta} + \frac{1}{\gamma} - \frac{1}{\delta} + etc.$$

arbitrary numerators and let

$$s = \frac{a}{\alpha} - \frac{b}{\beta} + \frac{c}{\gamma} - \frac{d}{\delta} + etc.$$

and it is proper to write $\frac{\alpha}{a}$, $\frac{\beta}{b}$, $\frac{\gamma}{c}$, $\frac{\delta}{d}$, etc. in place of the letters α, β, γ, δ, etc. in the first Theorem. Because of this the continued fraction will be

$$\frac{1}{s} = \frac{\alpha}{a} + \cfrac{\frac{\alpha^2}{a^2}}{\frac{\beta}{b} - \frac{\alpha}{a} + \cfrac{\frac{\beta^2}{b^2}}{\frac{\gamma}{c} - \frac{\beta}{b} + \cfrac{\frac{\gamma^2}{c^2}}{\frac{\delta}{d} - \frac{\gamma}{c} + etc.}}}.$$

Now let the fraction be multiplied above and below by ab, next by bc, third by cd, and so on; then multiplying by a, this will be obtained

$$\frac{a}{s} = \alpha + \cfrac{\alpha^2 b}{a\beta - b\alpha + \cfrac{ac\beta^2}{b\gamma - c\beta + \cfrac{bd\gamma^2}{c\delta - d\gamma + etc.}}}$$

From here the following is concluded.

THEOREM III

28. If the proposed infinite series is of this form

$$s = \frac{a}{\alpha} - \frac{b}{\beta} + \frac{c}{\gamma} - \frac{d}{\delta} + etc.,$$

then the following continued fraction will be formed from it

$$\frac{a}{s} = \alpha + \cfrac{\alpha^2 b}{a\beta - b\alpha + \cfrac{ac\beta^2}{b\gamma - c\beta + \cfrac{bd\gamma^2}{c\delta - d\gamma + etc.}}}.$$

29. In order to illustrate the proposition, consider this series

$$\frac{1}{1} - \frac{2}{2} + \frac{3}{3} - \frac{4}{4} + \frac{5}{5} - etc. = \frac{1}{2},$$

so that $s = \frac{1}{2}$; then the continued fraction will be

$$2 = 1 + \cfrac{2}{0 + \cfrac{3 \cdot 4}{0 + \cfrac{8 \cdot 9}{0 + \cfrac{15 \cdot 16}{0 + etc.}}}}$$

which reduces to this infinite product

$$2 = 1 + \frac{2 \cdot 1^2 \cdot 2 \cdot 4 \cdot 3^2 \cdot 4 \cdot 6 \cdot 5^2 \cdot 6 \cdot 8 \cdot 7^2 \cdot etc.}{1 \cdot 3 \cdot 2^2 \cdot 3 \cdot 5 \cdot 4^2 \cdot 5 \cdot 7 \cdot 6^2 \cdot 7 \cdot 9 \cdot 8^2 \cdot etc.},$$

this is not easily observed to be true, since the number of factors in the numerator and denominator are not able to be set as equal, and although they both are infinite. No one is able to be doubtful, in fact, that the value of that product is $= 1$.

30. Let us consider now this series

$$s = \frac{1}{2} - \frac{2}{3} + \frac{3}{4} - \frac{4}{5} + \frac{5}{6} - etc.,$$

the sum is $s = \log 2 - \frac{1}{2}$. Because

$$\begin{array}{lllll} a = 1, & b = 2, & c = 3, & d = 4, & \text{etc.} \\ \alpha = 2, & \beta = 3, & \gamma = 4, & \delta = 5, & \text{etc.,} \end{array}$$

the continued fraction will be

$$\frac{1}{\log 2 - \frac{1}{2}} = 2 + \cfrac{1 \cdot 2 \cdot 2^2}{-1 + \cfrac{1 \cdot 3 \cdot 3^2}{-1 + \cfrac{2 \cdot 4 \cdot 4^2}{-1 + \cfrac{3 \cdot 5 \cdot 5^2}{-1 + etc.}}}}$$

31. If we take this series

$$s = \frac{2}{1} - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + \frac{6}{5} - etc.,$$

the value of which is $\frac{1}{2} + \log 2$, we will have

$$\begin{array}{lllll} a = 2, & b = 3, & c = 4, & d = 5, & \text{etc.} \\ \alpha = 1, & \beta = 2, & \gamma = 3, & \delta = 4, & \text{etc.,} \end{array}$$

here the continued fraction emerges

$$\frac{2}{\frac{1}{2} + \log 2} = 1 + \cfrac{1 \cdot 3 \cdot 1^2}{1 + \cfrac{2 \cdot 4 \cdot 2^2}{1 + \cfrac{3 \cdot 5 \cdot 3^2}{1 + \cfrac{4 \cdot 6 \cdot 4^2}{1 + etc.}}}}$$

or

$$\frac{4}{2 \log 2 + \frac{1}{2}} = 1 + \cfrac{1^3 \cdot 3}{1 + \cfrac{2^3 \cdot 4}{1 + \cfrac{3^3 \cdot 5}{1 + etc.}}}$$

PROBLEM II

Transform this infinite series

$$s = \frac{x}{\alpha} - \frac{x^2}{\beta} + \frac{x^3}{\gamma} - \frac{x^4}{\delta} + etc.$$

into a continued fraction.

