Generalized Local Cohomology and the Canonical Element Conjecture

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Abstract

We study a generalization of the Canonical Element Conjecture. In particular we show that given a nonregular local ring $(A, \mathfrak{m})$ and an $i > 0$, there exist finitely generated A-modules M such that the canonical map from $\mathrm{Ext}^i_A(M/\mathfrak{m}M, \mathrm{Syz}_i(M/\mathfrak{m}M))$ to $\mathrm{H}^i_{\mathfrak{m}}(M, \mathrm{Syz}_i(M/\mathfrak{m}M))$ is nonzero. Moreover, we show that even when M has infinite projective dimension and $i > \dim(A)$, studying these maps sheds light on the Canonical Element Conjecture.

Key words: Homological Conjectures, Generalized local cohomology, Canonical Element Conjecture, Infinite projective dimension

1 Introduction

In this paper, we study a generalization of the Canonical Element Conjecture of Hochster:

Conjecture 1.1 (Canonical Element Conjecture). *Given a local ring $(A, \mathfrak{m})$ of dimension n, the canonical map*

$$\mathrm{Ext}^n_A(A/\mathfrak{m}, S_n) \to \mathrm{H}^n_{\mathfrak{m}}(S_n)$$

is nonzero, where $S_n = \mathrm{Syz}_n(A/\mathfrak{m})$.

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In [10], Hochster showed that the Canonical Element Conjecture implies the Improved New Intersection Conjecture:

Conjecture 1.2 (Improved New Intersection Conjecture)**.** *Let* $(A, \mathfrak{m})$ *be a local ring and*

$$F_\bullet : \qquad 0 \to F_n \to \cdots \to F_0 \to 0$$

be a complex of finitely generated free modules. If $\ell(H_i(F_\bullet)) < \infty$ *for* $i > 0$ *and* $H_0(F_\bullet)$ *has a nonzero minimal generator killed by a power of* $\mathfrak{m}$*, then* $\dim(A) \leqslant n$*.*

The Improved New Intersection Conjecture developed out of the work of Evans and Griffith in [5]. In [5], they used the existence of Hochster's Big Cohen-Macaulay modules to prove the Improved New Intersection Conjecture in the equicharacteristic case.

It was suspected that the Canonical Element Conjecture was stronger than the Improved New Intersection Conjecture. However in [1], Dutta showed that the Improved New Intersection Conjecture implies the Canonical Element Conjecture. More recently, the conjectures were shown to be true when the dimension of the ring is 3 by Heitmann in [7].

Instead of studying the Canonical Element Conjecture directly, we choose to study a generalization based on generalized local cohomology modules.

Definition 1.3. If $(A, \mathfrak{m})$ is a local ring with finitely generated A-modules M and S, the ith ***M*-local cohomology** of S with respect to $\mathfrak{m}$ is defined as:

$$\mathrm{H}^i_{\mathfrak{m}}(M, S) := \varinjlim_t \mathrm{Ext}^i_A(M/\mathfrak{m}^t M, S)$$

These modules were first studied by Grothendieck in [6, Exposé VI]. They were also studied by Herzog in his thesis [8]. Recently, these modules have attracted the interest of others as well. Of critical importance is that if the A-module M has infinite projective dimension, then $\mathrm{H}^i_{\mathfrak{m}}(M, S)$ might not vanish for $i > \dim(A)$. We direct the reader to the work of Suzuki [11] and the work of Herzog and Zamani [9] for details on the vanishing of generalized local cohomology. Since local cohomology modules vanish above the dimension of the ring, for a local ring $(A, \mathfrak{m})$ it is useless to study the maps

$$\mathrm{Ext}^i_A(A/\mathfrak{m}, \mathrm{Syz}_i(A/\mathfrak{m})) \to \mathrm{H}^i_{\mathfrak{m}}(\mathrm{Syz}_i(A/\mathfrak{m}))$$

when $i > \dim(A)$. However, since generalized local cohomology modules might not vanish above the dimension of the ring, studying the canonical maps

$$\mathrm{Ext}^i_A(M/\mathfrak{m}M, S_i) \to \mathrm{H}^i_{\mathfrak{m}}(M, S_i),$$

where M is some finitely generated A-module and S_i is either $\mathrm{Syz}_i(A/\mathfrak{m})$ or $\mathrm{Syz}_i(M/\mathfrak{m}M)$ give us a new angle from which to attack this problem.

Among other things, we prove two theorems which show the importance of studying the above maps. Namely in Section 2, we prove the following proposition:

Proposition 2.2. *Let $(A, \mathfrak{m})$ be a local ring, then the following are equivalent:*

(1) The canonical map

$$\mathrm{Ext}^i_A(A/\mathfrak{m}, \mathrm{Syz}_i(A/\mathfrak{m})) \to \mathrm{H}^i_{\mathfrak{m}}(\mathrm{Syz}_i(A/\mathfrak{m}))$$

is nonzero.

(2) For all nonzero finitely generated A-modules M, the canonical map

$$\mathrm{Ext}^i_A(M/\mathfrak{m}M, \mathrm{Syz}_i(A/\mathfrak{m})) \to \mathrm{H}^i_{\mathfrak{m}}(M, \mathrm{Syz}_i(A/\mathfrak{m}))$$

is nonzero.

(3) For all nonzero finitely generated A-modules M and N, the canonical map

$$\mathrm{Ext}^i_A(M/\mathfrak{m}M, \mathrm{Syz}_i(N/\mathfrak{m}N)) \to \mathrm{H}^i_{\mathfrak{m}}(M, \mathrm{Syz}_i(N/\mathfrak{m}N))$$

is nonzero.

When $n = \dim(A)$, we see that the validity of the Canonical Element Conjecture and the above proposition imply that the map

$$\mathrm{Ext}^n_A(M/\mathfrak{m}M, S_n) \to \mathrm{H}^n_{\mathfrak{m}}(M, S_n)$$

is nonzero for all nonzero finitely generated A-modules M.

Moreover, in Section 3 we show that merely studying modules of infinite projective dimension can shed light on the Canonical Element Conjecture:

Theorem 3.1. *Let $(A, \mathfrak{m})$ be a local ring of dimension n and depth $n-1$. If the Canonical Element Conjecture does not hold for A, that is, if for some t_0*

$$\mathrm{Ext}^n_A(A/\mathfrak{m}, \mathrm{Syz}_n(A/\mathfrak{m})) \to \mathrm{Ext}^n_A(A/\mathfrak{m}^{t_0}, \mathrm{Syz}_n(A/\mathfrak{m}))$$

is zero, then for all $i > n$ and all finitely generated A-modules S, the canonical maps

$$\vartheta_i : \mathrm{Ext}^i_A(M/\mathfrak{m}M, S) \to \mathrm{H}^i_{\mathfrak{m}}(M, S)$$

are zero whenever M is of the form $A/\mathfrak{a}$ where $\mathfrak{a} \subset \mathfrak{m}^{t_0}$.

Thus studying modules for which the maps ϑ_i are nonzero when $i > n$ may lead to a a possible proof of the Canonical Element Conjecture for rings of dimension n and depth $n-1$. In light of Theorem 3.1, we find the following theorem interesting:

Theorem 3.2. *Let $(A, \mathfrak{m})$ be a local ring which is not regular and let $(P_\bullet, \rho_\bullet)$ be a minimal free resolution of some finitely generated A-module Q of infinite projective dimension. If $M = \operatorname{Coker}(\rho_i^*)$ for some $i > 2$, then the canonical map*

$$\operatorname{Ext}^i_A(M/\mathfrak{m}M, S_i) \to \operatorname{H}^i_{\mathfrak{m}}(M, S_i)$$

is nonzero, where $S_i = \operatorname{Syz}_i(M/\mathfrak{m}M)$.

