Gaming the Law of Large Numbers

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Many of us view mathematics as a rich and wonderfully elaborate game. In turn, games can be used to illustrate mathematical ideas. In this article we discuss an adaptation of the game *Liar's Dice* that we will call *Fibber's Dice*. Not only is *Fibber's Dice* a fast-paced game which rewards gutsy moves and favors the underdog, it also brings to life concepts arising in the study of probability. In particular, Fibber's dice nicely shows the connection between counting, probabilities, relative frequency, and the law of large numbers.

The Rules of Fibber's Dice

We developed the rules for *Fibber's Dice* by directly adapting the rules of *Liar's Dice* found at [3]. These modifications make the game more suitable for the classroom.

Setup and object of the game Each player starts with a cup and six dice. The winner is the first player to lose all of their dice through bidding, bluffing, and luck.

Game play Each round starts with the players rolling their dice while using the cup to conceal the values. After each player examines their own dice, the first player makes a bid. This bid should represent an estimate of the total number of dice showing a particular value under all the cups on the table. However, there are two catches:

- All ones are wild and are counted as the value of whichever bid is made.
- You cannot bid on the number of ones showing.

Thus a call of "7 fours" is based on a prediction that there will be a total of at least 7 dice with a value of either four or one. Bids continue to the left increasing each time. There are two ways to increase the bid:

- One can increase the number of dice with any face value, two through six.
- One can keep the same number of dice and increase the face value.

So a bid of “7 fours” can be followed by, “7 fives” or by “8 twos,” but not by “6 sixes” or by “7 twos.”

Eventually, the bid of a previous player will seem too high to be representative of the dice on the table. At this point, instead of bidding, the current player says “No way!”

All players then uncover their dice and the relevant dice are counted and added to the number of ones revealed. If the call was, “7 twos” and there are fewer than 7 dice showing either two or a one, then the player who said “No way!” loses one die. If there are 7 (or more) twos and ones showing, then the player who made the bid loses a die. This die is permanently taken out of the game.

The next round begins with the player to the left of the player who lost the die making the opening bid. Play continues until a player loses all of his or her dice, thus winning the game.

In the classroom

Before playing Fibber’s Dice, we introduce some basic notions of probability, including definitions and examples. Much of our discussion involves coin flipping and dice. We end this class session with the question “If I flip a coin ten times in a row and eight of the outcomes are heads, is there something wrong with the coin?” This leads the students into a discussion of probability versus observed outcomes. As we shall see, this motivates playing Fibber’s Dice.

We usually spend two 50-minute periods playing this game. On the first day, we start by breaking the students into groups of four or five and give them 5–10 minutes to read over the rules. Next we play a quick example round as a class. Once the students understand the rules, they play the game in their groups. During this time we walk around checking to see that the game is understood and we start to ask leading questions about their strategies. Here are some examples:

- How many dice are on the table? How might knowing this be important?
- How many dice of a particular value do you expect to see?
- At this point, what would be a safe bid or good bid? Will good and safe be the same for all players?
- Does your bid depend on the previous bids? How?

These questions generally lead the students toward strategies involving probabilities rather than blind guessing. We end the first day by giving the students a homework assignment of writing up a brief description of their strategy.

On the second day, we ask the students about the strategies they wrote up. In particular, we use questions like the following to lead the discussion.

- What aspects of the game did you consider in your strategy?

- Is the opening bid different from other bids? How? Is this difference important for your strategy?
- Do previous bids matter in your strategy? How might you use this information?
- How do the dice in your cup affect your bid?

During this discussion, we listen to the students and work to guide them to ideas connected to probability. We take care to help the students use appropriate terminology, and lead them through the basic probabilities of dice rolling and how this can be used to develop strategies in the game. After discussing probabilities, we ask the students if they would now change their strategies, and if so, how? Many of our students will adjust their strategies as their understanding increases; seeing these changes helps us to assess their comprehension.

Once the students have some comfort with probabilities, we give them the following task:

> Suppose you are in a game with 30 dice. For any given die value, the possible bids for the number of dice showing are from 1 up to 30. Since we are just considering the probabilities of different bids, whether the bid is 17 threes or 17 sixes does not matter. Imagine a graph of a function where the possible bids are given on the x-axis, the y-axis ranges from 0 to 1, and the function values are the probabilities of a number of dice, showing a particular value, winning if challenged. What can be said about this graph?