32. Consider the following series, formed from the proposed series:

$$t = \frac{x}{\beta} - \frac{x^2}{\gamma} + \frac{x^3}{\delta} - \frac{x^4}{\epsilon} + etc.$$

and continuing

$$u = \frac{x}{\gamma} - \frac{x^2}{\delta} + \frac{x^3}{\epsilon} - \frac{x^4}{\zeta} + etc.$$

$$v = \frac{x}{\delta} - \frac{x^2}{\epsilon} + \frac{x^3}{\zeta} - \frac{x^4}{\eta} + etc.$$

etc.

and it will be

$$s = \frac{x}{\alpha} - tx = \frac{x(1-\alpha t)}{\alpha};$$

from which it becomes

$$\frac{x}{s} = \frac{\alpha}{1-\alpha t} = \alpha + \frac{\alpha^2 t}{1-\alpha t} = \alpha + \frac{\alpha^2}{-\alpha + \frac{1}{t}}.$$

Therefore, here it will be

$$\frac{x}{s} = \alpha + \frac{\alpha^2 x}{-\alpha x + \frac{x}{t}};$$

and in the same way it will be

$$\frac{x}{t} = \beta + \frac{\beta^2 x}{-\beta x + \frac{x}{u}}.$$

Therefore, if we substitute all these values in order, this continued fraction emerges

$$\frac{x}{s} = \alpha + \cfrac{\alpha^2 x}{\beta - \alpha x + \cfrac{\beta^2 x}{\gamma - \beta x + \cfrac{\gamma x}{\delta - \gamma x + etc.}}}.$$

33. If we write $\frac{x}{y}$ in place of x here, so that we have this series

$$s = \frac{x}{\alpha y} - \frac{x^2}{\beta y^2} + \frac{x^3}{\gamma y^3} - \frac{x^4}{\delta y^4} + etc.$$

then the continued fraction will be

$$\frac{x}{sy} = \alpha + \cfrac{\alpha^2 \frac{x}{y}}{\beta - \frac{\alpha x}{y} + \cfrac{\beta^2 \frac{x}{y}}{y - \frac{\beta x}{y} + etc.}}$$

which gives one free from the fractions in the numerators and denominotors

$$\frac{x}{sy} = \alpha + \cfrac{\alpha^2 x}{\beta y - \alpha x + \cfrac{\beta^2 xy}{\gamma y - \beta x + \cfrac{\gamma^2 xy}{\delta y - \beta x + etc.}}},$$

from which the next part emerges.

THEOREM IV

34. If the infinite series will have been of this form

$$s = \frac{x}{\alpha y} - \frac{x^2}{\beta y^2} + \frac{x^3}{\gamma y^3} - \frac{x^4}{\delta y^4} + etc.,$$

it is possible to form the following continued fraction from this series

$$\frac{x}{s} = \alpha y + \cfrac{\alpha^2 xy}{\beta y - \alpha x + \cfrac{\beta^2 xy}{\gamma y - \beta x + \cfrac{\gamma^2 xy}{\delta y - \gamma x + \cfrac{\delta^2 xy}{\epsilon y - \delta x + etc.}}}}.$$

35. Since

$$\log{(1+\frac{x}{y})} = \frac{x}{y} - \frac{x^2}{2y^2} + \frac{x^3}{3y^3} - \frac{x^4}{4y^4} + etc.,$$

having set

$$s = \log{(1+\frac{x}{y})}$$

it will be $\alpha = 1$, $\beta = 2$, $\gamma = 3$, $\delta = 4$, etc.
and hence this continued fraction emerges

$$\frac{x}{\log{(1+\frac{x}{y})}} = y + \cfrac{xy}{2y - x + \cfrac{4xy}{3y - 2x + \cfrac{9xy}{4y - 3x + etc.}}}.$$

36. Since an arc, with tangent t, is expressed by this series

$$\arctan t = t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \frac{t^9}{9} - etc.,$$

and it will be

$$t \arctan t = \frac{t^2}{1} - \frac{t^4}{3} + \frac{t^6}{5} - \frac{t^8}{7} + \frac{t^{10}}{9} - etc.$$