As a corollary, we show that in the case where A is a Gorenstein ring that is not regular, there are infinitely many isomorphism classes of A-modules M such that the map

$$\operatorname{Ext}^i_A(M/\mathfrak{m}M, \operatorname{Syz}_i(M/\mathfrak{m}M)) \to \operatorname{H}^i_{\mathfrak{m}}(M, \operatorname{Syz}_i(M/\mathfrak{m}M))$$

is nonzero for all i. We are currently working to expand the class of modules for which the canonical maps discussed above are nonzero.

2 Basic Results

It is clear that $\operatorname{H}^0_{\mathfrak{m}}(M, -)$ is a covariant left exact functor with an associated long exact sequence of generalized local cohomology modules. However, it is also true that $\operatorname{H}^0_{\mathfrak{m}}(-, N)$ is a contravariant left exact functor with an associated long exact sequence of generalized local cohomology modules. This is stated in [9], but here we provide a different proof.

Proposition 2.1. *Let A be a Noetherian ring, I be an ideal, and N be an A-module. Given a short exact sequence of finitely generated A-modules*

$$0 \to M' \to M \to M'' \to 0$$

we obtain the long exact sequence:

$$0 \to \operatorname{H}^0_I(M'', N) \to \operatorname{H}^0_I(M, N) \to \operatorname{H}^0_I(M', N) \to \operatorname{H}^1_I(M'', N) \to \cdots$$

Proof. For two natural numbers s and t, apply $- \otimes_A A/I^{s+t}$ to the short exact sequence of finitely generated A-modules

$$0 \to M' \to M \to M'' \to 0$$

to get the exact sequence

$$0 \to K \to M'/I^{s+t}M' \to M/I^{s+t}M \to M''/I^{s+t}M'' \to 0,$$

where K is the kernel of the map from $M'/I^{s+t}M'$ to $M/I^{s+t}M$. Note that

$$K = \frac{M' \cap I^{s+t}M}{I^{s+t}M'}.$$

Break off the following exact sequences

$$\begin{gathered}0 \to K \to M'/I^{s+t}M' \to Z \to 0,\\ 0 \to Z \to M/I^{s+t}M \to M''/I^{s+t}M'' \to 0,\end{gathered}$$

to see that:

$$Z \simeq \frac{M'/I^{s+t}M'}{(M' \cap I^{s+t}M)/I^{s+t}M'} \simeq \frac{M'}{M' \cap I^{s+t}M}$$

By the Artin-Rees Lemma we see that for some s and all t we have

$$M' \cap I^{s+t}M = I^t(M' \cap I^s M).$$

Additionally, for any given t, there exists some r such that

$$I^r M' \subset M' \cap I^{s+t}M = I^t(M' \cap I^s M) \subset I^t M'.$$

Hence the sets $I^t(M' \cap I^s M)$ and $I^t M'$ are cofinal with respect to t. Thus we see that

$$\begin{aligned}\varinjlim_t \operatorname{Ext}^i_A(Z, N) &\simeq \varinjlim_t \operatorname{Ext}^i_A(M'/I^t(M' \cap I^s M), N)\\ &\simeq \varinjlim_t \operatorname{Ext}^i_A(M'/I^t M', N),\end{aligned} \tag{2.1}$$

and so the short exact sequence

$$0 \to Z \to M/I^{s+t}M \to M''/I^{s+t}M'' \to 0$$

leads to:

$$\begin{aligned}&0 \to \operatorname{Hom}_A(M''/I^{s+t}M'', N) \to \operatorname{Hom}_A(M/I^{s+t}M, N) \to \operatorname{Hom}_A(Z, N) \to\\ &\to \operatorname{Ext}^1_A(M''/I^{s+t}M'', N) \longrightarrow \operatorname{Ext}^1_A(M/I^{s+t}M, N) \longrightarrow \operatorname{Ext}^1_A(Z, N) \longrightarrow \cdots\end{aligned}$$

Taking the direct limit with respect to t of each term above, and applying (2.1) we are done. □

Now we show a strong connection between the Canonical Element Conjecture and the maps $\operatorname{Ext}^i_A(M/\mathfrak{m}M, S_i) \to \mathrm{H}^i_{\mathfrak{m}}(M, S_i)$.

Proposition 2.2. *Let $(A, \mathfrak{m})$ be a local ring, then the following are equivalent:*

(1) The canonical map

$$\operatorname{Ext}^i_A(A/\mathfrak{m}, \operatorname{Syz}_i(A/\mathfrak{m})) \to \mathrm{H}^i_{\mathfrak{m}}(\operatorname{Syz}_i(A/\mathfrak{m}))$$

is nonzero.

(2) For all nonzero finitely generated A-modules M, the canonical map

$$\mathrm{Ext}^i_A(M/\mathfrak{m}M, \mathrm{Syz}_i(A/\mathfrak{m})) \to \mathrm{H}^i_{\mathfrak{m}}(M, \mathrm{Syz}_i(A/\mathfrak{m}))$$

is nonzero.

(3) For all nonzero finitely generated A-modules M and N, the canonical map

$$\mathrm{Ext}^i_A(M/\mathfrak{m}M, \mathrm{Syz}_i(N/\mathfrak{m}N)) \to \mathrm{H}^i_{\mathfrak{m}}(M, \mathrm{Syz}_i(N/\mathfrak{m}N))$$

is nonzero.

Proof. (1) ⇒ (2) For any finitely generated A-module M we may write

$$A^r \to M \to 0,$$

where r is the number of generators in a minimal generating set of M. Applying $- \otimes_A A/\mathfrak{m}$, we see that $A^r/\mathfrak{m}A^r \simeq M/\mathfrak{m}M$. Thus we have the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Ext}^i_A(M/\mathfrak{m}M, \mathrm{Syz}_i(A/\mathfrak{m})) \arrow[r, equal] \arrow[d] & \mathrm{Ext}^i_A(A^r/\mathfrak{m}A^r, \mathrm{Syz}_i(A/\mathfrak{m})) \arrow[d, "\neq 0"] \\
\mathrm{H}^i_{\mathfrak{m}}(M, \mathrm{Syz}_i(A/\mathfrak{m})) \arrow[r] & \mathrm{H}^i_{\mathfrak{m}}(A^r, \mathrm{Syz}_i(A/\mathfrak{m}))
\end{tikzcd}$$

By assumption

$$\mathrm{Ext}^i_A(A/\mathfrak{m}, \mathrm{Syz}_i(A/\mathfrak{m})) \to \mathrm{H}^i_{\mathfrak{m}}(\mathrm{Syz}_i(A/\mathfrak{m}))$$

is nonzero. From the definition of generalized local cohomology and since we may pull finite direct sums out of Ext modules, we see that the right vertical map is nonzero. Hence the left vertical map is nonzero.