While this question is vague, it allows the students freedom to explore avenues we may not foresee. We then take various suggestions from the students about what this graph might look like and analyze them. If the students get stuck with this discussion, we ask leading questions starting with the extreme bids "How often will 1 two win?" or "Which is more likely to win, 10 fives or 11 fives." We try to impress upon the students that the graph does not need to be exact, we are more interested its general shape and their thought process when constructing this graph.

Next, using 30 dice, or a random number generator, students can roll these dice and track the maximum number of dice showing a given value. Plotting these relative frequencies, students can check if these values match the graph they constructed above. We also ask the students to think about approximately how many times they need to repeat this experiment to get their relative frequencies close to the probabilities.

In the classroom, we discuss three different strategies, which will be described later. These strategies are introduced separately, allowing the students to use each in a game. Advanced students should try to model their strategies and see how they compare to the ones given here. The modeling can be done by successively rolling and collecting data or using the random number generator available in most spreadsheets.

Assessment

Since we use this project in classes with varied mathematical backgrounds, we have tried several different assessment methods:

- Writing assignments where the students describe their strategies, including cases for both initial bids and trailing bids.
- Given a description of a specific game situation, the students should be able to discuss the merits of a particular bid. They should also be able to give an argument for either challenging a certain bid, or if not, what the following bid should be. Often we provide a table similar to Table 1 to aid in their reasoning.

While the level of the students determines the computational complexity of the problems, most of the assessments we use are not entirely based on numerical answers. We look for student's understanding when using probabilities and expected values, which are measured by requiring the students to properly justify their answers through a description of their thought process.

Connections to probability

When introduced to probability, you are often presented with three notions: probability, relative frequency, and the law of large numbers. *Fibber's Dice* illustrates the connection between these ideas.

The notions of probability and relative frequency are glued together via the *law of large numbers*. This states that the relative frequency of an event approaches the probability of the event as the number of outcomes observed approaches infinity. The idea is that if a coin is flipped a few times, you may not experience the same number of heads as tails but if you keep flipping the coin long enough, the ratio of heads (or tails) to the total number of flips will approach $\frac{1}{2}$.

While many games of chance such as poker and roulette can be analyzed using probabilities, the number of events occurring in a given game is too few for one to really make use of these computations. An unlikely hand in poker can still appear, and the seemingly unlikely event of 8 reds in a row in roulette can still happen, but you may play for a long time without witnessing this event. Roulette can easily be modeled in the classroom using a deck of cards (in place of the ball and wheel) and poker chips for betting. Unfortunately, the number of rounds you can get in an hour is small and the only lesson the students are sure to walk away with is that gambling only pays off for the casino. The probability of 8 reds in a row in roulette is approximately $1/400$. Playing 10 rounds an hour, students would need to play for approximately 40 hours to see this event.

What makes *Fibber's Dice* different from most games involving probability is that the effect of the law of large numbers can readily be seen while playing the game. When five people are playing *Fibber's Dice*, the game starts with 30 dice on the table. This game will include at least 165 dice rolls, and often games have several hundred dice rolls. The event of rolling six dice and having them all come up sixes and ones has probability 1/729, but with the high number of rolls in Fibber's Dice we expect to see this even every 4 or 5 games. We feel this allows our students to *experience* the law of large numbers.

Bidding Strategies

In the following, we discuss and compare three bidding strategies. Depending on the level of the students, they might develop their own strategies which could be compared to those described here.