Now letting $t^2 = \frac{x}{y}$, so that $t = \sqrt{\frac{x}{y}}$, and it becomes

$$\sqrt{\frac{x}{y}} A \tan \sqrt{\frac{x}{y}} = \frac{x}{y} - \frac{x^4}{3y^2} + \frac{x^3}{5y^3} - \frac{x^4}{7y^4} + etc.$$

Therefore, here

$$s = \sqrt{\frac{x}{y}} A \tan \sqrt{\frac{x}{y}},$$

then

$$\alpha = 1, \quad \beta = 3, \quad \gamma = 5, \quad \delta = 7, \quad \text{etc.};$$

from which the continued fraction will emerge

$$\frac{\sqrt{xy}}{A \tan \sqrt{\frac{x}{y}}} = y + \cfrac{xy}{3y - x + \cfrac{9xy}{5y - 3x + \cfrac{25xy}{7y - 5x + etc.}}}.$$

Just as if $x = 1$ and $y = 3$,

$$Atan \frac{1}{\sqrt{3}} = \frac{\pi}{6}$$

this continued fraction will be

$$\frac{6\sqrt{3}}{\pi} = 3 + \cfrac{1 \cdot 3}{8 + \cfrac{3 \cdot 9}{12 + \cfrac{3 \cdot 25}{16 + etc.}}}.$$

37. If we write in place of the letters α, β, γ, δ, etc. in the theorem, the fractions

$$\tfrac{\alpha}{a}, \quad \tfrac{\beta}{b}, \quad \tfrac{\gamma}{c}, \quad \tfrac{\delta}{d}, \quad \text{etc.}$$

so that we have this series

$$s = \frac{ax}{\alpha y} - \frac{bx^2}{\beta y^2} + \frac{cx^2}{\gamma y^2} - \frac{dx^4}{\delta y^4} + etc.,$$

so the continued fraction takes this form

$$\frac{x}{s}=\frac{\alpha}{a}y+\cfrac{\alpha^2\frac{xy}{a^2}}{\frac{\beta}{b}y-\frac{\alpha}{a}x+\cfrac{\beta^2\frac{xy}{b^2}}{\frac{\gamma}{c}y-\frac{\beta}{b}x+\cfrac{\gamma^2\frac{xy}{c^2}}{\frac{\delta}{d}y-\frac{\gamma}{c}x+etc.}}}.$$

Here, it is first multiplied by a, then the numerator and denominator of the first fraction are multiplied by ab, then by bc, third by cd, etc. and the continued fraction takes on this form

$$\frac{ax}{s}=ay+\cfrac{\alpha^2bxy}{\alpha\beta y-b\alpha x+\cfrac{\beta acxy}{b\gamma y-c\beta x+\cfrac{\gamma^2bdxy}{c\delta y-d\gamma x+etc.}}}$$

The reward of this work will be given next.

Theorem V

38. If the proposed infinite series will be of this form

$$s=\frac{ax}{\alpha y}-\frac{bx^2}{\beta y^2}+\frac{cx^3}{\gamma y^3}-\frac{dx^4}{\delta y^4}+etc.$$

the following continued fraction will be formed from it

$$\frac{ax}{s}=ay+\cfrac{\alpha^2bxy}{\alpha\beta y-b\alpha x+\cfrac{\beta^2acxy}{b\gamma y-c\beta x+\cfrac{\gamma^2bdxy}{c\delta y-d\gamma x+etc.}}}$$

Problem III

Convert this proposed infinite series

$$s=\frac{1}{\alpha}-\frac{1}{\alpha\beta}+\frac{1}{\alpha\beta\gamma}-\frac{1}{\alpha\beta\gamma}+etc.$$

into a continued fraction.

Solution

39. From the proposed series we form the following series

$$t = \frac{1}{\beta} - \frac{1}{\beta\gamma} + \frac{1}{\beta\gamma\delta} - \frac{1}{\beta\gamma\delta\epsilon} + etc.$$

$$u = \frac{1}{\gamma} - \frac{1}{\gamma\delta} + \frac{1}{\gamma\delta\epsilon} - \frac{1}{\gamma\delta\epsilon\zeta} + etc.$$

etc.
and we will have

$$s = \tfrac{1-t}{\alpha}, \quad t = \tfrac{1-u}{\beta}, \quad u = \tfrac{1-v}{\gamma}, \quad \text{etc.}$$

therefore, we deduce

$$\frac{1}{s} = \frac{\alpha}{1-t} = \alpha + \frac{\alpha t}{1-t} = \alpha + \frac{\alpha}{-1+\frac{1}{t}}.$$

In the same way it will be

$$\frac{1}{t} = \beta + \cfrac{\beta}{-1 + \cfrac{1}{u}}, \quad \frac{1}{u} = \gamma + \cfrac{\gamma}{-1 + \cfrac{1}{v}}, \quad \text{etc.}$$

from which we obtain this continued fraction by substituting each expression into the previous one

$$\frac{1}{s} = \alpha + \cfrac{\alpha}{\beta - 1 + \cfrac{\beta}{\gamma - 1 + \cfrac{\gamma}{\delta - 1 + etc.}}}$$

from which we deduce the next theorem.