(2) ⇒ (3) Given any finitely generated A-module N, note that Ext commutes with finite direct sums and that we have

$$\mathrm{Syz}_i(N/\mathfrak{m}N) = \bigoplus_1^r \mathrm{Syz}_i(A/\mathfrak{m}).$$

(3) ⇒ (1) This is clear, as we may set $M = A$ and $N = A$. □

From [3, Theorem 3.2] and [4, Theorem 1.2], we see that given a local ring $(A, \mathfrak{m})$ of dimension n, the maps

$$\mathrm{Ext}^i_A(A/\mathfrak{m}, \mathrm{Syz}_i(A/\mathfrak{m})) \to \mathrm{H}^i_{\mathfrak{m}}(\mathrm{Syz}_i(A/\mathfrak{m}))$$

are nonzero for $0 \leqslant i \leqslant n-1$. Hence by Proposition 2.2, we see that for all finitely generated nonzero A-modules M the maps

$$\mathrm{Ext}^i_A(M/\mathfrak{m}M, \mathrm{Syz}_i(M/\mathfrak{m}M)) \to \mathrm{H}^i_{\mathfrak{m}}(M, \mathrm{Syz}_i(M/\mathfrak{m}M))$$

are nonzero for $0 \leqslant i \leqslant n-1$. We will now give a direct proof of this fact, using a similar line of reasoning as found in [3, Theorem 3.2] and [4, Theorem 1.2] for rings of dimension n and depth d where $0 < d < n$. The proof of this fact will illuminate connections that we will utilize later in this paper. Before we can get to the proof, we need some more tools. The following lemma, in a slightly less general form, appears explicitly as part of [1, Theorem 3.1] and implicitly in [3] and [4]:

Lemma 2.3. *Let $(A, \mathfrak{m})$ be a local ring, M be a finitely generated A-module, and $S_i = \operatorname{Syz}_i(M/\mathfrak{m}M)$. If for some integer i*

$$\operatorname{Ext}^i_A(M/\mathfrak{m}M, S_i) \to \operatorname{H}^i_{\mathfrak{m}}(M, S_i)$$

is the zero map, then for t sufficiently large the following holds: Given $(F_\bullet, \sigma_\bullet)$ and $(G_\bullet, \tau_\bullet)$, free resolutions of $M/\mathfrak{m}M$ and $M/\mathfrak{m}^tM$ respectively, one can construct a lift $\varphi_\bullet : G_\bullet \to F_\bullet$ of the canonical surjection from $M/\mathfrak{m}^tM$ to $M/\mathfrak{m}M$, such that $\varphi_j = 0$ for $j \geqslant i$.

Proof. Suppose that

$$\operatorname{Ext}^i_A(M/\mathfrak{m}M, S_i) \to \operatorname{H}^i_{\mathfrak{m}}(M, S_i)$$

is zero. Then for t sufficiently large, the canonical map

$$\operatorname{Ext}^i_A(M/\mathfrak{m}M, S_i) \to \operatorname{Ext}^i_A(M/\mathfrak{m}^tM, S_i)$$

is zero. Let $\varphi_\bullet : G_\bullet \to F_\bullet$ be a lift of the canonical surjection from $M/\mathfrak{m}^tM$ to $M/\mathfrak{m}M$ and consider the following commutative diagram:

$$\begin{tikzcd}
G_i \arrow[rr, "\tau_i"] \arrow[dr] \arrow[dd, "\varphi_i"] & & G_{i-1} \arrow[dd, "\varphi_{i-1}"] \arrow[ddl, dashed, "\delta"] \\
& T_i \arrow[ur, "\iota_T"] \arrow[dd] & \\
F_i \arrow[rr] \arrow[dr] & & F_{i-1} \\
& S_i \arrow[ur, "\iota_S"'] &
\end{tikzcd}$$

Here $T_i = \operatorname{Syz}_i(M/\mathfrak{m}^tM)$ and ι_T along with ι_S are canonical injections. Set $(-)^\vee = \operatorname{Hom}_A(-, S_i)$ and note that

$$\begin{aligned}
\operatorname{Ext}^i_A(M/\mathfrak{m}^tM, S_i) &= \frac{\operatorname{Ker}(\tau_{i+1}^\vee)}{\operatorname{Im}(\tau_i^\vee)} \\
&= \frac{\operatorname{Hom}_A(T_i, S_i)}{\operatorname{Im}(\tau_i^\vee)}.
\end{aligned}$$

By assumption, the class of φ_i is zero in $\operatorname{Ext}^i_A(M/\mathfrak{m}^tM, S_i)$. Thus the image of $\varphi_{i-1}|_{T_i}$ is zero in $\operatorname{Ext}^i_A(M/\mathfrak{m}^tM, S_i)$, by the commutativity of the diagram above. Hence the image of the class of $\varphi_{i-1}|_{T_i}$ is in $\mathrm{Im}(\tau_i^\vee)$. So there exists

$$\delta : G_{i-1} \to S_i$$

such that $\delta \circ \iota_T = \varphi_{i-1}|_{T_i}$. Now set $\widetilde{\varphi}_{i-1} := \varphi_{i-1} - \iota_S \circ \delta$. This map lifts to the zero map between T_i and S_i. Hence any further lift of φ will also be zero. $\square$

We will also need the following proposition, [1, Proposition 1.1]:

Proposition 2.4 (Dutta [1, Proposition 1.1]). *Let A be a local ring and let*

$$G_\bullet : \qquad \cdots \to G_i \to \cdots \to G_1 \to G_0$$

be a complex of finitely generated free A-modules with $H_0(G_\bullet) = M$. Let N be a submodule of M. Then we can find a complex $L_\bullet$ of finitely generated free modules and a map $\psi_\bullet : L_\bullet \to G_\bullet$ such that:

(1) $H_0(L_\bullet) = N$.
(2) $H_i(L_\bullet) \simeq H_i(G_\bullet)$ *for all positive* i.
(3) If $G_\bullet$ *is minimal, then so is* $L_\bullet$.
(4) $\mathrm{Cone}(\psi_\bullet)$ *is a resolution of* M/N.

The above proposition is useful for constructing maps of complexes from a single map of modules, even when the target complex is not exact. Finally, recall the following theorem [2, Theorem 1.1]:

Theorem 2.5 (Dutta [2, Theorem 1.1]). *Let A be a ring of dimension n and depth d where $0 < d < n$. Let $(L_\bullet, \lambda_\bullet)$ be a minimal complex such that:*

(1) $H_0(L_\bullet) \neq 0$.
(2) $\ell(H_i(L_\bullet)) < \infty$ *for all* i.
(3) $H_i(L_\bullet) = 0$ *for* $i \geqslant n - d$.

Then $\mathrm{Coker}(\lambda_i)$ *cannot have a free summand for $i > 1$ and $i \neq n$. Moreover* $\mathrm{Coker}(\lambda_n)$ *cannot have a free summand if and only if the Canonical Element Conjecture holds.*

We now state and prove the following:

Theorem 2.6. *Let $(A, \mathfrak{m})$ be a local ring of dimension n and depth d where $0 < d < n$. For all nonzero finitely generated A-modules M the canonical map*

$$\mathrm{Ext}^i_A(M/\mathfrak{m}M, S_i) \to \mathrm{H}^i_{\mathfrak{m}}(M, S_i)$$

is nonzero for $0 \leqslant i \leqslant n-1$, where $S_i = \mathrm{Syz}_i(M/\mathfrak{m}M)$.

Proof. Argue by way of contradiction. Suppose that

$$\operatorname{Ext}^i_A(M/\mathfrak{m}M, S_i) \to \mathrm{H}^i_{\mathfrak{m}}(M, S_i)$$

is the zero map. By Lemma 2.3, for $t \gg 0$ and minimal free resolutions $F_\bullet$ and $G_\bullet$ of $M/\mathfrak{m}M$ and $M/\mathfrak{m}^tM$ respectively, there exists a lift $\varphi_\bullet : G_\bullet \to F_\bullet$ of the canonical surjection $\varphi : M/\mathfrak{m}^tM \to M/\mathfrak{m}M$ such that φ_i is zero in the commutative diagram below:

$$\begin{tikzcd}
\cdots \arrow[r] & G_i \arrow[r, "\tau_i"] \arrow[d, "\varphi_i"] & G_{i-1} \arrow[r] \arrow[d, "\varphi_{i-1}"] & \cdots \arrow[r] & G_1 \arrow[r, "\tau_1"] \arrow[d, "\varphi_1"] & G_0 \arrow[r] \arrow[d, "\varphi_0"] & M/\mathfrak{m}^t M \arrow[r] \arrow[d, "\varphi"] & 0 \\
\cdots \arrow[r] & F_i \arrow[r, "\sigma_i"] & F_{i-1} \arrow[r] & \cdots \arrow[r] & F_1 \arrow[r, "\sigma_1"] & F_0 \arrow[r] & M/\mathfrak{m}M \arrow[r] & 0
\end{tikzcd}$$