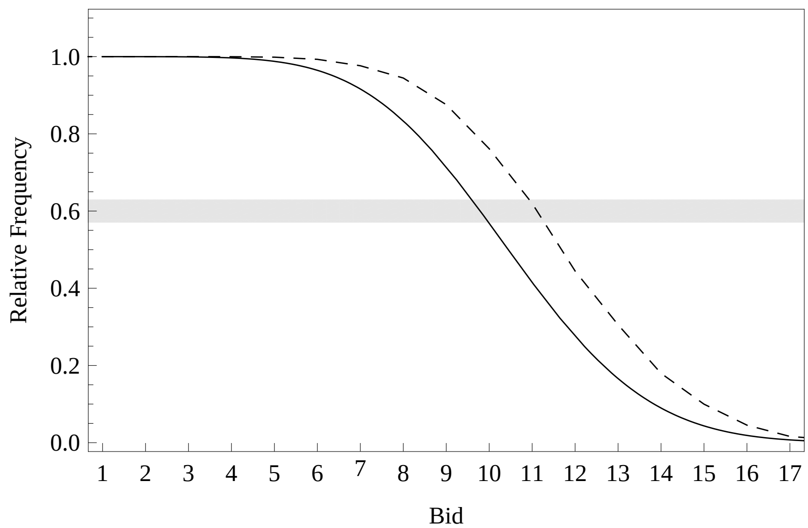
The most basic strategy for bidding is based on the idea of simple probabilities and expected value. Each round that there are n dice on the table, we expect on average to see $n/3$ dice of the same value. This comes from the fact that given any value, there is a $n/6$ probability that this value will arise, and there is a $n/6$ probability that a 1 (a wild value) will arise. Since these events are disjoint, the probabilities add and we arrive at a probability of $n/3$ for any single value. Hence one way to make bids would be to pick a die value at random and then bid $n/3$ of this value. As an example, suppose that there are 21 dice in play. A player using this strategy might bid: "7 twos," or "7 sixes." We'll call this sort of bidding *blind bidding* as players can do it without even examining their dice.

A player improves on blind bidding by looking under their cup. Another bidding strategy would be for the player bet $n/3$ dice, but choose the value by picking the value that occurs most often under their cup. We'll call this sort of bidding *value bidding*. So if there are 21 dice on the table, a player who finds the following dice under their cup ⚀ ⚂ ⚁ ⚅ ⚁ ⚁ would bid "7 twos," when using the value bidding strategy.

There is one last bidding strategy we will discuss, *advanced bidding*. With advanced bidding, take the total number of dice on the table, subtract the number of dice in your hand, and divide this number by three. This gives you an idea of how many dice of each value are under everyone else's cup. Now take what you have in your hand and add it to the value computed before, this is your bid. For example, suppose you have the following hand: ⚅ ⚁ ⚁, and there are 21 dice total on the table. You would expect to see 6 dice of any given value under everyone else's cup. While "8 twos" is a safe bid, "9 twos" might be better, pushing the bids higher and eliminating safe bids for following players. Advanced bidding uses nearly all the information available to the player. However, as we shall see, the advantages of advanced bidding are somewhat subtle.

Comparing the Bidding Strategies In the following comparisons, we used *Mathematica* and *Maple* for their programming capabilities. Students can run similar experiments using a spreadsheet or calculator if more advanced software is not available. They could even roll many dice and collate the data. In the following, we look into a couple of specific situations comparing our three strategies. Similar investigations could be completed by the students.

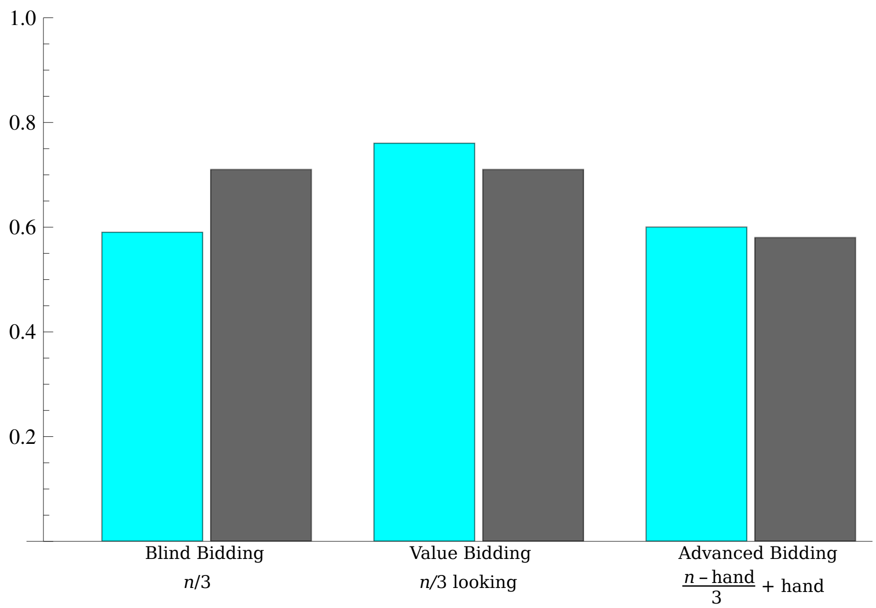
While value bidding is better than blind bidding, and advanced bidding seems like it should be the best, how can we be sure? Using *Mathematica*, we simulated 5000 rounds for each of these three bidding strategies. Each round was for five players with six dice each. We call a bid a *winning bid* if it would win if challenged. While the die value affects the strategy for bidding, the probability of a bid winning can be modeled using just the number of dice. We gathered data by looking at the winning and losing bids in these rounds. Our data is summarized in the following graph.