Theorem VI

40. If the proposed infinite series will be of this type

$$s = \frac{1}{\alpha} - \frac{a}{\alpha\beta} + \frac{1}{\alpha\beta\gamma} - \frac{1}{\alpha\beta\gamma} + etc.$$

then it is possible to form this continued fraction

$$\frac{1}{s} = \alpha + \cfrac{\alpha}{\beta - 1 + \cfrac{\beta}{\gamma - 1 + \cfrac{\gamma}{\delta - 1 + etc.}}}$$

41. If e denotes the number whose logarithm is one, then it is known that

$$\frac{1}{e} = 1 - \frac{1}{1} + \frac{1}{1 \cdot 2} - \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} - etc.$$

or

$$\frac{e-1}{e} = \frac{1}{1} - \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} - \frac{1}{a \cdot 2 \cdot 3 \cdot 4} + etc.$$

Therefore, here $s = \frac{e-1}{e}$, then

$$\alpha = 1, \quad \beta = 2, \quad \gamma = 3, \quad \delta = 4, \quad \text{etc.;}$$

from which this continued fraction arises

$$\frac{e}{e-1} = 1 + \cfrac{1}{1 + \cfrac{2}{2 + \cfrac{3}{3 + etc.}}}$$

42. Then

$$\cfrac{1}{1 + \cfrac{2}{2 + \cfrac{3}{3 + etc.}}} = \frac{1}{e-1}$$

since we are able to demonstrate without much difficulty, that if

$$\cfrac{a}{a + \cfrac{b}{b + \cfrac{c}{c + etc.}}} = s$$

then

$$a + \cfrac{a}{a + \cfrac{b}{b + \cfrac{c}{c + etc.}}} = \frac{s}{1-s},$$

and in this case it is

$$s = \tfrac{1}{e-1}, \quad a = 1, \quad b = 2, \quad c = 3, \quad \text{etc.}$$

it becomes

$$1 + \cfrac{1}{1 + \cfrac{2}{2 + \cfrac{3}{3 + \cfrac{3}{4 + etc.}}}} = \frac{1}{e-2}.$$

43. If in place of the letters α, β, γ, δ, etc. in the series in Theorem VI, the fractions $\frac{\alpha}{a}$, $\frac{\beta}{b}$, $\frac{\gamma}{c}$, $\frac{\delta}{d}$, etc. are written, so that

$$s = \frac{a}{\alpha} - \frac{ab}{\alpha\beta} + \frac{abc}{\alpha\beta\gamma} - \frac{abcd}{\alpha\beta\gamma\delta} + etc.,$$

here the continued fraction will be

$$\frac{1}{s} = \frac{\alpha}{a} + \cfrac{\frac{\alpha}{a}}{\frac{\beta}{b} - 1 + \cfrac{\frac{\beta}{b}}{\frac{\gamma}{c} - 1 + \cfrac{\frac{\gamma}{c}}{\frac{\delta}{d} - 1 + etc.}}}$$

If it is first multiplied by a, then the first fraction is multiplied above and below by b, the second by c, the third by d, etc., this form arises

$$\frac{a}{s} = \alpha + \cfrac{\alpha b}{\beta - b + \cfrac{\beta c}{\gamma - c + \cfrac{\gamma d}{\delta - d + etc.}}}$$

which is included in the next theorem.

Theorem VII

44. If the proposed infinite series will be of the form

$$s = \frac{a}{\alpha} - \frac{ab}{\alpha\beta} + \frac{abc}{\alpha\beta\gamma} - \frac{abcd}{\alpha\beta\gamma\delta} + etc.,$$

then this continued fraction arises

$$\frac{a}{s} = \alpha + \cfrac{\alpha b}{\beta - b + \cfrac{\beta c}{\gamma - c + \cfrac{\gamma d}{\delta - d + etc.}}}$$

45. Let us apply this to the next infinite series

$$s = \frac{1}{2} - \frac{1 \cdot 3}{2 \cdot 4} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} - \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} + etc.,$$

the sum of which is $s = \frac{\sqrt{2}-1}{\sqrt{2}}$; since

$$\begin{array}{lllll} a = 1 & b = 3 & c = 5 & d = 7 & \text{etc.} \\ \alpha = 2 & \beta = 4 & \gamma = 6 & \delta = 8 & \text{etc.;} \end{array}$$

therefore the continued fraction here will be

$$\frac{\sqrt{2}}{\sqrt{2}-1} = 2 + \cfrac{2 \cdot 3}{1 + \cfrac{4 \cdot 5}{1 + \cfrac{6 \cdot 7}{1 + etc.}}}.$$