Thus the map between $T_i = \operatorname{Syz}_i(M/\mathfrak{m}^tM)$ and S_i is zero and we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & T_i \arrow[r] \arrow[d, "0"] & G_{i-1} \arrow[r] \arrow[d] & T_{i-1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S_i \arrow[r] & F_{i-1} \arrow[r] & S_{i-1} \arrow[r] & 0
\end{tikzcd}$$

Apply $(-)^* = \operatorname{Hom}_A(-, A)$ to obtain:

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ext}^{i-1}_A(M/\mathfrak{m}M, A) \arrow[r] \arrow[d] & F^*_{i-1}/\operatorname{Im}(\sigma^*_{i-1}) \arrow[r] \arrow[d] \arrow[dl, dashed, "\ell"'] & \operatorname{Im}(\sigma^*_i) \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & \operatorname{Ext}^{i-1}_A(M/\mathfrak{m}^tM, A) \arrow[r] & G^*_{i-1}/\operatorname{Im}(\tau^*_{i-1}) \arrow[r] & \operatorname{Im}(\tau^*_i) \arrow[r] & 0
\end{tikzcd}$$

Since the far right vertical map is 0, we obtain a lift ℓ making the above diagram commute. Thus we see that the composition

$$F^*_{i-1}/\operatorname{Im}(\sigma^*_{i-1}) \to \operatorname{Ext}^{i-1}_A(M/\mathfrak{m}^tM, A) \to G^*_{i-1}/\operatorname{Im}(\tau^*_{i-1}) \to \operatorname{Im}(\tau^*_i) \tag{2.2}$$

is the zero map.

From Proposition 2.4, [1, Proposition 1.1], we obtain a minimal complex $L_\bullet$, such that $H_i(L_\bullet) = H_i(G^*_\bullet)$ and that the following diagram commutes:

$$\begin{tikzcd}
L_i \arrow[r, "\lambda_i"] \arrow[d] & L_{i-1} \arrow[r, "\lambda_{i-1}"] \arrow[d, "\psi_{i-1}"] & L_{i-2} \arrow[r] \arrow[d, "\psi_{i-2}"] & \cdots \arrow[r] & L_0 \arrow[r] \arrow[d, "\psi_0"] & \operatorname{Ext}^{i-1}_A(M/\mathfrak{m}^tM, A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] \arrow[d] & G^*_0 \arrow[r] \arrow[d] & G^*_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & G^*_{i-1} \arrow[r] \arrow[d] & G^*_{i-1}/\operatorname{Im}(\tau^*_{i-1}) \arrow[r] \arrow[d] & 0 \\
L_{i-1} \arrow[r] & L_{i-2} \oplus G^*_0 \arrow[r] & L_{i-3} \oplus G^*_1 \arrow[r] & \cdots \arrow[r] & G^*_{i-1} \arrow[r] & \operatorname{Im}(\tau^*_i) \arrow[r] & 0
\end{tikzcd} \tag{2.3}$$

Note that the bottom row is the cone of $\psi_\bullet$. Now consider the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & F_0^* \arrow[r, "\sigma_0^*"] \arrow[d, equal] \arrow[ddl, dashed, "h"'] & F_1^* \arrow[r] \arrow[d] \arrow[ddl, dashed, "h'"'] & \cdots \arrow[r] & F_{i-1}^* \arrow[r] \arrow[d] & F_{i-1}^*/\operatorname{Im}(\sigma_{i-1}^*) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] \arrow[d] & G_0^* \arrow[r] \arrow[d, "(0,\mathbb{1}_{G_0^*})"'] & G_1^* \arrow[r] \arrow[d] & \cdots \arrow[r] & G_{i-1}^* \arrow[r] \arrow[d] & G_{i-1}^*/\operatorname{Im}(\tau_{i-1}^*) \arrow[r] \arrow[d] & 0 \\
L_{i-1} \arrow[r] & L_{i-2} \oplus G_0^* \arrow[r] & L_{i-3} \oplus G_1^* \arrow[r] & \cdots \arrow[r] & G_{i-1}^* \arrow[r] & \operatorname{Im}(\tau_i^*) \arrow[r] & 0
\end{tikzcd}$$

By (2.2), the composition $F_{i-1}^*/\operatorname{Im}(\sigma_{i-1}^*) \to \operatorname{Im}(\tau_i^*)$ is the zero map. Hence we have produced a lift that is homotopic to zero. Thus there exist homotopy maps h, h' such that:

$$(-\lambda_{i-1}, -\psi_{i-1}) \circ h + h' \circ \sigma_0^* = (0, \mathbb{1}_{G_0^*})$$

However, $\operatorname{Im}(\sigma_0^*) \subset \mathfrak{m}F_1^*$, thus we see:

$$\operatorname{Im}(\psi_{i-1}) = \operatorname{Im}(-\psi_{i-1}) = G_0^*$$

This together with (2.3) shows that ψ_{i-1} induces a surjection $\operatorname{Coker}(\lambda_i) \twoheadrightarrow G_0^*$ and hence $\operatorname{Coker}(\lambda_i)$ has a free summand. We will now show that this cannot be the case. By construction, the homology of $L_\bullet$ is:

$$H_j(L_\bullet) = \begin{cases} \operatorname{Ext}_A^{i-j-1}(M/\mathfrak{m}^t M, A) & \text{for } j < i, \\ 0 & \text{for } j \geqslant i \end{cases}$$

Note that $\ell(H_j(L_\bullet)) < \infty$ for all j. Moreover, for $j \geqslant i - d$, $H_j(L_\bullet) = 0$. For $1 < i \leqslant n-1$, $L_\bullet$ satisfies the hypothesis of Theorem 2.5, [2, Theorem 1.1], implying again that the cokernel of λ_i cannot have free summand, a contradiction. □

While the validity of [3, Theorem 3.2] and [4, Theorem 1.2] both rely on the validity of the Improved New Intersection Conjecture in equicharacteristic, the argument above relies on Theorem 2.5, [2, Theorem 1.1]. This observation is interesting since the Improved New Intersection Conjecture is a statement about complexes which seems meaningful only in degree less than the dimension of the ring, while Theorem 2.5 is a statement that is meaningful in any degree. However despite this seeming advantage, it seems difficult to use it to obtain a stronger result.