In the graph above, the solid line is a continuous representation of the relative frequency of making a winning bid using blind bidding. The dashed line represents the relative frequency of making a winning bid using value bidding. Notice that when there are thirty dice on the table, value bidding allows the player to bid 1 die higher than blind bidding while maintaining a comfortable relative frequency of .6 of winning a challenge. It is somewhat of a surprising fact that from this perspective, advanced bidding gives *no significant advantage* over value bidding. To see how advanced bidding gives an advantage we look at the game from a new viewpoint.

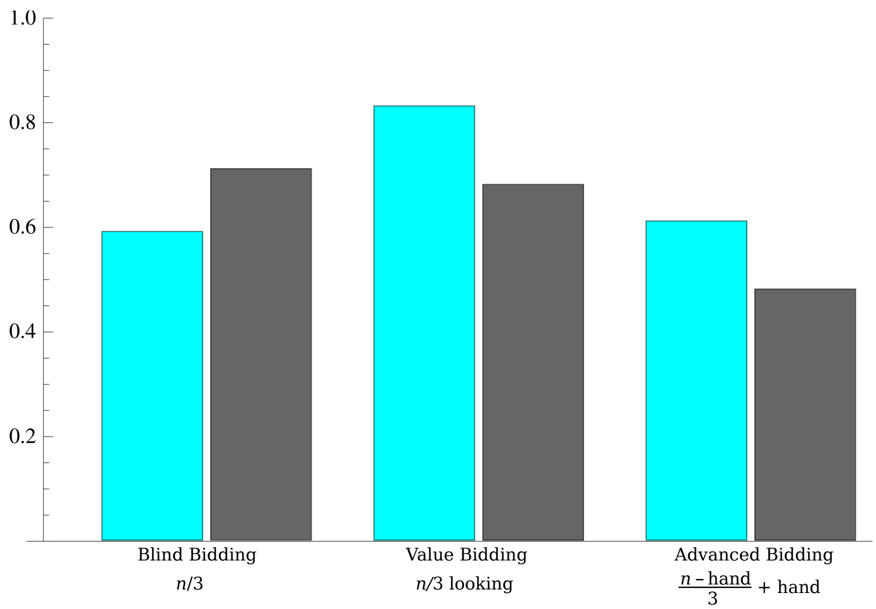
Favors the Underdog As shown in the situations described above, the advantage in *Fibber's Dice* goes to the players with the most dice, since these players have the most information available to them. However, these players are furthest from winning! As a player loses dice (which is good!), value bidding becomes less effective. This makes *Fibber's Dice*'s more fun to play as it gives the underdog a chance to catch up.

We witness this advantage to the underdog through computer experiments. Suppose that there are five people playing, four of which are doing quite well, say they have only three dice each. Suppose further that the fifth person is doing considerably less well, and still has all six of their starting dice. Let's suppose that someone with three dice is bidding. What are their prospects? Consider the following bar chart:



The teal bar represents the player's probability of making a winning bid, via the three strategies, blind bidding, value bidding, and advanced bidding. The gray bar represents the probability of the next player making a higher winning bid *assuming that the first player was correct.* In this case we assume that both players have three dice. From our data we see that value bidding gives the greatest probability of stating a winning bid. However, if you are using value bidding and if you are correct, then the next player also has a high probability of being correct with their next bid. Value bidding does not put much pressure on the next player. If the next player does not challenge—you cannot lose dice, meaning you won't win!