So subtracting one it will be

$$\frac{1}{\sqrt{2}-1} = 1 + \cfrac{2 \cdot 3}{1 + \cfrac{4 \cdot 5}{1 + \cfrac{6 \cdot 7}{1 + etc.}}},$$

which produces

$$\sqrt{2} = 1 + \cfrac{1 \cdot 1}{1 + \cfrac{2 \cdot 3}{1 + \cfrac{4 \cdot 5}{1 + \cfrac{6 \cdot 7}{1 + etc.}}}}.$$

Problem IV

Convert the proposed infinite series of this form

$$s = \frac{x}{\alpha} - \frac{x^2}{\alpha\beta} + \frac{x^3}{\alpha\beta\gamma} - \frac{x^4}{\alpha\beta\gamma\delta} + etc.$$

into a continued fraction.

Solution

Let us state that

$$t = \frac{x}{\beta} - \frac{x^2}{\beta\gamma} + \frac{x^3}{\beta\gamma\delta} - \frac{x^4}{\beta\gamma\delta\epsilon} + etc.$$

and

$$u = \frac{x}{\gamma} - \frac{x^2}{\gamma\delta} + \frac{x^3}{\gamma\delta\epsilon} - \frac{x^4}{\gamma\delta\epsilon\zeta} + etc.$$

so that

$$s = \frac{x - tx}{\alpha};$$

and then

$$\frac{x}{s} = \frac{\alpha}{1-t} = \alpha + \frac{\alpha t}{1-t}.$$

So it is

$$\frac{\alpha t}{1-t} = \frac{\alpha}{-1 + \frac{1}{t}} = \frac{\alpha x}{-x + \frac{x}{t}}$$

and it will be

$$\frac{x}{s} = \alpha + \frac{\alpha x}{-x + \frac{x}{t}}.$$

In the same way it will be discovered that

$$\frac{x}{t} = \beta + \frac{\beta x}{-x + \frac{x}{u}},$$

$$\frac{x}{u} = \gamma + \frac{\gamma x}{-x + \frac{x}{v}}, etc.$$

If each of these values are substituted in turn into the preceding fraction, the following continued fraction will be obtained

$$\frac{x}{s} = \alpha + \cfrac{\alpha x}{\beta - x + \cfrac{\beta x}{\gamma - x + \cfrac{\gamma x}{\delta - x + etc.}}}$$

and from this arises

Theorem VIII

47. If the proposed infinite series will be of this kind

$$s = \frac{x}{\alpha} - \frac{x^2}{\alpha\beta} + \frac{x^3}{\alpha\beta\gamma} - \frac{x^4}{\alpha\beta\gamma\delta} + etc.$$

then the next continued fraction will be formed from it

$$\frac{x}{s} = \alpha + \cfrac{\alpha x}{\beta - x + \cfrac{\beta x}{\gamma - x + \cfrac{\gamma x}{\delta - x + etc.}}}$$

48. If in place of x we will write $\frac{x}{y}$, so that we have this infinite series

$$s = \frac{x}{\alpha y} - \frac{x^2}{\alpha\beta y^2} + \frac{x^3}{\alpha\beta\gamma y^3} - \frac{x^4}{\alpha\beta\gamma\delta y^4} + etc.$$

the next continued fraction arises

$$\frac{x}{s} = \alpha y + \cfrac{\alpha xy}{\beta y - x + \cfrac{\beta xy}{\gamma y - x + \cfrac{\gamma xy}{\delta y - x + etc.}}}$$

49. Let us suppose $\alpha = 1, \quad \beta = 2, \quad \gamma = 3, \quad \delta = 4,$ etc. so that

$$s = \frac{x}{y} - \frac{x^2}{1 \cdot 2 \cdot y^2} + \frac{x^3}{1 \cdot 2 \cdot 3y^3} - \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4y^4} + etc.$$

where

$$s = 1 - e^{\frac{-x}{y}},$$

and here this continued fraction is formed

$$\frac{x}{1 - e^{\frac{-x}{y}}} = y + \cfrac{xy}{2y - x + \cfrac{2xy}{3y - x + \cfrac{3xy}{4y - x + etc.}}} = \frac{xe^{\frac{x}{y}}}{e^{\frac{x}{y}} - 1}.$$

The next special formulas will be obtained from this assuming $x = 1$, and the numbers $1, 2, 3, 4, 5$, etc. in succession in place of y:

$$\frac{e}{e-1} = 1 + \cfrac{1}{1 + \cfrac{2}{2 + \cfrac{3}{3 + etc.}}}$$

$$\frac{\sqrt{e}}{\sqrt{e}-1} = 2 + \cfrac{2}{3 + \cfrac{4}{5 + \cfrac{6}{7 + etc.}}}$$

$$\frac{\sqrt[3]{e}}{\sqrt[3]{e}-1} = 3 + \cfrac{3}{5 + \cfrac{6}{8 + \cfrac{9}{11 + etc.}}}$$

$$\frac{\sqrt[4]{e}}{\sqrt[4]{e}-1} = 4 + \cfrac{4}{7 + \cfrac{8}{11 + \cfrac{12}{15 + etc.}}}$$

etc.