3 Infinite Projective Dimension

In the case of standard local cohomology, the local cohomology modules all vanish above the dimension of the ring. This is not necessarily the case for generalized local cohomology. In [1], Dutta discovered a beautiful interpretation of the Canonical Element Conjecture which he reiterates in [3] and [4]. Here we present a similar interpretation involving generalized local cohomology over local rings which are not regular. Consider a minimal free resolution of $M/\mathfrak{m}M$:

$$\cdots \longrightarrow F_i \xrightarrow{\sigma_i} F_{i-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\sigma_1} F_0 \longrightarrow M/\mathfrak{m}M \longrightarrow 0$$

Since A is not regular, we see that $\mathrm{pd}_A(M/\mathfrak{m}M)$ is not finite. For each i, let $S_i = \mathrm{Syz}_i(M/\mathfrak{m}M)$ and break this resolution into short exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & S_i & \longrightarrow & F_{i-1} & \longrightarrow & S_{i-1} & \longrightarrow & 0 \\
0 & \longrightarrow & S_{i-1} & \longrightarrow & F_{i-2} & \longrightarrow & S_{i-2} & \longrightarrow & 0 \\
 & & \vdots & & \vdots & & \vdots & & \\
0 & \longrightarrow & S_1 & \longrightarrow & F_0 & \longrightarrow & M/\mathfrak{m}M & \longrightarrow & 0
\end{array}$$

Apply the functors $\mathrm{Hom}_A(M/\mathfrak{m}M, -)$ and $\mathrm{H}^0_{\mathfrak{m}}(M, -)$ to each of the short exact sequences above. Looking at the connecting homomorphisms of the long exact sequences of the corresponding derived functors, one obtains the following commutative diagram:

$$\begin{tikzcd}
E^0 \arrow[r, "\delta_0"] \arrow[d, "\vartheta_0"] & E^1 \arrow[r, "\delta_1"] \arrow[d, "\vartheta_1"] & \cdots \arrow[r, "\delta_{i-2}"] & E^{i-1} \arrow[r, "\delta_{i-1}"] \arrow[d, "\vartheta_{i-1}"] & E^i \arrow[r, "\delta_i"] \arrow[d, "\vartheta_i"] & \cdots \\
H^0 \arrow[r, "\bar{\delta}_0"] & H^1 \arrow[r, "\bar{\delta}_1"] & \cdots \arrow[r, "\bar{\delta}_{i-2}"] & H^{i-1} \arrow[r, "\bar{\delta}_{i-1}"] & H^i \arrow[r, "\bar{\delta}_i"] & \cdots
\end{tikzcd} \tag{3.1}$$

Here $E^i = \mathrm{Ext}^i_A(M/\mathfrak{m}M, S_i)$, $H^i = \mathrm{H}^i_{\mathfrak{m}}(M, S_i)$, $S_i = \mathrm{Syz}_i(M/\mathfrak{m}M)$, and the vertical maps are from the definition of $\mathrm{H}^i_{\mathfrak{m}}(M, -)$. In this case, the δ_i's map the image of $\mathbb{1}_{M/\mathfrak{m}M}$ in E^0 to the image of $\mathbb{1}_{S_i}$ in E^i. We will briefly sketch why this is so. Consider the commutative diagram with exact rows:

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
0 \arrow[r] & \mathrm{Hom}_A(F_1, S_1) \arrow[r] \arrow[u, "\sigma_2'"] & \mathrm{Hom}_A(F_1, F_0) \arrow[r] \arrow[u, "\sigma_2''"] & \mathrm{Hom}_A(F_1, M/\mathfrak{m}M) \arrow[r] \arrow[u, "\sigma_2'''"] & 0 \\
0 \arrow[r] & \mathrm{Hom}_A(F_0, S_1) \arrow[r] \arrow[u, "\sigma_1'"] & \mathrm{Hom}_A(F_0, F_0) \arrow[r] \arrow[u, "\sigma_1''"] & \mathrm{Hom}_A(F_0, M/\mathfrak{m}M) \arrow[r] \arrow[u, "\sigma_1'''"] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Consider $\mathbb{1}_{M/\mathfrak{m}M} \in \mathrm{Hom}_A(M/\mathfrak{m}M, M/\mathfrak{m}M) \simeq E^0$. This element has a preimage, call it $\mathbb{1}'''_{M/\mathfrak{m}M} \in \mathrm{Ker}(\sigma'''_1)$. By the exactness of the rows, we may pull this element back to $\mathbb{1}_{F_0} \in \mathrm{Hom}_A(F_0, F_0)$ and then apply σ''_1. By the commutativity of the diagram, we see that this element pulls back to an element of $\mathrm{Hom}_A(F_1, S_1)$, namely σ_1. One can show that this element is a nonzero element in E^1 represented by the image of $\mathbb{1}_{S_1}$. This diagram chase is precisely the construction of δ_1. Working inductively, we see that each δ_i maps the image of $\mathbb{1}_{S_i}$ in E^i to the image of $\mathbb{1}_{S_{i+1}}$ in E^{i+1}.

In [10], Hochster shows that when $M = A$, there are canonical elements associated to the maps ϑ_i above. The same is true in our more general setting. Let ε_i be the image of $\mathbb{1}_{S_i}$ in $\mathrm{Ext}^i_A(M/\mathfrak{m}M, S_i)$ and suppose that there exists $\zeta \in \mathrm{Ext}^i_A(M/\mathfrak{m}M, S_i)$ such that $\vartheta_i(\zeta) \neq 0$. Since ζ is represented by some element $f \in \mathrm{Hom}_A(S_i, S_i)$, we have two maps induced by the functors $(-)_* = \mathrm{Hom}_A(M/\mathfrak{m}M, -)$ and $(-)_\vee = \mathrm{H}^i_\mathfrak{m}(M, -)$, forming the commutative diagram below:

$$\begin{tikzcd}
\mathrm{Ext}^i_A(M/\mathfrak{m}M, S_i) \arrow[r, "\vartheta_i"] \arrow[d, "f_*"] & \mathrm{H}^i_\mathfrak{m}(M, S_i) \arrow[d, "f_\vee"] \\
\mathrm{Ext}^i_A(M/\mathfrak{m}M, S_i) \arrow[r, "\vartheta_i"] & \mathrm{H}^i_\mathfrak{m}(M, S_i)
\end{tikzcd}$$

By the commutativity of the above diagram

$$\begin{aligned}
\vartheta_i(\zeta) &= \vartheta_i(f_*(\varepsilon_i)) \\
&= f_\vee(\vartheta_i(\varepsilon_i)).
\end{aligned}$$

Since $\vartheta_i(\zeta) \neq 0$, we see that $\vartheta_i(\varepsilon_i) \neq 0$. The element $\vartheta_i(\varepsilon_i)$ is our canonical element. In particular, we see that ϑ_i is nonzero if and only if the element $\vartheta_i(\varepsilon_i)$ is nonzero.

The commutativity of diagram (3.1) above shows that if ϑ_i is nonzero, then ϑ_j is nonzero for $0 \leqslant j \leqslant i$. Moreover, studying the nature of ϑ_i when $i > \dim(A)$ has direct implications for the Canonical Element Conjecture:

Theorem 3.1. *Let $(A, \mathfrak{m})$ be a local ring of dimension n and depth $n-1$. If the Canonical Element Conjecture does not hold for A, that is, if for some t_0*

$$\mathrm{Ext}^n_A(A/\mathfrak{m}, \mathrm{Syz}_n(A/\mathfrak{m})) \to \mathrm{Ext}^n_A(A/\mathfrak{m}^{t_0}, \mathrm{Syz}_n(A/\mathfrak{m}))$$

is zero, then for all $i \geqslant n$ and all finitely generated A-modules S, the canonical maps

$$\vartheta_i : \mathrm{Ext}^i_A(M/\mathfrak{m}M, S) \to \mathrm{H}^i_\mathfrak{m}(M, S)$$

are zero whenever M is of the form $A/\mathfrak{a}$ where $\mathfrak{a} \subset \mathfrak{m}^{t_0}$.