Now suppose that the underdog, the person with six dice is the first player, and someone with 3 dice is following them:



Using blind bidding leaves the probability unchanged. We see an improvement using value bidding. When the underdog uses advanced bidding the second players probability of stating a winning bid falls dramatically.

In both of the above graphs, it is difficult to see the difference between advanced bidding and blind bidding. Recall, the extra information used to make an advanced bid allows the player to make a *higher bid with the same probability* as the other bidding techniques. This is how the second player's probability of a winning bid is being forced down. This higher bid will cause the second player to either challenge the first player because the bid is above $n/3$, or make a high bid with which they are uncomfortable.

Challenging Strategies The question now becomes, "When should I challenge?" A basic strategy would be to challenge whenever a bid is made that is greater than $n/3$. How much higher should you bid? If there are n dice on the table, the probability that exactly a quantity of q of any value are showing, including wilds is:

$$P(q) = ({}_nC_q)\left(\frac{1}{3}\right)^q\left(\frac{2}{3}\right)^{n-q}$$

The second factor gives us the probability of q dice rolling the chosen value or a wild. The third factor gives the probability of $n - q$ dice not rolling the chosen value or a wild. To see the role of the first factor, consider the following diagram:

$$\underbrace{\square \quad \square \quad \square \quad \bullet\bullet\bullet \quad \square}_{n \text{ dice}}$$

We know that q of the dice in this picture are showing the chosen value or wilds. The factor ${}_nC_q$ gives us all the ways to choose q of the n positions for these dice.

In *Fibber's Dice*, we are more interested in the probability that q or more dice show a given value. This probability is given by:

$$P_{\text{sum}}(q) = \sum_{x=q}^{n} ({}_nC_x) \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{n-x}$$

While playing *Fibber's Dice*, we are looking for a *comfort zone*. A comfort zone is a range in the probabilities which seem "good," not too risky and not too safe. Remember that we lose dice by being challenged and winning.

As the round progresses, the bidding continues increasing, moving farther from the value $n/3$. Each player may have their own comfort zone, but eventually the bids get too high, or the probabilities get too low, and the player whose turn it is will prefer to challenge than make a bid of their own. Unfortunately, $P_{\text{sum}}(q)$ is not a calculation most people can do while playing. Of course, students could easily compute the values using a spreadsheet. We used Maple 11, [2], to construct the approximate probabilities. The following table is a portion of what we give to students so they can find their comfort zone as a bid quantity.

(n) number of dice	$n/3$	$P_{\text{sum}}(q)$		
		.4	.25	.16
36	12	13	14	15
39	13	14	15	16
42	14	15	16	17
45	15	16	17	18
48	16	17	18	19
51	17	18	19	20
54	18	19	20	21
57	19	20	21	23
60	20	21	22	24

Table 1: Some q values for specific n and $P_{\text{sum}}(q)$

For example, when 54 dice are on the table, there is a .4 probability that some face value or 1 appears on 19 or more dice. For a more conservative player, this might be the upper limit of their comfort zone. A more adventurous player might be comfortable with the .25 probability associated with 20 or more dice.

Conclusion

It is our hope that you try out *Fibber's Dice*, have fun, be daring, and experience the law of large numbers. In our classes, we like to ask the question, "Does knowing the mathematical theory behind the game make the game more or less fun?" We know our answer.

References

[1] J.S. Milton and J.C. Arnold, *Introduction to probability and statistics: Principles and applications for engineering and the computing sciences*, 3rd ed., Mcgraw Hill, 1995.

[2] Waterloo Maplesoft Ontario, *Maple 11*, 2007.

[3] Wikipedia, *Liar's dice — wikipedia, the free encyclopedia*, 2008, [Online; accessed 8-August-2008].

[4] ______, *Perudo — wikipedia, the free encyclopedia*, 2008, [Online; accessed 8-August-2008].