Problem V

If the proposed series will be of this form

$$s = \frac{ax}{\alpha y} - \frac{abx^2}{\alpha\beta y^2} + \frac{abcx^3}{\alpha\beta\gamma y^3} - \frac{abcdx^4}{\alpha\beta\gamma\delta y^4} + etc.,$$

convert this into a continued fraction.

Solution

50. From the proposed series let us next form

$$t = \frac{bx}{\beta y} - \frac{bcx^2}{\beta\gamma y^2} + \frac{bcdx^3}{\beta\gamma\delta y^3} - \frac{bcdex^4}{\beta\gamma\delta\epsilon y^4} + etc.,$$

$$u = \frac{cx}{\gamma y} - \frac{cdx^2}{\gamma\delta y^2} + \frac{cdex^3}{\gamma\delta\epsilon y^3} - \frac{cdefx^4}{\gamma\delta\epsilon\zeta y^4} + etc.,$$

so that

$$s = \frac{ax}{\alpha y}(1-t)$$

and here

$$\frac{ax}{s} = \frac{\alpha y}{1-t} = \alpha y + \frac{\alpha y t}{1-t}.$$

It is

$$\frac{\alpha y t}{1-t} = \frac{\alpha y}{-1+\frac{1}{t}} = \frac{abxy}{-bx+\frac{bx}{t}},$$

from which it becomes

$$\frac{ax}{s} = \alpha y + \frac{abxy}{-bx+\frac{bx}{t}};$$

in the same way from the relation

$$t = \frac{bx}{\beta y}(1-u)$$

it becomes

$$\frac{bx}{t} = \beta y = \frac{\beta cxy}{-cx+\frac{cx}{u}}$$

and so again. Having substituted these values, this continued fraction is produced

$$\frac{ax}{s} = \alpha y + \cfrac{\alpha bxy}{\beta y - bx + \cfrac{\beta cxy}{\gamma y - cx + \cfrac{\gamma dxy}{\delta y - dx + etc.}}}.$$

THEOREM IX

51. If the proposed series will be

$$s = \frac{ax}{\alpha y} - \frac{abx^2}{\alpha\beta y^2} + \frac{abcx^3}{\alpha\beta\gamma y^3} - \frac{abcdx^4}{\alpha\beta\gamma\delta y^4} + etc.,$$

then the continued fraction

$$\frac{ax}{s} = \alpha y + \cfrac{\alpha bxy}{\beta y - bx + \cfrac{\beta cxy}{\gamma y - cx + \cfrac{\gamma dxy}{\delta y - dx + etc.}}}.$$

will be formed.

52. Let us give an example from this theorem worth noting. Let us consider this integral formula:

$$Z = \int z^{m-1} dz (1+z^n)^{\frac{k}{n}-1}.$$

It is supposed that the integral vanishes when $z = 0$, and we put

$$Z = v(1+z^n)^{\frac{k}{n}}$$

and differentiating it is

$$dZ = z^{m-1}dz(1+z^n)^{\frac{k}{n}-1} = dv(1+z^n)^{\frac{k}{n}} + kvz^{n-1}dz(1+z^n)^{\frac{k}{n}-1},$$

which divided by $(1+z^n)^{\frac{k}{n}-1}$ is

$$z^{m-1}dz = dv(1+z^n) + kvz^{n-1}dz$$

and, therefore,

$$\frac{dv}{dz}(1+z^n) + kvz^{n-1} - z^{m-1} = 0.$$

53. Since z is assumed to be very small, it becomes

$$Z = \frac{z^m}{m} = v,$$

then we seek to express the quantity v by an infinite series whose the first term is the power z^m, the exponents of the terms in the sequence increase by the number n; let us now find the infinite series for v

$$v = Az^m - Bz^{m+n} + Cz^{m+2n} - Dz^{m+3m} + etc,$$

let us substitute that value into the differential equation and write the same powers of z in the following way

$$\begin{array}{rcccccccccl}
\frac{dv}{dz} & = & mAz^{m-1} & - & (m+n)Bz^{m+n-1} & + & (m+2n)Cz^{m+2n-1} & - & (m+3n)Dz^{m+3m-1} & & \text{+etc.} \\
\frac{z^n dv}{dz} & = & & + & mA & - & (m+n)B & + & (m+2n)C & & \text{-etc.} \\
+kvz^{n-1} & = & & + & kA & - & kB & + & kC & & \text{-etc} \\
-z^{m-1} & = & -1 & & & & & & & &
\end{array}$$