Proof. Suppose that the canonical map

$$\mathrm{Ext}^n_A(A/\mathfrak{m}, \mathrm{Syz}_n(A/\mathfrak{m})) \to \mathrm{Ext}^n_A(A/\mathfrak{m}^{t_0}, \mathrm{Syz}_n(A/\mathfrak{m}))$$

is zero. Following the proof of Theorem 2.6 with A substituted for M and n substituted for i, we obtain the following diagram:

$$\begin{tikzcd} L_n \arrow[r, "\lambda_n"] \arrow[d] & L_{n-1} \arrow[r, "\lambda_{n-1}"] \arrow[d, "\psi_{n-1}"] & L_{n-2} \arrow[r] \arrow[d, "\psi_{n-2}"] & \cdots \arrow[r] & L_0 \arrow[r] \arrow[d, "\psi_0"] & \operatorname{Ext}_A^{n-1}(A/\mathfrak{m}^{t_0}, A) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & G_0^* \arrow[r] & G_1^* \arrow[r] & \cdots \arrow[r] & G_{n-1}^* \arrow[r] & G_{n-1}^*/\operatorname{Im}(\tau_{n-1}^*) \arrow[r] & 0 \end{tikzcd} \tag{3.2}$$

One should note that since the depth of A is $n-1$, the above diagram has exact rows. Again using the same technique as in the proof of Theorem 2.6, one can show that ψ_{n-1} is onto. Now we will use the same basic technique as used in [1, Proposition 2.1]. Since ψ_{n-1} is onto, (3.2) above shows that

$$L_{n-1}/\operatorname{Im}(\lambda_n) = \operatorname{Syz}_{n-1}(\operatorname{Ext}_A^{n-1}(A/\mathfrak{m}^{t_0}, A))$$

has a free summand. Since $(\operatorname{Syz}_{n-1}(\operatorname{Ext}_A^{n-1}(A/\mathfrak{m}^{t_0}, A)))^* = \operatorname{Ker}(\lambda_n^*)$, we see that $\operatorname{Ker}(\lambda_n^*)$ has a free summand. Hence there exists a minimal generator of

$$\frac{\operatorname{Ker}(\lambda_n^*)}{\operatorname{Im}(\lambda_{n-1}^*)} = \operatorname{Ext}_A^{n-1}(\operatorname{Ext}_A^{n-1}(A/\mathfrak{m}^{t_0}, A), A)$$

mapping to a minimal generator of $L_{n-1}^*/\operatorname{Im}(\lambda_{n-1}^*)$. Thus $L_{n-1}^*/\operatorname{Im}(\lambda_{n-1}^*)$ has a minimal generator killed by

$$\mathfrak{m}^{t_0} \subset \operatorname{Ann}_A(\operatorname{Ext}_A^{n-1}(\operatorname{Ext}_A^{n-1}(A/\mathfrak{m}^{t_0}, A), A)).$$

Let M be any finitely generated A-module of the form $A/\mathfrak{a}$ where $\mathfrak{a} \subset \mathfrak{m}^{t_0}$. Now $M/\mathfrak{m}^t M \simeq A/(\mathfrak{m}^t + \mathfrak{a})$. In particular,

$$M/\mathfrak{m}^{t_0}M = A/\mathfrak{m}^{t_0}.$$

Write:

$$\begin{tikzcd} 0 \arrow[r] & \frac{\mathfrak{m}}{\mathfrak{m}^{t_0}} \arrow[r] \arrow[d] & \frac{A}{\mathfrak{m}^{t_0}} \arrow[r] \arrow[d] & \frac{A}{\mathfrak{m}} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & Z \arrow[r] & \frac{L_{n-1}^*}{\operatorname{Im}(\lambda_{n-1}^*)} \arrow[r] & \frac{A}{\mathfrak{m}} \arrow[r] & 0 \end{tikzcd} \tag{3.3}$$

In the above diagram, the middle vertical map is defined by mapping the image of 1 in $A/\mathfrak{m}^{t_0}$ to a minimal generator of $L_{n-1}^*/\operatorname{Im}(\lambda_{n-1}^*)$ which is killed by $\mathfrak{m}^{t_0}$. The map $L_{n-1}^*/\operatorname{Im}(\lambda_{n-1}^*) \to A/\mathfrak{m}$, is defined as to make the right hand square commute by mapping the minimal generator mentioned above to the image of 1 in $A/\mathfrak{m}$. The module Z is the kernel of this map.

Apply $(-)^*$ to the top row of (3.2) to obtain:

$$0 \longrightarrow L_0^* \longrightarrow L_1^* \longrightarrow \cdots \longrightarrow L_{n-1}^*/\operatorname{Im}(\lambda_{n-1}^*) \longrightarrow 0$$

Thus for $0 \leqslant j \leqslant n-1$, we have that:

$$H^j(L_\bullet^*) = \operatorname{Ext}_A^j(\operatorname{Ext}_A^{n-1}(A/\mathfrak{m}^{t_0}, A), A)$$

Since $\operatorname{Ext}_A^{n-1}(A/\mathfrak{m}^{t_0}, A)$ is killed by $\mathfrak{m}^{t_0}$ and $\operatorname{depth}(A) = n-1$, we see that the above sequence is exact, hence

$$\operatorname{pd}_A(L_{n-1}^*/\operatorname{Im}(\lambda_{n-1}^*)) \leqslant n-1.$$

Thus $\operatorname{Ext}_A^i(L_{n-1}^*/\operatorname{Im}(\lambda_{n-1}^*), S) = 0$ for all $i \geqslant n$ and all A-modules S. Applying $\operatorname{Hom}_A(-, S)$ to (3.3), we obtain the commutative diagram:

$$\begin{tikzcd}
\operatorname{Ext}_A^{i-1}(Z, S) \arrow[r] \arrow[d] & \operatorname{Ext}_A^i(A/\mathfrak{m}, S) \arrow[r] \arrow[d, equal] & 0 \arrow[d] \\
\operatorname{Ext}_A^{i-1}(\mathfrak{m}/\mathfrak{m}^{t_0}, S) \arrow[r] & \operatorname{Ext}_A^i(A/\mathfrak{m}, S) \arrow[r, "0"] & \operatorname{Ext}_A^i(A/\mathfrak{m}^{t_0}, S)
\end{tikzcd}$$

Since $A/\mathfrak{m} \simeq M/\mathfrak{m}M$ and $A/\mathfrak{m}^{t_0} \simeq M/\mathfrak{m}^{t_0}M$, we see that for $t = t_0$,

$$\operatorname{Ext}_A^i(M/\mathfrak{m}M, S) \to \operatorname{Ext}_A^i(M/\mathfrak{m}^t M, S)$$

is zero. Thus if we take the direct limit over t of the right hand side, the induced canonical map is zero. □

Consider a ring A of dimension n and depth $n-1$. If for every integer t we could find a module of the form $A/\mathfrak{a}$ where $\mathfrak{a} \subset \mathfrak{m}^t$ such that

$$\vartheta_i : \operatorname{Ext}_A^i(M/\mathfrak{m}M, S_i) \to \mathrm{H}_\mathfrak{m}^i(M, S_i)$$

is nonzero for some $i > n$, then the above theorem shows that the Canonical Element Conjecture holds for A. Unfortunately we cannot yet show this. However, our next theorem gives a construction for modules such that ϑ_i is nonzero, even when the validity of the Canonical Element Conjecture is unknown.

Theorem 3.2. *Let $(A, \mathfrak{m})$ be a local ring which is not regular and let $(P_\bullet, \rho_\bullet)$ be a minimal free resolution of some finitely generated A-module Q of infinite projective dimension. If $M = \operatorname{Coker}(\rho_i^*)$ for some $i > 2$, then the canonical map*

$$\operatorname{Ext}_A^i(M/\mathfrak{m}M, S_i) \to \mathrm{H}_\mathfrak{m}^i(M, S_i)$$

is nonzero, where $S_i = \operatorname{Syz}_i(M/\mathfrak{m}M)$.

Proof. Argue by way of contradiction. Suppose that for some natural number i, the map

$$\operatorname{Ext}_A^i(M/\mathfrak{m}M, S_i) \to \mathrm{H}_\mathfrak{m}^i(M, S_i)$$

is the zero map. By Lemma 2.3, for $t \gg 0$ and minimal free resolutions $F_\bullet$ and $G_\bullet$ of $M/\mathfrak{m}M$ and $M/\mathfrak{m}^tM$ respectively, there exists a lift $\varphi_\bullet : G_\bullet \to F_\bullet$ of the canonical surjection $\varphi : M/\mathfrak{m}^tM \to M/\mathfrak{m}M$ such that φ_i is zero in the commutative diagram below:

$$
\begin{tikzcd}
\cdots \arrow[r] & G_i \arrow[r, "\tau_i"] \arrow[d, "\varphi_i"] & G_{i-1} \arrow[r] \arrow[d, "\varphi_{i-1}"] & \cdots \arrow[r] & G_1 \arrow[r, "\tau_1"] \arrow[d, "\varphi_1"] & G_0 \arrow[r] \arrow[d, "\varphi_0"] & M/\mathfrak{m}^tM \arrow[r] \arrow[d, "\varphi"] & 0 \\
\cdots \arrow[r] & F_i \arrow[r, "\sigma_i"] & F_{i-1} \arrow[r] & \cdots \arrow[r] & F_1 \arrow[r, "\sigma_1"] & F_0 \arrow[r] & M/\mathfrak{m}M \arrow[r] & 0
\end{tikzcd}
$$

Thus the map between $T_i = \mathrm{Syz}_i(M/\mathfrak{m}^tM)$ and S_i is zero and we have the following commutative diagram with exact rows:

$$
\begin{tikzcd}
0 \arrow[r] & T_i \arrow[r] \arrow[d, "0"] & G_{i-1} \arrow[r] \arrow[d] & T_{i-1} \arrow[r] \arrow[d, "\varphi_{i-2}"] & 0 \\
0 \arrow[r] & S_i \arrow[r] & F_{i-1} \arrow[r] & S_{i-1} \arrow[r] & 0
\end{tikzcd}
$$

Apply the functor $(-)^* = \mathrm{Hom}_A(-, A)$ and write:

$$
\begin{tikzcd}
0 \arrow[rr] & & \mathrm{Ker}(\sigma_i^*) \arrow[r] \arrow[dd] & F_{i-1}^* \arrow[r] \arrow[dd, "\varphi_{i-1}^*"] \arrow[ddl, "\varepsilon"'] & \mathrm{Im}(\sigma_i^*) \arrow[r] \arrow[dd, "0"] & 0 \\
& \mathrm{Im}(\sigma_{i-1}^*) \arrow[ur] \arrow[dd, "\varphi^*"] & & & & \\
0 \arrow[rr] & & \mathrm{Ker}(\tau_i^*) \arrow[r] & G_{i-1}^* \arrow[r] & \mathrm{Im}(\tau_i^*) \arrow[r] & 0 \\
& \mathrm{Im}(\tau_{i-1}^*) \arrow[ur, "\psi"'] & & & &
\end{tikzcd}
$$

Setting $\varphi^* = \varphi_{i-1}^*|_{\mathrm{Im}(\sigma_{i-1}^*)}$ and letting $\iota_F : \mathrm{Im}(\sigma_{i-1}^*) \to F_{i-1}^*$ and ψ be the canonical injections, we have that:

$$\psi \circ \varphi^* = \varepsilon \circ \iota_F \tag{3.4}$$

Let

$$\cdots \longrightarrow L_i \xrightarrow{\lambda_i} L_{i-1} \longrightarrow \cdots \longrightarrow L_1 \xrightarrow{\lambda_1} L_0 \longrightarrow \mathrm{Ker}(\tau_i^*) \longrightarrow 0$$

be a minimal free resolution of $\mathrm{Ker}(\tau_i^*)$. We may now lift ψ to obtain the diagram below:

$$
\begin{tikzcd}
0 \arrow[r] \arrow[d] & F_0^* \arrow[r, "\sigma_1^*"] \arrow[d, equal, "\varphi_0^*"] & F_1^* \arrow[r] \arrow[d, "\varphi_1^*"] & \cdots \arrow[r] & F_{i-2}^* \arrow[r, two heads] \arrow[d, "\varphi_{i-2}^*"] & \mathrm{Im}(\sigma_{i-1}^*) \arrow[d, "\varphi^*"] & \\
0 \arrow[r] \arrow[d] & G_0^* \arrow[r, "\tau_1^*"] \arrow[d, "\psi_{i-2}"] & G_1^* \arrow[r] \arrow[d, "\psi_{i-3}"] & \cdots \arrow[r] & G_{i-2}^* \arrow[r, two heads] \arrow[d, "\psi_0"] & \mathrm{Im}(\tau_{i-1}^*) \arrow[d, "\psi"] & \\
L_{i-1} \arrow[r, "\lambda_{i-1}"] & L_{i-2} \arrow[r, "\lambda_{i-2}"] & L_{i-3} \arrow[r] & \cdots \arrow[r] & L_0 \arrow[r] & \mathrm{Ker}(\tau_i^*) \arrow[r] & 0
\end{tikzcd}
$$

However, we also have the following commutative diagram:

$$
\begin{tikzcd}
0 \arrow[r] \arrow[d] & F_0^* \arrow[r, "\sigma_1^*"] \arrow[d] & F_1^* \arrow[r] \arrow[d] & \cdots \arrow[r] & F_{i-3}^* \arrow[r, "\sigma_{i-2}^*"] \arrow[d] & F_{i-2}^* \arrow[r, two heads] \arrow[d, "\sigma_{i-1}^*"] & \operatorname{Im}(\sigma_{i-1}^*) \arrow[d, "\iota_F"] & \\
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & F_{i-1}^* \arrow[r] \arrow[d] & F_{i-1}^* \arrow[r] \arrow[d, "\varepsilon"] & 0 \\
L_{i-1} \arrow[r, "\lambda_{i-1}"] & L_{i-2} \arrow[r, "\lambda_{i-2}"] & L_{i-3} \arrow[r] & \cdots \arrow[r] & L_1 \arrow[r, "\lambda_1"] & L_0 \arrow[r] & \operatorname{Ker}(\tau_i^*) \arrow[r] & 0
\end{tikzcd}
$$

By (3.4) above we see that we have two lifts of the map $\psi \circ \varphi^* = \varepsilon \circ \iota_F$. Hence we have homotopy maps g and g' such that

$$
\begin{aligned}
\psi_{i-2} \circ \varphi_0^* &= \lambda_{i-1} \circ g + g' \circ \sigma_1^*, \qquad \text{hence} \\
\operatorname{Im}(\psi_{i-2} \circ \varphi_0^*) &= \operatorname{Im}(\lambda_{i-1} \circ g + g' \circ \sigma_1^*),
\end{aligned}
$$

with right hand side of the above equation contained in $\mathfrak{m}L_{i-2}$ as the maps λ_{i-1} and σ_1^* are minimal. Thus, if we assume that the canonical map from $\operatorname{Ext}_A^i(M/\mathfrak{m}M, S_i)$ to $\operatorname{H}_{\mathfrak{m}}^i(M, S_i)$ is zero, we must conclude that:

$$\operatorname{Im}(\psi_{i-2}) \subset \mathfrak{m}L_{i-2}$$

This leads to a contradiction as we will now show that:

$$\operatorname{Im}(\psi_{i-2}) \not\subset \mathfrak{m}L_{i-2}$$

Recall the construction of M and lift the canonical surjection θ to a map of complexes to obtain:

$$
\begin{tikzcd}
P_0^* \arrow[r] \arrow[d, "\theta_i"] & P_1^* \arrow[r] \arrow[d, "\theta_{i-1}"] & P_2^* \arrow[r] \arrow[d, "\theta_{i-2}"] & \cdots \arrow[r] & P_{i-1}^* \arrow[r] \arrow[d, "\theta_1"] & P_i^* \arrow[r] \arrow[d, equal, "\theta_0"] & M \arrow[r] \arrow[d, "\theta"] & 0 \\
G_i \arrow[r] & G_{i-1} \arrow[r] & G_{i-2} \arrow[r] & \cdots \arrow[r] & G_1 \arrow[r] & G_0 \arrow[r] & M/\mathfrak{m}^t M \arrow[r] & 0
\end{tikzcd}
$$