54. Now, if the individual powers of each z are reduced to zero, the next values will be obtained:

$$\begin{aligned}
mA - 1 = 0, &\quad \text{therefore} \quad A = \frac{1}{m}, \\
-(m+n)B + (m+k)A = 0, &\quad \text{therefore} \quad B = \frac{(m+k)A}{m+n}, \\
(m+2n)C - (m+n+k)B = 0, &\quad \text{therefore} \quad C = \frac{(m+n+k)B}{m+2n}, \\
-(m+3n)D + (m+2n+k)C = 0, &\quad \text{therefore} \quad D = \frac{(m+2n+k)C}{m+3n}, \\
\text{etc.} &\quad \qquad\qquad\quad \text{etc.}
\end{aligned}$$

55.Therefore, let us substitute these obtained values and now find the infinite series for v:

$$v = \frac{1}{m}z^m - \frac{m+k}{m(m+n)}z^{m+n} + \frac{(m+k)(m+n+k)}{m(m+n)(m+2n)}z^{m+2n}$$

$$-\frac{(m+k)(m+n+k)(m+2n+k)}{m(m+n)(m+2n)(m+3n)}z^{m+3n} + etc.,$$

Let us reduce this series to the form of our theorem, let us represent the series in this way

$$v = \frac{z^{m-n}}{m}\{z^n - \frac{m+k}{m+n}z^{2n} + \frac{(m+k)(m+n+k)}{(m+n)(m+2n)}z^{3n} - \frac{(m+k)(m+n+k)(m+2n+k)}{(m+n)(m+2n)(m+3n)}z^{4n} + etc.\}$$

56. Since

$$Z = \int z^{m-1}dz(1+z^n)^{\frac{k}{n}-1},$$

let us set

$$V = \frac{mZ}{z^{m-n}(1+z)^{\frac{k}{n}}},$$

so that

$$V = z^n - \frac{m+k}{m+n}z^{2n} + \frac{(m+k)(m+n+k)}{(m+n)(m+2n)}z^{3n} - \frac{(m+k)(m+n+k)(m+2n+k)}{(m+n)(m+2n)(m+3n)}z^{4n} + etc.,$$

therefore, the function V of z must annihilate the differential form by integration, which for whatever value of z, the determined value is found. So we are able to evaluate the integral so that it vanishes when $z = 0$. We are able to assign the values to v more easily when the value of the variable z is a fraction, let us assume in general

$$z^n = \frac{x}{y},$$

so that this form arises

$$V = \frac{x}{y} = \frac{(m+k)x^2}{(m+n)y^2} + \frac{(m+k)(m+n+k)x^2}{(m+n)(m+2n)y^2} - \frac{(m+k)(m+n+k)(m+2n+k)x^4}{(m+n)(m+2n)(m+3n)y^4} + etc,$$

which produces the series discussed in our theorem when $s = V$, then

$$a = 1, \quad b = m+k, \quad c = m+n+k, \quad \text{etc.},$$
$$\alpha = 1, \quad \beta = m+n, \quad \gamma = m+2n, \quad \text{etc.}$$

57. Now from this integral the formula supplies us with the continued fraction:

$$\frac{x}{V} = y + \cfrac{(m+k)xy}{(m+n)y - (m+k)x + \cfrac{(m+n)(m+n+k)xy}{(m+2n)y - (m+n+k)x + \cfrac{(m+2n)(m+2n+k)xy}{(m+3n)y - (m+2n+k)x + etc.}}}$$

the value of it is

$$\frac{xz^{m-n}(1+z^n)^{\frac{k}{n}}}{mZ}.$$

58. Let us consider this form for the sake of an example

$$Z = \int \frac{dz}{\sqrt{1+z^2}} = \log(z + \sqrt{1+z^2});$$

therefore, $m = 1, \quad n = 2, \quad k = 1$, and it will become

$$V = \frac{z\log(z+\sqrt{1+z^2})}{\sqrt{1+z^2}},$$

the value of which is equal to this series

$$z^2 - \frac{2}{3}z^4 + \frac{2\cdot 4}{3\cdot 5}z^6 - \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}z^8 + etc.,$$

if we write $\frac{x}{y}$ in place of z^2, it becomes

$$V = \frac{\sqrt{x}}{\sqrt{x+y}}\log(\frac{\sqrt{x}+\sqrt{x+y}}{\sqrt{y}}) = \frac{x}{y} - \frac{2}{3}\cdot\frac{x^2}{y^2} + \frac{2\cdot 4}{3\cdot 5}\frac{x^3}{y^3} - \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\frac{x^4}{y^4} + etc.;$$

this form will be the continued fraction

$$\frac{\sqrt{x}(x+y)}{\log(\frac{\sqrt{x}+\sqrt{x+y}}{\sqrt{y}})} = y + \cfrac{1\cdot 2xy}{3y - 2x + \cfrac{3\cdot 4xy}{5y - 4x + \cfrac{5\cdot 6xy}{7y - 6x + \cfrac{7\cdot 8xy}{9y - 8x + etc.}}}}$$