Apply $(-)^*$ and look at:

$$
\begin{tikzcd}
G_{i-2}^* \arrow[rrr, "\tau_{i-1}^*"] \arrow[dr] \arrow[dd] & & & G_{i-1}^* \arrow[r, "\tau_i^*"] \arrow[dd, "\theta_{i-1}^*"] & G_i^* \arrow[dd] \\
& \operatorname{Im}(\tau_{i-1}^*) \arrow[r, hook, "\psi"] & \operatorname{Ker}(\tau_i^*) \arrow[ur] \arrow[dd, "\gamma"] & & \\
P_2 \arrow[rrr] \arrow[drr] & & & P_1 \arrow[r] & P_0 \\
& & U \arrow[ur] & &
\end{tikzcd}
$$

where $U = \operatorname{Syz}_2(Q)$ and $\gamma = \theta_{i-1}^*|_{\operatorname{Ker}(\tau_i^*)}$. Letting $\widetilde{\theta} = \gamma \circ \psi$, we have the following commutative triangle:

$$\begin{tikzcd}
\mathrm{Im}(\tau^*_{i-1}) \arrow[dr, "\psi"] \arrow[dd, "\widetilde{\theta}"'] & \\
& \mathrm{Ker}(\tau^*_i) \arrow[dl, "\gamma"] \\
U &
\end{tikzcd}$$

Abusing notation slightly, from our work above, we have three complexes $G^*_\bullet \to \mathrm{Im}(\tau^*_{i-1})$, $L_\bullet \to \mathrm{Ker}(\tau^*_i)$, and the complex formed by truncating $P_\bullet$ at P_2, which we will denote by $P_\bullet \to U$. In this case, $\psi_\bullet$ is lift of ψ, $\theta^*_\bullet$ lifts $\widetilde{\theta}$, and since $P_\bullet$ is exact we may lift γ to a map of complexes $\gamma_\bullet$. We may put these lifts and complexes together into a long diagram with commutative squares which we examine near degree $i-2$:

$$\begin{tikzcd}
0 \arrow[rr] \arrow[dr] \arrow[dd] & & G^*_0 \arrow[rr, "\tau^*_1"] \arrow[dr, "\psi_{i-2}"] \arrow[dd, equal, "\theta^*_0"'] & & G^*_1 \arrow[rr] \arrow[dr] \arrow[dd] & & \cdots & \\
& L_{i-1} \arrow[rr] \arrow[dl] & & L_{i-2} \arrow[rr] \arrow[dl, "\gamma_{i-2}"] & & L_{i-3} \arrow[rr] \arrow[dl] & & \cdots \\
P_{i+1} \arrow[rr, "\rho_{i+1}"] & & P_i \arrow[rr] & & P_{i-1} \arrow[rr] & & \cdots &
\end{tikzcd}$$

Since $\theta^*_\bullet$ and $\gamma_\bullet \circ \psi_\bullet$ are both lifts of $\widetilde{\theta} = \gamma \circ \psi$, we see that there are homotopy maps, h and h', such that:

$$\theta^*_0 - \gamma_{i-2} \circ \psi_{i-2} = \rho_{i+1} \circ h + h' \circ \tau^*_1$$

However, $\mathrm{Im}(\rho_{i+1} \circ h + h' \circ \tau^*_1) \subset \mathfrak{m}P_i$, and so

$$(\theta^*_0 - \gamma_{i-2} \circ \psi_{i-2}) \otimes_A A/\mathfrak{m} = 0.$$

Hence θ^*_0 and $\gamma_{i-2} \circ \psi_{i-2}$ agree on minimal generators modulo $\mathfrak{m}$. Thus we see that $\mathrm{Im}(\psi_{i-2}) \not\subset \mathfrak{m}L_{i-2}$, yielding a contradiction. □

By the discussion given at the beginning of this section, we see that if we construct an A-module M such that for some $i > 2$

$$\mathrm{Ext}^i_A(M/\mathfrak{m}M, \mathrm{Syz}_i(M/\mathfrak{m}M)) \to \mathrm{H}^i_{\mathfrak{m}}(M, \mathrm{Syz}_i(M/\mathfrak{m}M))$$

is nonzero, then

$$\mathrm{Ext}^j_A(M/\mathfrak{m}M, \mathrm{Syz}_j(M/\mathfrak{m}M)) \to \mathrm{H}^j_{\mathfrak{m}}(M, \mathrm{Syz}_j(M/\mathfrak{m}M))$$

is nonzero for all $0 \leqslant j \leqslant i$. One may wonder if there are rings and modules such that the map

$$\mathrm{Ext}^i_A(M/\mathfrak{m}M, \mathrm{Syz}_i(M/\mathfrak{m}M)) \to \mathrm{H}^i_{\mathfrak{m}}(M, \mathrm{Syz}_i(M/\mathfrak{m}M))$$

is nonzero for all $i \geqslant 0$. Our next corollary answers this question in the affirmative:

Corollary 3.3. *Let $(A, \mathfrak{m})$ be a nonregular Gorenstein ring of dimension n, then there exist infinitely many isomorphism classes of finitely generated A-modules M such that the canonical map*

$$\operatorname{Ext}^i_A(M/\mathfrak{m}M, \operatorname{Syz}_i(M/\mathfrak{m}M)) \to \mathrm{H}^i_{\mathfrak{m}}(M, \operatorname{Syz}_i(M/\mathfrak{m}M))$$

is nonzero for all $i \geqslant 0$.

Proof. Consider a module M constructed as in Theorem 3.2 where $i = n+1$. Let $F_\bullet$ be a minimal free resolution of $M/\mathfrak{m}M$ and let $S_i = \operatorname{Syz}_i(M/\mathfrak{m}M)$. Write

$$0 \longrightarrow S_{n+2} \longrightarrow F_{n+1} \longrightarrow S_{n+1} \longrightarrow 0$$

and look at the corresponding long exact sequences of Ext and generalized local cohomology modules:

$$\begin{tikzcd}
 & \operatorname{Ext}^{n+1}_A(M/\mathfrak{m}M, S_{n+1}) \arrow[r] \arrow[d, "\neq 0"] & \operatorname{Ext}^{n+2}_A(M/\mathfrak{m}M, S_{n+2}) \arrow[d] \\
\mathrm{H}^{n+1}_{\mathfrak{m}}(M, F_{n+1}) \arrow[r] & \mathrm{H}^{n+1}_{\mathfrak{m}}(M, S_{n+1}) \arrow[r, "\delta"] & \mathrm{H}^{n+2}_{\mathfrak{m}}(M, S_{n+2})
\end{tikzcd}$$

Since A is Gorenstein, A has injective dimension n. Thus

$$\mathrm{H}^{n+1}_{\mathfrak{m}}(M, F_{n+1}) = \varinjlim \operatorname{Ext}^{n+1}_A(M/\mathfrak{m}^t M, F_{n+1}) = 0.$$

Thus δ above is injective, and so

$$\operatorname{Ext}^{n+2}_A(M/\mathfrak{m}M, S_{n+2}) \xrightarrow{\neq 0} \mathrm{H}^{n+2}_{\mathfrak{m}}(M, S_{n+2}).$$

A similar proof will work for any $i > n$. □

Remark 3.4. It would be interesting to know if there are non-Gorenstein rings such that $\mathrm{H}^i_{\mathfrak{m}}(M, A) = 0$ for $i \gg 0$, where M has infinite projective dimension.

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