59. Therefore, if we let $x = 1$ and $y = 1$, we will have this continued fraction

$$\frac{\sqrt{2}}{\log((1+\sqrt{2})} = 1 + \cfrac{1\cdot 2}{1 + \cfrac{3\cdot 4}{1 + \cfrac{5\cdot 6}{1 + etc.}}}$$

the same value appearing as an infinite series is

$$\frac{\log(1+\sqrt{2})}{\sqrt{2}} = 1 - \frac{2}{3} + \frac{2\cdot 4}{3\cdot 5} - \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7} + \frac{2\cdot 4\cdot 6\cdot 8}{3\cdot 5\cdot 7\cdot 9} - etc.$$

But allowing $x = 1$ to remain, and letting $y = 2$, the infinite series will be

$$\frac{1}{2} - \frac{2}{3}\cdot\frac{1}{4} + \frac{2\cdot 4}{3\cdot 5}\cdot\frac{1}{8} - \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\cdot\frac{1}{16} + etc.,$$

and the continued fraction will be

$$\frac{\sqrt{3}}{\log(\frac{1+\sqrt{3}}{\sqrt{2}})} = 2 + \cfrac{1\cdot 2\cdot 2}{4 + \cfrac{3\cdot 4\cdot 2}{6 + \cfrac{5\cdot 6\cdot 2}{8 + etc.}}}.$$

The next form is deduced from it

$$\frac{\sqrt{3}}{2\log(\frac{1+\sqrt{3}}{\sqrt{2}})} = 1 + \cfrac{1}{2 + \cfrac{6}{3 + \cfrac{15}{4 + \cfrac{28}{5 + etc.}}}}$$

where the numerators are alternating triangular numbers.

60. We come to this case, in which $x = 1$ and $y = 3$, since irrational numbers arise by this; in this case it will be

$$\frac{1}{4}\log 3 = \frac{1}{3} - \frac{2}{3}\cdot\frac{1}{9} + \frac{2\cdot 4}{3\cdot 5}\cdot\frac{1}{27} - \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\cdot\frac{1}{81} + etc.;$$

the continued fraction will be

$$\frac{4}{\log 3} = 3 + \cfrac{1\cdot 2\cdot 3}{7 + \cfrac{3\cdot 4\cdot 3}{11 + \cfrac{5\cdot 6\cdot 3}{15 + \cfrac{7\cdot 8\cdot 3}{19 + etc.}}}}.$$

61. Therefore, since I plainly explained this new method of transforming any infinite series into a continued fraction, I rightly seem to me to have enriched the study of continued fractions by no little means. Therefore, may I unite the theorems having observed them worthy, by the fraction in the above §42

$$\cfrac{1}{1 + \cfrac{2}{2 + \cfrac{3}{3 + \cfrac{4}{4 + etc.}}}} = \frac{1}{e-1}$$

let us transform it into this

$$1 + \cfrac{1}{2 + \cfrac{2}{3 + \cfrac{3}{4 + etc.}}} = \frac{1}{e-2}.$$

Theorem X

62. If

$$s = \cfrac{aA}{\alpha A + \cfrac{bB}{\beta B + \cfrac{cC}{\gamma C + etc.}}}$$

then

$$\frac{bs}{a - \alpha s} = \beta A + \cfrac{cA}{\gamma B + \cfrac{dB}{\delta C + \cfrac{eC}{\epsilon D + etc.}}}.$$

Demonstration

Since

$$s = \cfrac{aA}{\alpha A + \cfrac{bB}{\beta B + \cfrac{cC}{\gamma C + etc.}}}$$

If we divide the first fraction by A, the next by B, the third by C, etc, it produces

$$s = \cfrac{a}{\alpha + \cfrac{\frac{b}{A}}{\beta + \cfrac{\frac{c}{B}}{\gamma + \cfrac{\frac{d}{C}}{\delta + etc.}}}}.$$

Now, we multiply the second fraction of the form above and below by A, the third by B, the forth by C, and continuing so we will obtain this form

$$\cfrac{a}{\alpha + \cfrac{b}{\beta A + \cfrac{cA}{\gamma B + \cfrac{dB}{\delta C + etc.}}}}.$$

If we let

$$t = \beta A + \cfrac{cA}{\gamma B + \cfrac{dB}{\delta C + etc.}}$$

then

$$s = \frac{a}{\alpha + \frac{b}{t}} = \frac{at}{\alpha t + b},$$

from which is found

$$t = \frac{bs}{a - \alpha s}.$$

q. e. d.